



Louis J. Appell
George Appell
Helen Appell

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MILNE-DOWNEY STANDARD ALGEBRA

BY

WILLIAM J. MILNE, PH.D., LL.D.

Late President of New York State College for Teachers
Albany, N. Y.

AND

WALTER F. DOWNEY, A.B., ED.M.

Head Master English High School
Boston, Mass.

Member of the National Committee on
Mathematical Requirements



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MILNE-DOWNEY STANDARD ALGEBRA

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PREFACE

IN the preparation of this "Standard Algebra," careful consideration has been given to the most modern courses of study, including the "Report of the National Committee on Mathematical Requirements" and the requirements of the "College Entrance Examination Board."

Attention is called to the following outstanding features of the book:

1. There is a gradual transition from *arithmetic* to *algebra*, especially in the first part of the book.

2. The *practical applications* of the subject have been emphasized by laying much stress on the equation, the problem, the formula, and the graph.

3. Many exercises in *algebraic representation* are included, in order to familiarize the pupil with the language of algebra and to help him in making the statements of problems.

4. The *problems* deal with interesting facts gathered from a variety of sources, including science, geometry, and the business world. As far as possible, they involve situations of everyday life.

5. The *graph* is presented with a view to giving pupils an adequate understanding of the various types in common use. Exercises are provided to give experience in representing statistical data by approved methods and in interpreting correctly graphic representations in general. The data on which these exercises are based are interesting as well as practical.

6. The *formula* has received due recognition. Numerous formulas from *geometry* and *science*, with which the pupil

will come in contact, have been introduced to illustrate algebraic processes and to emphasize the notion of the *dependence* of one variable on another. The "functional relation" emphasized by the National Committee is introduced early and is further developed in Chapter XVII.

7. The *simplification* of subject matter has been secured by the omission of all topics recommended for exclusion by the best courses of study and by the thoughtful selection of the material used in the development of each topic.

8. The chapter on *logarithms* precedes that on *progressions*, so that practical applications of the former may be made in the latter.

9. A chapter on the *numerical trigonometry* of the right triangle has been included. This topic is interesting to pupils; it is not difficult; and it provides an opportunity for many significant and practical applications.

10. *Self-reliance* and *accuracy* are developed by repeated insistence on the numerical *check* of algebraic solutions. The exercise of judgment and common sense is encouraged by studying the *approximate character of measurement* and the *degree of accuracy* of results.

11. The *time tests* used furnish an interesting device for developing skill in algebraic technique. The *general review* and *examination questions* at the end of the book provide the teacher with an abundance of additional drill material on all important topics, to supplement the exercises in the body of the book.

Grateful acknowledgment is made to Ralph P. Bliss of the Commercial High School, Brooklyn, New York, for valuable assistance in the preparation of the manuscript.

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STANDARD ALGEBRA

CHAPTER I

INTRODUCTION TO ALGEBRA

From arithmetic to algebra

If you are familiar with the principles of arithmetic, you should have no difficulty with algebra. Algebra, indeed, is simply an extension of arithmetic. In algebra you add, subtract, multiply, and divide integers and fractions, using the signs $+$, $-$, $\times$, and $\div$ to indicate these operations; you factor products, raise numbers to powers and find their roots, just as you do in arithmetic.

Algebra, to be sure, makes a wider application of the principles learned in arithmetic, gives an extended meaning to the signs $+$ and $-$, and introduces new symbols and topics; but you will pass so naturally from familiar to new ideas that your horizon will be enlarged without great effort on your part.

One of the chief distinctions between arithmetic and algebra is the extensive use in algebra of *letters to represent numbers*. Even in arithmetic, where the ten digits, 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0, are generally used, numbers are sometimes represented by letters. Thus, X in Roman numbers represents 10; C, 100; M, 1000, etc. The difference is that in algebra x , c , m , etc., under different conditions may represent different values. Thus, in $x = 6 + 3$, x represents 9, while in $x = 8 - 4$, x represents 4.

In arithmetic, letters are used also in formulas. Thus, the formula for the area of a rectangle, $A = b \times h$, means

that the *number* of square units in the area equals the product of the *number* of units in the base by the *number* of units in the altitude. If b represents the number 5 and h the number 4, then A represents the number 20.

In this chapter you will learn to use letters to simplify the statement and the solution of problems and to shorten the work.

If you wish to represent the value of several coins, you may use the letter q to represent the word *quarters*, d for *dimes*, and c for *cents*. If the number of coins consists of 5 quarters, 4 dimes, and 3 cents, you may write it $5q + 4d + 3c$. How could you write the sum of 2 quarters, 3 dimes, and 4 cents?

It is logical to use initial letters in algebraic expressions as in the above exercise. How could you write eight gross plus ten dozen?

EXERCISES

1. If $h = 100$ and $t = 10$, what is the value of $h + t$?
2. If $q = 25$ cents and $d = 10$ cents, what is the value of $q - d$?
3. If $p = 16$ ounces and $o = 1$ ounce, how many ounces are represented by $p - o$?
4. When $a = 10$ and $b = 6$, what is the value of $a + b$? of $a - b$?
5. In algebra, **multiplication** may be indicated by the raised dot or without any sign.

ILLUSTRATIONS

$a \cdot b$ and ab both mean $a \times b$.

$3a \cdot b$ and $3ab$ both mean $3 \times a \times b$.

- (1) What does bc mean? $2bc$?
- (2) If $b = 2$ and $c = 3$, what is the value of $2bc$? What is the value of $3bc$?
6. What is the value of $2xy$ if $x = 4$ and $y = 5$?

7. Division may be indicated in two ways.

ILLUSTRATIONS

$a \div b$ and $\frac{a}{b}$ mean the same. The expression $\frac{a}{b}$ is called a **fraction**.

$\frac{3a}{b}$ means $3 \times a \div b$.

If $a = 6$ and $b = 3$, what is the value of $\frac{a}{b}$? of $\frac{3a}{b}$?

8. We may write $\frac{1}{2}$ of n either $\frac{1}{2}n$ or $\frac{n}{2}$. What is the value of $\frac{1}{2}n$ when $n = 4$? when $n = 8$?

9. We may write $\frac{1}{4}$ of a either $\frac{1}{4}a$ or $\frac{a}{4}$. What is the value of $\frac{a}{4}$ when $a = 16$? What is the value of $\frac{a}{4}$ when $a = 12$?

10. A boy has m marbles and some one gives him n marbles. How many marbles has he then? If $m = 3$ and $n = 7$, how many marbles has he?

11. C is often used for 100. What is the value of $10C + 6C + 3C$?

12. If w stands for 7 days, how many days are represented by $8w + 12w + 5w$?

13. Printers frequently use the letter M for 1000. How many sheets of paper are there in $5M + 10M - 2M$?

14. Using d for 12 oranges, how many oranges are there in $8d - 3d + 6d$?

15. If x equals 10, what is the value of $6x + 2x - x$? of $5x - 5$?

16. If $x = 6$, what is the value of $\frac{1}{2}x + 4x - 8$? of $3x + 5x + 2x - x$?

17. If R stands for 500 sheets of paper (a ream), how many sheets are there in $10R + 8R - 5R$?

18. Edith is 14 years old. How old was she 4 years ago? a years ago? How old will she be in 3 years? in y years?

19. At c cents each, how much will 5 oranges cost?

20. George won a race by running the distance in t seconds. Represent the time John took if he ran 2 seconds longer.

21. James weighs p pounds. Represent Edward's weight if he weighs 10 pounds less than James.

22. A boy had d dimes. How many dollars did he have? how many cents?

23. If m represents the number of miles a boy can walk in a certain time, indicate the distance his father, who walks twice as fast, can walk in the same time.

24. Mary paid c cents for a pin and half as much for a belt. Represent the number of cents she paid for the belt.

25. What two whole numbers are nearest to 9? to x , if x is a whole number? to a , if a is a whole number?

26. If y is an even number, what are the two nearest even numbers? the two nearest odd numbers?

Abbreviating statements about numbers

Notice the following three ways of stating the same problem:

1. If 5 is added to a certain number, the result is 14. What is the number?

In this method the problem is *stated in words*.

2. What number added to 5 equals 14?

This is a shorter method of stating the problem in words.

3. $5 + n = 14$

This is the briefest way of stating the problem.

In each case it is obvious that the number is 9. Which method do you prefer? Why?

In algebra, we shall see that word statements of problems may be expressed very briefly by *using letters to represent numbers*.

Here are two illustrations :

1. If 5 is added to a certain number, the result is 8. This statement may be condensed as follows :

$$n + 5 = 8$$

2. If 6 is subtracted from a number, the result is 4.

$$n - 6 = 4$$

In the preceding exercises you have been translating word statements into **abbreviated (algebraic)** statements.

An algebraic statement that two quantities are equal is called an **equation**. $n + 5 = 8$ and $n - 6 = 4$ are equations.

The quantity on the left side of the equality sign is called the **first member** of the equation, and the quantity on the right is called the **second member**. Note that here we think of $n + 5$ or of $n - 6$ as one quantity.

EXERCISES

1. Write each of the following statements in the shortest algebraic form, as suggested above :

(a) Martha has some stamps. If you give her 5 stamps more, she will then have 18 stamps.

(b) If 8 is added to a number, the result is 15.

(c) Henry has some marbles. If you give him 7 marbles more, he will then have 22 marbles.

(d) Mary bought 2 books. She then had 12 books.

(e) The sum of a number and 6 is 25.

(f) Four times a number is 28.

(g) Five times a number is 35.

(h) If a number is increased by 50, the result is 180.

(i) The number of pins on a paper plus 60 equals 120.

- (j) Two times a number plus 1 equals 21.
- (k) Three times a number less 1 equals 20.
- (l) If 7 is taken from a number, the result is 8.

2. Write a word statement for $12 - n = 5$.

3. Which of the two word statements below refers to the equation $2m = 28$?

(a) The sum of 2 and a certain number equals 28.

(b) The product of 2 and a certain number equals 28.

Write an equation for the statement above that does not refer to $2m = 28$.

4. Make a word statement that can be expressed by $n + 17 = 35$.

5. Make a word statement for the equation $5p = 35$.

6. Translate or state in words: $n + 5 = 18$.

7. Write as an equation: Three fourths of a certain number is 20.

8. Express as an equation: One half of a certain number is 16.

Solving equations by inspection

The equation is a very important tool in mathematics, as well as in the sciences, where mathematics is widely used. The equation gives us a new method of attacking a problem. Sometimes an equation is so simple that the value of the unknown number can be found by inspection.

ILLUSTRATIONS

(1) $2n = 28$

This equation states that twice a certain number equals 28; hence if we divide by 2, we shall find that $n = 14$.

(2) $3n + 5 = 20$

We subtract 5 from each side of the equation and get $3n = 15$. Next we divide each member by 3 and have $n = 5$.

$$(3) \quad \frac{w}{4} = 5$$

If $\frac{1}{4}$ of w equals 5, we find the value of w by multiplying by 4, which gives us $w = 20$.

EXERCISES

Find the value of the unknown number in each of the following equations:

1. $2x = 20$ (Two times what number equals 20?)

2. $\frac{n}{2} = 8$ (One half of what number equals 8?)

3. $3w + 2 = 11$ (Three times what number plus 2 equals 11?)

4. $4y - 3 = 9$ (Four times what number minus 3 equals 9?)

Translate the following into word statements as in examples 1 to 4 above and find the values of the unknown numbers:

5. $3b + 2 = 11$

6. $2a - 1 = 13$

7. $3 + x = 17$

8. $x - 5 = 15$

9. $18 = 3x$

10. $9r = 45$

11. $2x + 3 = 15$

12. $x - 4 = 8$

13. $5a + 4 = 24$

14. $4a - 1 = 15$

15. $\frac{1}{2}x = 3$

16. $\frac{1}{2}x + 1 = 11$

17. $\frac{2z}{3} = 10$

18. $\frac{3r}{4} = 12$

19. $20 - x = 16$

20. $1 + 2x = 17$

21. $18 = 5 + x$

22. $16 - x = 2$

23. $5 + 2m = 25$

24. $4x - 3 = 29$

25. $\frac{2}{3}x + \frac{1}{3}x = 8$

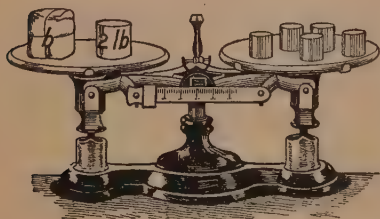
26. $26 = 5a + 1$

27. $4 + 2x = 24$

28. $6 = \frac{x}{100}$

Changing the form of an equation

A study of the balanced scales below will help you to understand the method of handling equations.



BALANCED SCALES

In the figure, you will observe that b (a bag of sugar), with a two-pound weight, balances 5 one-pound weights, or 5 pounds, on the other side. What is the weight of the bag of sugar?

If you take the known weight of 2 pounds from the left side, and wish to keep the scale balanced, what must you take from the right side?

Note that the figure suggests the equation:

$$b + 2 = 5$$

If you take the number 2 from the left-hand side of this equation, what must you take from the right-hand side in order that you may still have an equation?

An **axiom** is a statement so evident that it is accepted without proof.

Notice that in getting from $b + 2 = 5$ the equation $b = 3$, you apply the following axiom:

Subtraction axiom. *The same number may be subtracted from both members of an equation without destroying the equality.*

Apply the subtraction axiom in solving the following equations:

1. $x + 5 = 8$

6. $10 = x + 4$

2. $y + 7 = 12$

7. $12 = c + 2$

3. $z + 9 = 16$

8. $x + 1.5 = 4.5$

4. $a + 12 = 15$

9. $x + 4.1 = 10.1$

5. $30 = m + 5$

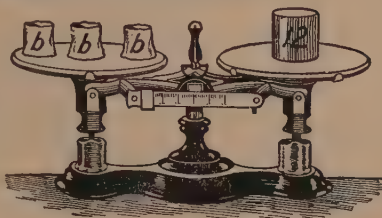
10. $3x + 3 = 7$

Notice that the figure suggests the equation

$$b + b + b = 12$$

If $3b = 12$, $b = ?$

If you divide one member of an equation by some number, what must you do to the other member to preserve the equality?



BALANCED SCALES

In getting from $3b = 12$ the equation $b = 4$, you apply the following axiom:

Division axiom. *Both members of an equation may be divided by the same number without destroying the equality.*

NOTE. Division by zero is excepted.

Apply the division axiom in solving the following:

1. $2x = 10$

5. $8 = 2y$

2. $4x = 12$

6. $42 = 10b$

3. $2r = 26$

7. $3.0 = 1.5n$

4. $4a = 18$

8. $4.5 = 1.5c$

A grocer had some sugar in a sugar bucket. After he had sold 1 pound, he decided to weigh the remainder. He found that the sugar left in the bucket weighed 9 pounds.

If you let p represent the number of pounds the grocer had at first, you may express the above operation by the equation

$$p - 1 = 9$$

Now if you replace the pound of sugar sold, you may express the operation by adding 1 to each member of the above equation, obtaining

$$p = 10$$

Notice that in getting from $p - 1 = 9$ the equation $p = 10$, you apply the following axiom:

Addition axiom. *The same number may be added to both members of an equation without destroying the equality.*

Compare these two equations:

$$(1) \frac{n}{4} = 5$$

$$(2) n = 20$$

What has been done to each member of the first equation to get the second equation? Translate the two equations into complete sentences.

Multiply each member of the equation $\frac{n}{2} = 5$ by 2.

What new equation do you get?

Notice that in getting from $\frac{n}{2} = 5$ the equation $n = 10$, you apply the following axiom:

Multiplication axiom. *Both members of an equation may be multiplied by the same number without destroying the equality.*

Which axiom should you use in solving each of the following equations?

$$1. x - 3 = 6$$

$$4. \frac{n}{2} = 7$$

$$2. t - 5 = 16$$

$$3. x + 4 = 8$$

$$5. 3x = 12$$

Solve the following equations and tell what axiom you are using:

ILLUSTRATION $\frac{x}{3} = 4$

Multiply both members by 3: $3 \times \frac{x}{3} = x$; $3 \times 4 = 12$; $x = 12$
(Multiplication axiom)

$$6. \frac{x}{2} = 3$$

$$9. \frac{1}{2}s = 1.6$$

$$7. \frac{n}{5} = 2$$

$$10. \frac{n}{4} = 2.5$$

$$8. \frac{a}{6} = 3$$

$$11. \frac{1}{5}a = 3.1$$

Summary of axioms used in solving equations

1. *The same number may be added to both members of an equation without destroying the equality.*

2. *The same number may be subtracted from both members of an equation without destroying the equality.*

3. *Both members of an equation may be multiplied by the same number without destroying the equality.*

4. *Both members of an equation may be divided by the same number (except zero) without destroying the equality.*

EXERCISES (Examples 1 to 8, Oral ; 9 to 30, Written)

Find the value of the unknown number in each of the following equations, telling exactly what you do to each member of the equation:

1. $a + 6 = 10$

2. $3x = 18$

3. $36 = 4m$

4. $2y + 3y = 45$

5. $40 = 5x + 4x + x$

6. $x - 2 = 8$

7. $\frac{x}{2} = 3$

8. $\frac{x}{3} = 5$

9. $56 + z = 98$

10. $29 = 2a - 11$

11. $\frac{2x}{3} = 12$

12. $8a - 3 = 29$

13. $16 = 2y + 9$

14. $6t + 9 = 27$

15. $13 = 4m - 7$

16. $2.5 = \frac{1}{2}x$

17. $27 = 6s - 9$

18. $2\frac{1}{2}x = 10$

19. $6b + b - 2 = 12$

20. $8y + 11 = 51$

21. $3x - x - 5 = 15$

22. $5m + 2m - 6 = 50$

23. $6a + 2a - 1 = 15$

24. $5x - 11 = 19$

25. $35 = 8n + 15$

26. $4s + 2\frac{1}{2} = 6\frac{1}{2}$

27. $2\frac{1}{3}s + \frac{2}{3}s = 6$

28. $5.2y - 3.2y = 4$

29. $1\frac{1}{2}y - 1 = 2$

30. $1.5x + 1.0x = 7.5$

Checking the solution of an equation

An equation is said to be **solved** when the correct value of the unknown number is found.

$$\begin{array}{rcl} \text{Let } x + 3 = 9 & \text{be the equation} \\ \quad \quad \quad 3 = 3 & \text{Subtract 3 from each member} \\ \text{Then } x & = 6, & \text{and the equation is said to be solved} \end{array}$$

CHECK. In order that you may be self-reliant, you should *check* or test the correctness of your results. If you substitute the result 6 for x in the equation $x + 3 = 9$, you obtain $6 + 3 = 9$. Since, when 6 is substituted for x , the members of the equation are equal, 6 *satisfies* the condition expressed by the equation $x + 3 = 9$.

A number that satisfies an equation is called the **root** of the equation. Hence 6 is the *root* of $x + 3 = 9$.

EXERCISES (Examples 1 to 4, Oral; 5 to 16, Written)

Solve the following equations and check the results:

1. $3x = 12$

9. $\frac{4a}{5} = 20$

2. $6 = x + 2$

10. $3n - 2 + 7 = 8$

3. $a - 2 = 5$

11. $2p + \frac{1}{3} = \frac{7}{3}$

4. $10 = x + 2$

12. $5n + \frac{1}{2} = 10\frac{1}{2}$

5. $3x + 2 = 8$

13. $8x - \frac{1}{4} = 7\frac{3}{4}$

6. $5x + 6 = 21$

14. $\frac{3a}{4} = 15$

7. $4y - 3 = 13$

15. $7n + 1.3 = 22.3$

8. $\frac{2y}{3} = 10$

16. $3y + 2.5 - 1.5 = 13$

Practice in making algebraic statements

The exercises that follow will give further practice in translating statements about numbers into algebraic statements.

EXERCISES

1. What is the cost of 6 books at 90 cents each? at c cents each?

2. If a man drives a car at the rate of 30 miles an hour, how far does he go in 3 hours? in 4 hours? in h hours?

3. If 4 pencils cost 8 cents, how much does 1 pencil cost? How much do 6 pencils cost? n pencils?

4. A boy has n bats and buys m bats more. How many bats has he then?

5. Express in another form: $\frac{1}{3}$ of m ; $\frac{1}{3}$ of t ; $\frac{1}{3}$ of $2t$.

6. The difference between two numbers is 2. The smaller number is s . Express the larger number.

7. The difference between two numbers is d . The larger number is 1. Express the smaller number.

8. Ruth is c years old. How old was she 7 years ago? How old will she be 10 years hence?

9. A man gave \$300 to his two sons. He gave n dollars to one. Express the amount which he gave to the other.

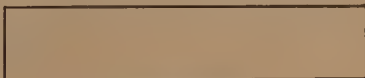
10. A class officer was elected by 15 more votes than his opponent received. If the latter received v votes, express the total number of votes cast.

11. Martha is n years old. If Walter is twice as old, what is his age? What will his age be 5 years hence?

12. There are 4 times as many boys as girls in a certain English class. If there are n girls, how many boys are there? how many pupils? State by an equation that there are 30 pupils in the class. Find the number of boys.

13. An agent sold 4 times as many books in March as in April. Express the number he sold in each of these months.

14. One day Arthur sold half as many papers as his brother sold. Express the number of papers each boy sold.



15. A rectangle is 5 times as long as it is wide. If it is w feet wide, how long is it? What is its **perimeter** (the distance around it)?

RECTANGLE



SQUARE

16. If one side of a square is s inches long, what is the perimeter of the square? State by an equation that the perimeter is 48 inches.

Directions for solving problems

1. Represent one of the unknown numbers by some letter, as n for number.
2. From the conditions of the problem find an expression for each of the other unknown numbers.
3. Find from the conditions two expressions that are equal and write the equation of the problem.
4. Solve the equation.
5. Check the result by reading the problem and substituting the result in the conditions of the problem to see if these conditions are met. Remember that it is not sufficient to check the equation, since you may have written a wrong equation.

EXERCISES

1. One number is 3 times another number and their sum is 8. Find the numbers.

SOLUTION

Let n = the smaller number

Then $3n$ = the larger number

$$(1) \quad n + 3n = 8$$

$$(2) \quad 4n = 8 \quad \text{From (1)}$$

$$(3) \quad n = 2 \quad (2) \div 4$$

$$(4) \quad 3n = 6 \quad 3 \times (3)$$

CHECK. The larger number, 6, is 3 times the smaller number, 2; and the sum of 6 and 2 is 8.

In forming the equation $n + 3n = 8$ from the conditions that the sum of the two numbers equals $n + 3n$ and also 8, we use the following axiom:

Equality axiom. *If two numbers are equal to the same number, they are equal to each other.*

2. Separate 116 into two parts, one of which shall be 3 times the other.

3. Separate 72 into two parts, one of which shall be $\frac{1}{3}$ of the other.

4. What number increased by $\frac{1}{2}$ of itself equals 54?

5. What number decreased by $\frac{1}{5}$ of itself equals 84?

6. Of the steam vessels built on the Great Lakes one year, 21, or 5 less than $\frac{1}{3}$ of all, were of steel. How many steam vessels were built on the Lakes that year?

SOLUTION

Let n = the number of steam vessels built

Then $\frac{n}{3} - 5$ = the number of steel vessels built

Also 21 = the number of steel vessels built

$$(1) \quad \frac{n}{3} - 5 = 21 \quad \text{Equality axiom}$$

$$(2) \quad \frac{n}{3} + 5 - 5 = 21 + 5 \quad (1) + 5$$

$$(3) \quad \frac{n}{3} = 26 \quad \text{Like terms in (2) united}$$

$$(4) \quad n = 78 \quad 3 \times (3)$$

$\therefore 78$ = the number of steam vessels built

Check this problem to see whether it is correct.

NOTE. The sign $\therefore$ means *therefore*.

7. Frank Alden's wages doubled each year for 3 years. At the end of the third year he was getting \$48 a week. What were his wages during the first year?

SOLUTION

Let d = the number of dollars a week that Frank received during the first year

Then $2d$ = the number of dollars a week that he received during the second year

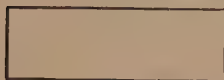
Also $4d$ = the number of dollars a week that he received during the third year

$$(1) \quad 4d = 48 \quad \text{Equality axiom}$$

$$(2) \quad d = 12 \quad (1) \div 4$$

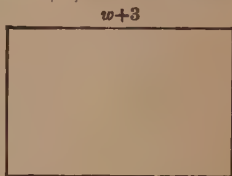
CHECK. The first year he received \$12 a week, the second year, \$24 a week, and the third year, \$48 a week.

8. The rectangle at the left is 3 times as long as it is wide. State algebraically that the perimeter is 3 inches.



What are the dimensions of the rectangle?

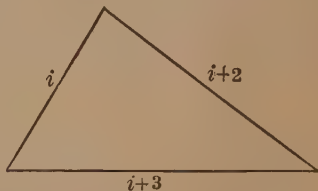
9. The rectangle at the right is 3 units longer than it is wide. Its perimeter is 30 units. What are its dimensions?



10. Susan has \$8.50 in a savings account and is depositing \$.50 a week. Edna has \$12.25 and is depositing \$.25 a week. In how many weeks will Susan have as much as Edna?

SUGGESTION. Let n = the number of weeks in which Susan and Edna will have the same amount of money. Then $8.50 + .50n$ equals what?

11. The perimeter of a triangle is 20 inches. The second side is 2 inches longer than the first side and the third side is 3 inches longer than the first side. Find the length of each side of the triangle.



SOLUTION

Let i = the number of inches in the first side of the triangle

Then $i + 2$ = the number of inches in the second side of the triangle

Also $i + 3$ = the number of inches in the third side of the triangle

$$(1) \quad i + i + 2 + i + 3 = 20 \quad \text{Equality axiom}$$

$$(2) \quad 3i + 5 = 20 \quad \text{Like terms in (1) united}$$

$$(3) \quad 3i = 15 \quad (2) - 5$$

$$(4) \quad i = 5, \text{ the number of inches in the first side} \\ (3) \div 3$$

$$i + 2 = 7, \text{ the number of inches in the second side}$$

$$i + 3 = 8, \text{ the number of inches in the third side}$$

CHECK. 5 inches (the length of the first side) + 7 inches (the length of the second side) + 8 inches (the length of the third side) = 20 inches (the perimeter).

12. At Portland, Oregon, recently vessels were loaded with 50 million feet of lumber for home and foreign ports. Find the foreign shipment, if it was 4 times the shipment to home ports.

13. In a recent baseball game between Detroit and St. Louis American League teams, there was a total of 6 runs scored. Detroit scored twice as many as St. Louis. What was the score?

14. The sum of the angles of a triangle is 180° . If two of them are equal to each other and the third angle is 55° , how many degrees are there in each angle?

15. Mary and her father took an all-day automobile ride, going 3 times as far in the afternoon as in the forenoon. In all they covered 120 miles. How far did they go in the morning?

16. John's radio set cost him 4 times as much as Henry's. Both sets together cost \$125. What was the cost of each?

17. In a high school in Indiana there were 48 more freshmen than seniors. In both classes together there were 144 students. How many freshmen were there?

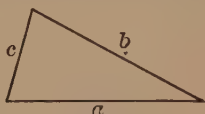
18. In a recent season, the girls' basket-ball team of Denning High School won 4 times as many games as they lost. In all they played 20 games. What was the season's record?

19. A high-school play was given for the benefit of the athletic association. The total sum received from the sale of tickets was twice the amount received from advertisements in the printed program. The amount received from both sources was \$147. How much was received from each source?

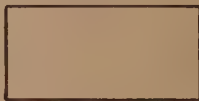
20. The sides of any square are equal in length. How long is one side of a square if the perimeter is 36 inches?



SQUARE



TRIANGLE



RECTANGLE

21. The length of each of the sides a and b of a triangle is twice the length of the side c . If the perimeter is 40 inches, what is the length of each side?

22. The opposite sides of any rectangle are equal. If a rectangle is twice as long as it is wide and its perimeter is 48 inches, how wide is it? how long?

23. Divide 21 into three parts such that the first is twice the second, and the second is twice the third.

SUGGESTION. Let p = the third part; then $2p$ = the second part, and $2 \cdot 2p$ = the first part; that is, $p + 2p + 2 \cdot 2p = 21$.

24. Three newsboys sold 60 papers. If the first sold twice as many papers as the second, and the third sold 3 times as many as the second, how many papers did each sell?

25. The perimeter of a triangle is 37 inches. The second side is 4 inches more than the first side and the third side is 3 inches less than the first side. Find the length of each side.

26. A plumber and two helpers together earned \$15 a day. How much did each earn in a day if the plumber earned 4 times as much as each helper?

27. James bought a pony and a saddle for \$120. If the saddle cost $\frac{1}{3}$ as much as the pony, find the cost of each.

SUGGESTION. Let s = the number of dollars the saddle cost.

28. Find two parts of 92, one of which is $\frac{1}{3}$ of the other.

29. Among 400 persons in a skating rink $\frac{1}{3}$ as many were skaters as spectators. Find the number of each.

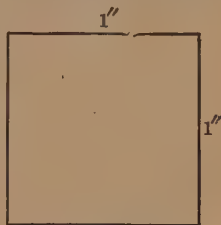
CHAPTER II

THE NATURE OF ALGEBRAIC NUMBERS

Observe the square shown at the right. It is drawn so that each side is 1 inch long. What is the perimeter of this square?

Draw on the blackboard a square whose sides are each 5 inches long. What is its perimeter?

What is the perimeter of a square whose sides are each 3 inches long? 5 feet long? 8 yards long? s inches long?



ONE-INCH SQUARE

An algebraic expression whose parts are not separated by the sign $+$ or $-$ is called a **term**.

Thus, s , $4s$, a , $4ab$, and $\frac{x}{y}$ are terms.

When a term like $4s$ or $4ab$ is made up of a numerical and a literal part, the numerical part, 4, is called the **numerical coefficient** or simply the **coefficient**.

Name the numerical coefficients in $4x$, $5ab$, $6xy$.

In a broader sense any factor or factors of a product may be called the **coefficient** of the other factor or factors. Thus, in $4ab$, $4a$ may be considered the coefficient of b . In aby , a may be considered the coefficient of by , or ab the coefficient of y .

NOTE. Since $1x$ means the same as x , it is not necessary to write the coefficient 1.

EXERCISES

Read the coefficient in each of the following :

1. $2x$

3. $1.5b$

5. $\frac{3}{4}z$

2. $3a$

4. $\frac{1}{2}y$

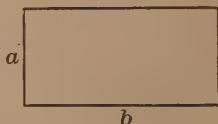
6. $.8x$

Monomials, binomials, trinomials, polynomials

The numbers $2x$, $3y$, and $4s$ and all the numbers in the preceding exercises consist of only one term and are called **monomials**.

1. The rectangle at the right has a base of 1 inch and an altitude of $\frac{1}{2}$ inch. What is its perimeter?

2. What is the perimeter of a rectangle whose base is 5 inches and whose altitude is 3 inches? of another whose dimensions are 6 feet by 5 feet?



RECTANGLE

3. If b represents the base of a rectangle and a the altitude, what does $2a + 2b$ represent? State in words the rule for finding the perimeter of any rectangle.

Notice that $2a + 2b$ is made up of two terms connected by the sign $+$. This sign tells us to add as soon as we have found the values of $2a$ and $2b$.

A number with two terms, like $2a + 2b$, $3x + 2y$, $5z - 3$, $4 + k$, is called a **binomial**.

EXERCISES

Using the expression $p = 2a + 2b$ to represent the perimeter of a rectangle, find the perimeter of each of the following rectangles :

1. $a = 4$ inches

$b = 6$ inches

SOLUTION

$$\begin{aligned} p &= 2a + 2b = 2 \times 4 \text{ inches} + 2 \times 6 \text{ inches} \\ &= 8 \text{ inches} + 12 \text{ inches} = 20 \text{ inches} \end{aligned}$$

2. $a = 3$ feet

$b = 5$ feet

3. $a = 10$ rods

$b = 15$ rods

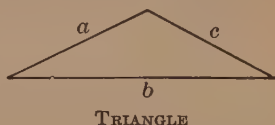
- | | |
|------------------------------|---------------------------|
| 4. $a = 5$ yards | $b = 7$ yards |
| 5. $a = 1\frac{1}{2}$ inches | $b = 2\frac{1}{2}$ inches |
| 6. $a = 2\frac{1}{4}$ feet | $b = 3\frac{1}{4}$ feet |
| 7. $a = 1.1$ inches | $b = 4.1$ inches |
| 8. $a = 2$ feet | $b = 4$ feet |
| 9. $a = 3$ inches | $b = 6$ inches |
| 10. $a = 4$ rods | $b = 8$ rods |

1. What is the perimeter of a triangle whose sides are 6 inches, 8 inches, and 10 inches? of another triangle whose sides are 5 units, 9 units, and 12 units?

2. Write a rule for finding the perimeter of any triangle.

3. What is the perimeter of a triangle whose sides are a , b , and c ?

4. What is the value of the algebraic number $a + b + c$ if a is 3 inches, b is 8 inches, and c is 10 inches?



The expression $a + b + c$ is a number of *three* terms, hence it is called a **trinomial**.

An algebraic number of two or more terms is called a **polynomial**. The word *polynomial* means *containing many terms*. Note that binomials and trinomials are special kinds of polynomials.

EXERCISES

Using the statement $p = a + b + c$ to represent the perimeter of a triangle, find the perimeter of each of the following triangles:

- | | | |
|-------------------|----------------|----------------|
| 1. $a = 2$ inches | $b = 3$ inches | $c = 4$ inches |
|-------------------|----------------|----------------|

SOLUTION

$$p = a + b + c = 2 \text{ inches} + 3 \text{ inches} + 4 \text{ inches} = 9 \text{ inches}$$

- | | | |
|-----------------|--------------|--------------|
| 2. $a = 5$ feet | $b = 6$ feet | $c = 3$ feet |
|-----------------|--------------|--------------|

Find the perimeter of each of the following triangles:

- | | | |
|------------------------------|---------------------------|--------------------------|
| 3. $a = 12$ rods | $b = 10$ rods | $c = 8$ rods |
| 4. $a = 15$ yards | $b = 13$ yards | $c = 11$ yards |
| 5. $a = 1\frac{1}{2}$ inches | $b = 2\frac{1}{2}$ inches | $c = 3$ inches |
| 6. $a = 3.1$ feet | $b = 4.1$ feet | $c = 2.1$ feet |
| 7. $a = 8$ inches | $b = 9$ inches | $c = 10$ inches |
| 8. $a = 12.1$ meters | $b = 13.1$ meters | $c = 10.1$ meters |
| 9. $a = 6\frac{1}{3}$ yards | $b = 7\frac{1}{3}$ yards | $c = 8\frac{1}{3}$ yards |

10. Classify each of the following algebraic numbers as a monomial, a binomial, or a trinomial:

- | | |
|--------------------|-----------------------|
| (a) $3x$ | (d) $\frac{1}{2} + x$ |
| (b) $3m + 2n + 5x$ | (e) $2a + 2b + 8$ |
| (c) $5x + 2y$ | (f) $.2 + 3y$ |

11. Read the coefficient in each term in the algebraic numbers used in exercise 10.

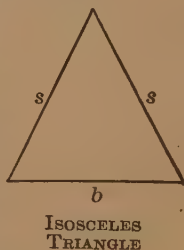
Uniting terms

The triangle at the right is **isosceles** because two of its sides are equal. Show that its perimeter is $s + s + b$.

Terms that differ only in their numerical coefficients are called **like terms** or **similar terms**. In the trinomial $s + s + b$, the first and second terms are similar. Similar terms may be added or united, hence the perimeter of this isosceles triangle may be written $2s + b$. We cannot unite b and s because these are **unlike terms** or **dissimilar terms**.

Are $2x$, $3y$, and $7a$ similar or dissimilar terms? If you are told to add or unite $2x$, $3y$, and $7a$, the only thing you can do is to indicate the sum thus:

$$2x + 3y + 7a$$



EXERCISES

Write the following in the briefest form by uniting like terms:

1. $3a + 2a + a$

5. $2a + b + 3b$

2. $5x + 7x + 2x$

6. $3x + 2y + 5x - y$

3. $8x - 3x + 2x$

7. $s + 2t + 3t + 4s$

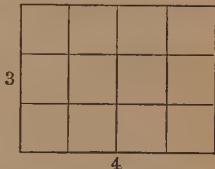
4. $6x - 8x + 2x$

8. $2b + 3c - b - b + c$

Meaning of a number of the form ab

In algebra, we often use such numbers as ab , $2xy$, $3mn$, etc. We need to learn the meaning of such numbers.

The rectangle shown at the right is divided into small squares or square units. We may imagine this figure to be a photograph of a rectangle 4 feet long and 3 feet wide. There are 3 rows of squares with 4 squares in a row. How many square units are there? This number represents the **area** of the rectangle in square feet.



AREA OF RECTANGLE

1. What is the area of a rectangle whose length is 10 inches and whose width is 6 inches? of another whose length is 8 feet and width 4 feet?

2. State in words a rule for finding the area of any rectangle. If b represents the length of the base and a the length of the altitude of a rectangle, what does $a \times b$, or ab , represent?

3. What is the value of ab when a is 8 and b is 3? Ask this question using the words *area*, *base*, and *altitude*.

4. Find the area of a rectangle whose base, b , is 9 feet and whose altitude, a , is 7 feet.

5. Find the value of xy when x is 20 and y is 5; when x is 6 and y is $\frac{1}{3}$.

6. Show by a drawing that xy is the same as yx ; that ab is the same as ba .

7. The expression ab represents a **product** composed of two factors, a and b . How is the product affected if either a or b is increased in value? if either a or b is decreased in value? If a is decreased to zero, what is the value of ab , or $0 \times b$? If b is decreased to zero, what is the value of ab ?

Notice that any product having a zero factor = 0.

8. What is the value of xyz if $x = 1$, $y = 1$, $z = 2$?

9. What is the value of xyz if $x = 0$, $y = 1$, $z = 2$?

10. What is the value of $3mn$ if $m = 0$ and $n = 5$?

Meaning of an exponent

The square shown at the right is divided into square units. There are 4 rows and 4 units in a row; that is, there are as many units in a row as there are rows. What is the area of the square? What is the area of a square whose sides are each 5 units? of one with sides 10 feet long? 8 feet long? What is the general rule in arithmetic for finding the area of any square?



SQUARE

The area of a square, $s \times s$, may be written s^2 . The "2" is called an *exponent*. An **exponent** is a small figure or letter written a little to the right of a number and above it, to show how many times the number is taken as a *factor*.

NOTE. We do not write the exponent 1, since s^1 is the same as s .

1. What is the value of n^2 when n is 8? 6? 5? 1.5? 10?

2. What is the value of $5n^2$ when n is 2? What is the value of $(5n)^2$?

3. Point out the exponents in the following:

$$a^2 \quad a^2b^2 \quad (a)^2 \quad 2x^2 \quad (2x)^2$$

A **cubic unit** is a cube (like the diagram) which is one unit in length, width, and height. How many such units could you place in the cube shown here?

We say its **volume** is 27 cubic units.

4. What is the volume of a cubical box each of whose edges is 5 inches long?

5. Show why the rule for finding the volume of a cube may be written $s \times s \times s$, or s^3 .

6. The exponent 3 in s^3 shows how many times s is taken as a factor. What is the value of s^3 when s is 4 feet? 3 inches? 10 yards?

7. What is the difference between an exponent and a coefficient?

8. Point out the numerical coefficients and the exponents in the following:

$5x^3$

$3a^2b^4$

$4m^3n^5$

$8a^2b^2c^2$

$10x^3y^2z^4$

Illustration of the meaning of a trinomial of the form $x^2 + 2xy + y^2$

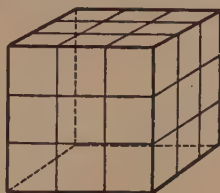
Observe the first square. Each of its sides is 7 units long. If we were to calculate its area, we should get: $5^2 + 2 \times 2 \times 5 + 2^2$, or 49, square units.

How many rectangles are there in the figure? How many squares are there in the figure?

Observe the second square. Each of its sides is $(x + y)$ units long. What is the area of each of the four figures which make up the large square? Show that the area of the second square shown is:

$$x^2 + 2xy + y^2.$$

Read the exponents and the coefficients in each term of the trinomial $x^2 + 2xy + y^2$.



CUBE

5		2	
5 ²		2x5	5
2x5		2 ²	2

x	y	
x^2	xy	x
xy	y^2	y

REVIEW EXERCISES

Read and tell the meaning of each of the following expressions :

1. $8 \div 4$

3. $\frac{m}{n}$

5. $\frac{a}{b} + \frac{r}{s}$

2. $w \div v$

4. $\frac{ab}{3x}$

6. $\frac{a-r}{b+s}$

Indicate :

7. The sum of 5 and 2 ; of x and y .

8. The difference* of 9 and 6 ; of m and n .

9. The product of 3 and 4, in two ways ; the product of r and s , in three ways.

10. The quotient of 8 divided by 5, in two ways ; the quotient of p divided by q , in two ways.

11. The sum of 5 times d and 2 times c .

12. The difference of a times b and 2 times 4.

13. The product of 3 m and n .

14. The quotient of $v - w$ divided by c times d .

15. The product of 2 and $a + b$.

The product of 2 and $a + b$ is written $2(a + b)$ and equals $2a + 2b$.

NOTE. **Parentheses**, (), are used to group numbers when the numbers in each group are to be considered as a single term. Thus, $2(a + b)$ is a single term.

16. The product of $2x + 7$ and $3y - 2$.

The product of $2x + 7$ and $3y - 2$ is indicated thus :

$$(2x + 7)(3y - 2)$$

17. The product of $a - b$ and $5m + 2$.

18. The product of a and $a + b$ divided by the product of b and $a - b$.

* In this book, the *difference* of two numbers means the first number mentioned less the second.

Factors and powers

The product of 2 and 6 or of 3 and 4 is 12, and each of the numbers 2, 6, 3, and 4 is a **factor** of 12. So also, each of the numbers 3, a , and b is a factor of $3ab$, as are also $3a$, $3b$, and ab .

The product obtained by using a number a certain number of times as a factor is called a **power** of that number.

Thus, $2 \times 2 \times 2 \times 2$, briefly written 2^4 , is the fourth power of 2; $a \times a$, or a^2 , is the square of a .

a^2 is read " a square" or " a to the second power"; a^3 is read " a cube" or " a to the third power" or " a exponent 3"; and so on. a^n means that a is to be taken n times as a factor.

EXERCISES

Read, and tell the meaning of:

- | | | | |
|----------|-------------|--------------|---------------------|
| 1. x^6 | 4. x^2y^2 | 7. $3zw^2$ | 10. $9ab^3c^2d^4$ |
| 2. y^4 | 5. a^3b^3 | 8. $4p^3q^5$ | 11. $5p^2q^3s^4t^5$ |
| 3. z^n | 6. r^ns^m | 9. $2m^3n^2$ | 12. $7x^3ym^7n^4$ |

Express in abbreviated form by using exponents:

- | | | |
|---------------------------------|--------------|---|
| 13. $2 \cdot 2$ | 16. $3aaa$ | 19. $2 \cdot 2 \cdot 2 \cdot x \cdot x$ |
| 14. $3 \cdot 3 \cdot 3$ | 17. $8lll$ | 20. $7 \cdot z \cdot z \cdot z \cdot z$ |
| 15. $5 \cdot 5 \cdot 5 \cdot 5$ | 18. $9rrrss$ | 21. $3 \cdot a \cdot a \cdot b \cdot b \cdot b$ |

22. What is the coefficient of x in $3x$? in ax ? in $3ax$?

NOTE. A coefficient is *numerical*, *literal*, or *mixed* according as it is composed of figures, letters, or both. Thus, in $4b$, 4 is the numerical coefficient of b ; in ab , a is the literal coefficient of b ; in $4ab$, $4a$ is the mixed coefficient of b . When not otherwise specified, *coefficient* is used to designate the *numerical coefficient*. (See page 27.)

23. What is the coefficient of t^2 in $5t^2$? in at^2 ? in gt^2 ? in $\frac{1}{2}gt^2$? What kind of coefficient is each?

Name the various factors of:

- | | | | |
|-----------|---------------|-------------|------------|
| 24. ax | 26. x^3 | 28. $6n$ | 30. $pqrs$ |
| 25. $3mn$ | 27. $5r^2s^2$ | 29. $15z^2$ | 31. $24vt$ |

32. From the algebraic expressions given below select :

- (1) Binomials (3) Monomials (5) Similar terms
(2) Trinomials (4) Polynomials (6) Dissimilar terms

$$2ax$$

$$3ax + 2y^2$$

$$3x^2y$$

$$2x^2y - xy + a^2x^2$$

$$2a + 3b$$

$$3a^2x^2 - 4ax + 2d - y^2$$

$$3x + 2b$$

$$6a - c + d + 3a - c^2 - 2a^2$$

33. Find the value of $3 + 4 - 2 + 3$; of $3 \times 4 \div 2 \times 3$.

SOLUTIONS

$$3 + 4 - 2 + 3 = 7 - 2 + 3 = 5 + 3 = 8$$

$$3 \times 4 \div 2 \times 3 = 12 \div 2 \times 3 = 6 \times 3 = 18$$

When only $+$ and $-$ occur in any expression, or only $\times$ and $\div$, the operations are performed in order from left to right.

34. Find the value of :

(a) $3 - 2 - 1 + 8 - 3 + 4$ (b) $5 + 1 - 4 + 3 - 2 + 6$

35. Find the value of $7 + 10 - 6 \div 3 \times 4$.

SOLUTION

$$7 + 10 - 6 \div 3 \times 4 = 7 + 10 - 2 \times 4 = 7 + 10 - 8 = 9$$

Unless otherwise indicated, as by the use of parentheses, when $\times$, $\div$, or both occur in connection with $+$, $-$, or both, the indicated multiplications and divisions are performed first.

Find the value of :

36. $5 \times 10 - 7$

41. $6 + 2 \times 8 - 4 \div 2$

37. $5 \times (10 - 7)$

42. $(6 + 2) \times 8 - 4 \div 2$

38. $2 \times 5 + 3 \times 4$

43. $(6 + 2 \times 8 - 4) \div 2$

39. $(25 - 13) \div 4 \times 2$

44. $6 + 2 \times (8 - 4) \div 2$

40. $16 - 2 \times 2 \times 12 \div 4$

45. $6 + 2 \times (8 - 4 \div 2)$

Read, and tell the meaning of each of these polynomials :

46. $2x^2 + y^2$

49. $3x^2 + 2y - 3z$

47. $3x - 4y$

50. $a^2x^2 - 3xy + 2z^2$

48. $4ab - c^3$

51. $5b^2y + x^2y^2 + 5cz^2$

Represent algebraically :

52. The sum of 5 times a and 3 times the square of x .

53. Three times b , less 3 times the fifth power of a .

54. The product of a , b , and the square of $a - c$.

55. Twice x , divided by 5 times the sum of a , b , and c .

56. Seven times the product of x and y , increased by 3 times the cube of z .

57. The cube of m subtracted from the product of m and n .

58. The product of a used five times as a factor, multiplied by the sum of b and c .

59. Twelve times the square of a , diminished by 5 times the cube of b .

60. Eight times the product of a and b , divided by 4 times the seventh power of c .

Numerical substitution

When a particular number takes the place of a letter or a general number, the process is called **substitution**.

EXERCISES

1. When $a = 2$ and $b = 3$, find the numerical value of $3ab$; of a^4 .

SOLUTIONS

$$3ab = 3 \cdot 2 \cdot 3 = 18$$

$$a^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$$

When $a = 5$, $b = 3$, $c = 10$, $m = 4$, $n = 2$, find the value of :

- | | | | |
|--------------|---------------------------|---------------------------------|-----------------------------------|
| 2. $10a$ | 12. a^2b^2 | 20. $\frac{2a+m}{ab-c}$ | 25. $\frac{a}{b} + \frac{b}{c}$ |
| 3. $2ab$ | 13. $a^b c^{2n}$ | 21. $\frac{ac-m^2}{bc+m}$ | 26. $\frac{m}{a} - \frac{10}{ab}$ |
| 4. $3cm$ | 14. $\frac{1}{3}ab^2$ | 22. $\frac{3a^2+m}{a+bc}$ | 27. $\frac{2}{a} - \frac{7}{abm}$ |
| 5. $6bc$ | 15. $\frac{1}{2}bm$ | 23. $\frac{abc}{ab+c}$ | 28. $\frac{m}{a} + \frac{2b}{c}$ |
| 6. $5m^2$ | 16. $\frac{1}{5}abc$ | 24. $\frac{1}{a} + \frac{1}{b}$ | 29. $\frac{4m}{c} - \frac{m}{a}$ |
| 7. $2a^2b^n$ | 17. $3b^2cm^2$ | | |
| 8. $3bm^3$ | 18. $\frac{a+b}{3a+c}$ | | |
| 9. $4a^3b$ | 19. $\frac{a^2-b^2}{a+b}$ | | |
| 10. a^nm^4 | | | |
| 11. $(ab)^2$ | | | |

30. When $m = 0$ and $n = 4$, find the value of $3 m^2 n$.

SOLUTION

$$3 m^2 n = 3 \cdot 0^2 \cdot 4 = 3 \cdot 0 \cdot 4 = 0$$

When any factor of a product is 0, the product is 0; therefore any power of 0 is 0.

When $a = 4$, $b = 2.5$, $c = 3.2$, and $r = 0$, evaluate:

31. $\frac{r(a+b)}{b+c}$

35. $\frac{b}{a} - \frac{c}{ab}$

39. $\frac{3a}{4b} + \frac{2b}{ac}$

32. $\frac{ab+ar}{b^2+c}$

36. $\frac{6b}{3a} - \frac{ac}{a+bc}$

40. $\frac{c}{2ab} + \frac{5r}{b^2}$

33. $\frac{ac+b}{5c-ac}$

37. $\frac{a}{4b} - \frac{2}{ab} + c$

41. $\frac{ab+c}{bc-a} + \frac{b}{a}$

34. $\frac{a^2+b^2}{a^2-b^2}$

38. $\frac{abr+c}{a+b-c}$

42. $\frac{b}{a} + \frac{a}{c} - \frac{c}{b}$

43. When $x = 3$ and $y = 2$, find the value of $(x+y)^2$.

SOLUTION

$$(x+y)^2 = (3+2)^2 = 5^2 = 25$$

44. Show that $2x + 3x = 5x$ when $x = 2$; when $x = 3$. Giving x any value you choose, find whether $2x + 3x = 5x$.

45. Show that $m(a+b) = ma + mb$ when $m = 5$, $a = 4$, and $b = 3$. Try other values of m , a , and b .

46. Show that $(a-b)^2 = a^2 - 2ab + b^2$ when $a = 4$ and $b = 2$.

When $a = 5$, $b = 3$, $m = 4$, $n = 1$, find the value of:

47. $a^2 + 2ab + b^2$

50. $(b-1)^5$

48. $(a-b)^2$

51. $ab - bn + mb^2 \div 3mn^2$

49. $n^5 - 1$

52. $(ab - bn + mb^2) \div 3mn^2$

REVIEW

Classify each expression according to the number of its terms. Find the value of each expression when $t = 10$.

1. $6t$

4. t^2

7. t^3

2. $7t$

5. $t^2 + 2t + 1$

8. $t^3 + 2t^2 + 5t + 4$

3. $9t + 9$

6. $3t^2 + 6t + 5$

9. $5t^3 + 3t^2 + 8t + 6$

10. (a) Write 25 as a polynomial in t , t representing 10.

(b) Letting t represent 10 and using exponents to represent powers of t , express in polynomial form :

732

523

893

4867

11. (a) What does $2a$ denote? a^2 ?

(b) Illustrate the difference in meaning between $2a$ and a^2 when $a = 1$; when $a = 2$; when $a = 3$; when $a = \frac{1}{2}$; when $a = \frac{1}{3}$.

(c) For what values of a are $2a$ and a^2 equal?

12. Which is the greater, 2^3 or 3^2 ? 4^2 or 2^4 ? 2^5 or 5^2 ? 2^3 or 2^2 ? $(\frac{1}{2})^3$ or $(\frac{1}{2})^2$? 1^3 or 1^2 ?

13. Is the value of a number less than 1 increased or decreased by raising it to any power? the value of a number greater than 1?

14. Find, for $x = 1$, the value of:

$$3x \quad 4x^2 \quad 6x^3 \quad 8x^5 - 4x^4 + 2x^3 - x + 5$$

Name the exponent of x in each term that contains x .

15. Name the coefficient of n in each of these monomials:

$$2n \quad n \quad \frac{1}{2}n \quad bn \quad 3b^2n \quad a^2b^3n$$

16. Write three similar monomials; four dissimilar monomials.

17. If n is a whole number greater than 1 and a is any number, what is the meaning of a^n ?

Find the value of each of the following expressions when $a = 5$, $b = 4$, $c = 3$, $d = 2$, $e = 1$, and $n = 3$:

18. $6ab$; $2cd$; $4nbd$; $\frac{1}{2}ea$; nd^{n+1}

19. $3a^2b$; $3ab^2$; $3(ab)^2$; d^2n^3 ; $(dn)^3$; d^{n-2}

20. $a + b \div d - n \div e$ 22. $10 \div d + 3 \div n - e$

21. $a(b - d) + a - n \div c$ 23. $10 \div (d + 3) + ac \div n$

24. $c^5 + c^4 + 2c^3 - 2c^2 - 3c + 3$

25. $a^4 + b^5 - c^2 - d^2 + e^6$

26. $d^7 + d^6 + 3d^5 - 5d^4 + 2d^3 - 4d^2 + 8d - 1$

27. For what value of x is $12x$ equal to 72?

Write “ $12x$ is equal to 72” as an equation. Solve the equation.

Express the following in algebraic form and solve the equations when you can :

28. Three times a certain number, n , is 21.

29. The sum of a certain number and 3 times the number is 40.

30. Six times a number less 4 times that number is 13.

31. The distance around a square lot, each side of which is a feet long, is 1280 feet.

32. Twice a certain number, less $\frac{1}{3}$ of the number, equals 24.

33. Mary had m books and James had twice as many, the two together having 18 books.

34. John had 50 cents, spent c cents, and earned d cents. How much money had he then?

35. Ida buys x articles at c cents each and offers \$5 in payment. How much change should she receive?

Time tests

(A) *Evaluating algebraic numbers*

Try to do correctly ten of these examples in four minutes.

1. If $x = 3$ and $y = 2$, what does $2x^2 - 3xy$ equal?

2. If $m = 4$ and $n = 3$, what does $3mn + nm^2$ equal?

3. If $a = 3$ and $b = 2$, what does $ab^2 - 2ab$ equal?

4. If $x = 2$ and $y = 5$, what does $\frac{x^2}{2} + \frac{4xy}{5}$ equal?

5. If $a = 2$ and $b = 4$, what does $a^2 + 3a^2b$ equal?

6. If $a = 2$ and $c = 3$, what does $5a^2 - 2ac$ equal?
7. If $a = 2$ and $b = 3$, what does $3ab + a^2b$ equal?
8. If $x = 3$ and $y = 5$, what does $5x^2 - y^2$ equal?
9. If $a = 3$ and $y = 4$, what does $\frac{a^2}{3} + \frac{4ay}{6}$ equal?
10. If $r = 5$ and $s = 3$, what does $r^2 + s^2$ equal?
11. If $x = 2$ and $y = 3$, what does $4x^2 - 2xy$ equal?
12. If $a = 4$ and $b = 3$, what does $ab^2 + 2ab$ equal?
13. If $c = 3$ and $d = 4$, what does $cd^2 - 3cd$ equal?
14. If $x = 5$ and $y = 3$, what does $(x + y)^2$ equal?
15. If $r = 4$ and $s = 5$, what does $3rs - 2s^2$ equal?

(B) *Uniting terms*

These exercises furnish drill in *uniting* or *collecting terms*.
Try to do correctly thirteen examples in four minutes.

1. $6a^2 + 4 - 3a^2 - 2 + 4a^2 + 5$
2. $3x + 8y + 6x - 3y - 4y - 5x$
3. $6x^3 + 3y + 5x^3 + 2y - 9x^3 + y$
4. $4 + 2x^2 + 3 + x^2 + 2x^2 - 5$
5. $8xy + 5 - 2xy + 12 - 3xy + 4$
6. $12a^2 + 4b - 6a^2 - 3b + 5a^2 + 8b$
7. $9x^2 + 10x - 6x^2 - 6x + 8x^2 + 16x$
8. $15x + 8 - 6x + 7 - 7x - 12$
9. $16 + 7b^2 - 12 + 6b^2 + 3 + 4b^2$
10. $20ab - 6ab + 5c - 3c + 5ab + 6c$
11. $12a + 9b^3 + 6a - 12a + 5b^3 - 8b^3$
12. $10 + 6r - 3 - 3r + 12 + 15r$
13. $4a^3 + 5 - a^3 + 4 + 6a^3 - 8$
14. $2a + 3b - a - 2b + 4a - b$

CHAPTER III

GRAPHS

How graphs are used to show number relations

The relations between numbers in a long table may often be made more vivid by a picture because it teaches through the eye. Such a picture is called a **graph**. Extensive use is made of graphs in daily newspapers, in industrial and commercial journals, and in textbooks.

Graphs are frequently used by statisticians to supplement their tables; by stock brokers to show the rise and fall of prices in a given time; by merchants to indicate sales, profits, etc.; by nurses and physicians to mark the progress of diseases; and by scientists to help in investigations.

It is therefore important for all pupils to learn how to interpret and how to construct graphs.

Paper ruled in small squares, called **squared paper**, is convenient for drawing graphs.

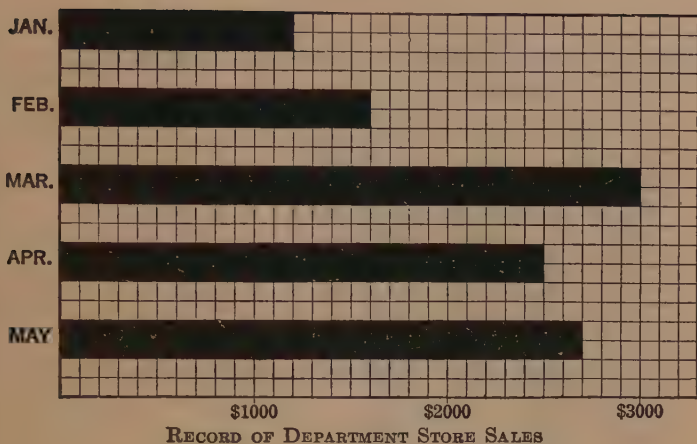
The relative proportion of the dimension of a graph to the corresponding objects represented is called the **scale** of the graph.

The scale of the first graph on page 43 is 1 inch to \$1000.

Reading a graph

(1) The accounting office of a department store sent the record of monthly sales to the manager of the store. With the record was the following **bar graph**:

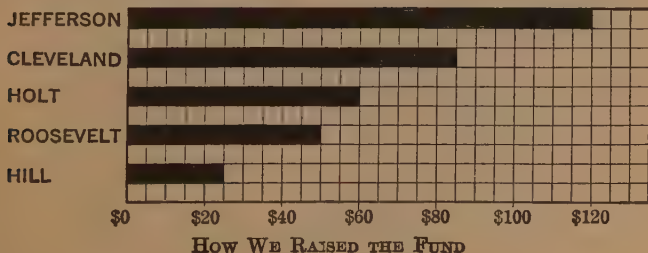
NOTE. The bar graph is often used to picture data without any connected sequence.



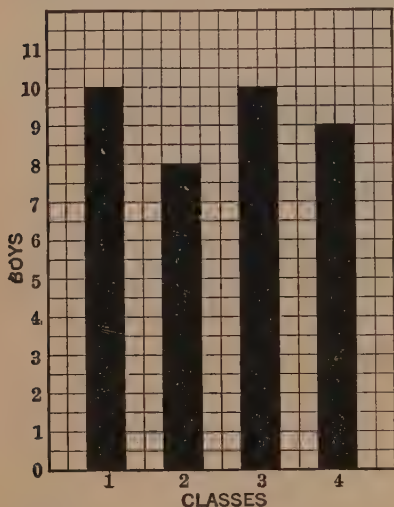
EXERCISES

1. What was the amount of sales in each month?
2. In what month were the sales the smallest? the largest?
3. How much greater were the February sales than those of January?
4. How much less were the April sales than those of March?
5. How much greater were the April sales than the January sales?
6. Compare the sales for May and for February.

(2) The graph below shows the amount raised by five schools for the Near-East Relief Fund.



Make a table of two columns. In one column write the names of the schools represented in the second graph on page 43 and in the other column show how much was collected by each of the schools.

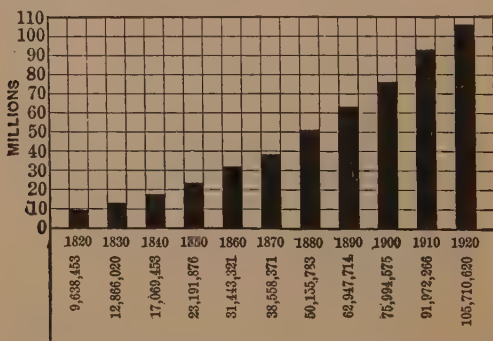


NUMBER OF BOYS ON THE SCHOOL TRACK TEAM

(3) The graph at the left was used by the coach of the school track team. He reported the number of boys from each class in the high school who were regular members of the track team. How many of each class were there on the team?

(4) By looking at the graph below, try to estimate the approximate percentage of increase of population in the United States during each 10-year period from 1820 to 1920.

Check your estimate from the figures given.



POPULATION IN THE UNITED STATES FROM 1820 TO 1920

The vertical measuring line (population line in the last graph) is sometimes called the **vertical axis**, and the horizontal measuring line (year line), the **horizontal axis**.

EXERCISES

1. What is meant by the scale of a graph?
2. What do you notice about the width of the bars? of the spaces between them? Compare the width of the bars and the width of the spaces.
3. How should the items be arranged? Why?
4. Where is the zero of the scale? Could it be omitted in a bar graph?
5. Where is the title? What is the purpose of a title?
6. What rules of neatness and accuracy can you suggest?

How to make a bar graph

Many graphs in newspapers and magazines are not attractive because they are not carefully made. The appearance of bar graphs will be improved by observing the following directions:

(1) The scale should appear on the graph. It should be planned with reference to the size of the paper, to show the smallest number accurately, without forcing the larger numbers to "run off" the page.

(2) The zero of the scale should appear on the graph.

(3) The bars should be of uniform width.

(4) The spaces between the bars should all be the same width as the bars themselves.

(5) The title should be carefully printed, or neatly written, where it is easily seen.

(6) Each bar should have its distinguishing mark or title.

(7) The graph, above all, should be *accurate*.

EXERCISES

Make a bar graph for each of the following :

1. The amount of money paid by the members of the Junior Red Cross in six different schools: \$100, \$50, \$75, \$125, \$45, \$30.

SUGGESTION. Let each space represent \$5.

2. The approximate area in square miles of these four states: Kentucky, 40,000; Nevada, 110,000; Oklahoma, 70,000; South Carolina, 30,000.

SUGGESTION. Let each space represent 5000 square miles.

3. The number of pupils with perfect attendance in a class of 35: January, 30; February, 20; March, 25; April, 32; May, 35.

4. The number of automobiles made in a factory during one week: Monday, 400; Tuesday, 425; Wednesday, 375; Thursday, 410; Friday, 450; Saturday, 225.

5. The growth of the freshman class of a certain high school from 1914 to 1923:

YEAR	NO. OF PUPILS	YEAR	NO. OF PUPILS
1914	90	1919	140
1915	95	1920	142
1916	112	1921	145
1917	100	1922	151
1918	110	1923	165

6. Results in an algebra examination given to 100 pupils:

RESULTS	NO. OF PUPILS
95%-100%	8
90%-94%	7
85%-89%	6
80%-84%	20
75%-79%	14
70%-74%	13
65%-69%	12
60%-64%	12
55%-59%	5
Less than 55%	3
No. Examined	100

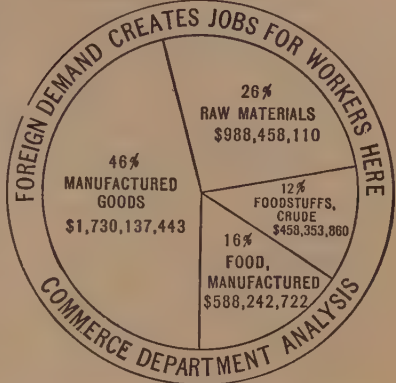
Use of the circle graph

It is sometimes helpful to use a **circle graph**. The diagrams on this page show the use of such graphs.

This type of graph should not be used unless the facts are recorded at the same time and the relations of *all* the parts to each other and to the whole are indicated.

These graphs depend on the fact that the sum of all angles about a given point equals 360° . The point chosen is the center of the circle. An angle of 60° would be drawn to represent $\frac{60}{360}$, or $\frac{1}{6}$, of the total amount, etc.

Shipments Valued at \$3,765,192,135



MANUFACTURES CONSTITUTE 46% OF UNITED STATES EXPORTS

(Reproduced from "The Outline of Business," edited by Ralph F. Couch)

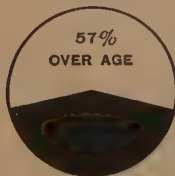
EXERCISES

1. Using the diagram showing the data relating to United States exports, compare the percentage of manufactured goods with the percentage of raw materials exported.

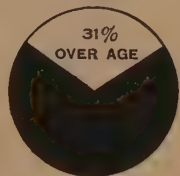
ONE-ROOM SCHOOLS



ALL NON-CITY SCHOOLS



ALL CITY SCHOOLS



PERCENTAGE OF OVER-AGE PUPILS IN A STATE SCHOOL SYSTEM

2. In the over-age chart above, compare the percentage of over-age children in the city schools with the percentage found in non-city schools.

Make a circle graph of each of the following :

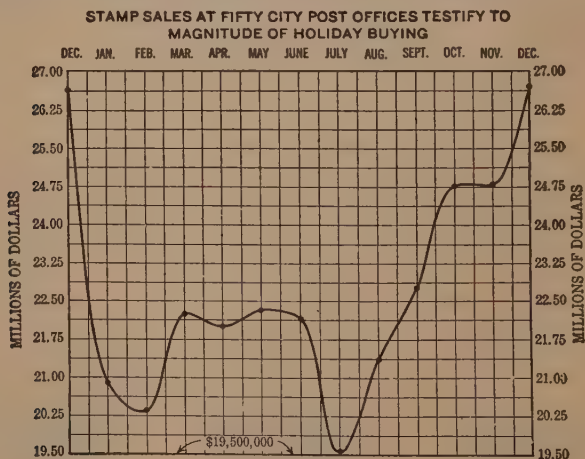
3. A family's monthly account shows the following: rent, \$50; milk, \$6; groceries, \$34; heat and light, \$5; all other expenses, \$25; savings, \$10.

4. Each day Edgar Jerome sleeps 9 hours, spends 2 hours at meals, studies or recites 7 hours, does chores 3 hours, and uses 3 hours for recreation.

5. Of every dollar which Malta City spends \$.23 goes to the schools and library, \$.15 to the parks and streets, \$.20 to the Police and Fire Departments, \$.19 to the Health Department, and the remainder to "overhead and upkeep."

Curved-line graphs

Whenever the data given represent gradual or continuous changes, **curved-line graphs** are found convenient.



December Postal Receipts Highest in History

Business Improvement Reflected in Postal Revenues

Dec. 26,677,571	Mar. 22,236,775	June 22,168,830	Sept. 22,764,414
Jan. 20,902,997	Apr. 22,098,847	July 19,543,153	Oct. 24,777,328
Feb. 20,339,220	May 22,316,240	Aug. 21,372,489	Nov. 24,812,404
			Dec. 26,727,276

(Reproduced from "The Outline of Business," edited by Ralph F. Couch)

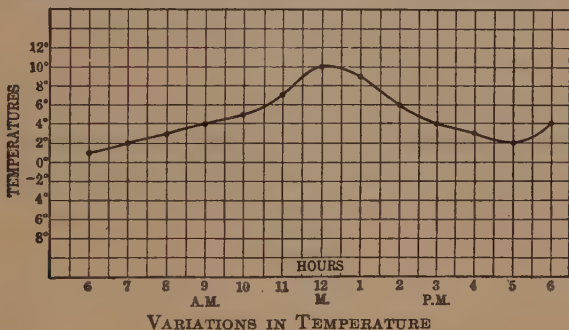
EXERCISES

1. From the graph on page 48 tell in what month the receipts were lowest; in what months there was the least change in receipts.

2. Trace a curve to indicate the hourly changes of temperature on a certain winter day from 6 A.M. to 6 P.M.

Time	6 A.M.	7 A.M.	8 A.M.	9 A.M.	10 A.M.	11 A.M.	12 M.
Readings	1°	2°	3°	4°	5°	7°	10°
Time	1 P.M.	2 P.M.	3 P.M.	4 P.M.	5 P.M.	6 P.M.	
Readings	9°	6°	4°	3°	2°	4°	

This diagram shows the graph for exercise 2. Each space on the vertical axis represents 2° and each space on the horizontal axis, $\frac{1}{2}$ hour.



At 6 A.M. the temperature was 1° and at 12 M., 10°. What was the rise? What was the maximum temperature? the minimum? the fall in temperature from noon to 5 P.M.? What was the temperature at 10.30 A.M.? At what other time was the temperature the same as at 10.30 A.M.?

Draw a graph for each of the following examples:

3. The hourly temperature readings one day were:

Time	6 A.M.	7 A.M.	8 A.M.	9 A.M.	10 A.M.	11 A.M.	12 M.
Readings	20°	25°	28°	34°	40°	42°	50°
Time	1 P.M.	2 P.M.	3 P.M.	4 P.M.	5 P.M.	6 P.M.	
Readings	46°	40°	39°	32°	37°	38°	

SUGGESTION. Let each space on the vertical axis represent 2°.

4. The hourly temperature readings for another day were :

Time	7 A.M.	8 A.M.	9 A.M.	10 A.M.	11 A.M.	12 M.	
Readings	2°	4°	6°	8°	9°	9°	
Time	1 P.M.	2 P.M.	3 P.M.	4 P.M.	5 P.M.	6 P.M.	7 P.M.
Readings :	10°	6°	2°	1°	3°	5°	5°

5. Represent by a curve the growth in population of a village, making use of the following :

Year	1908	1913	1918	1923
Population	1500	1850	2300	2700

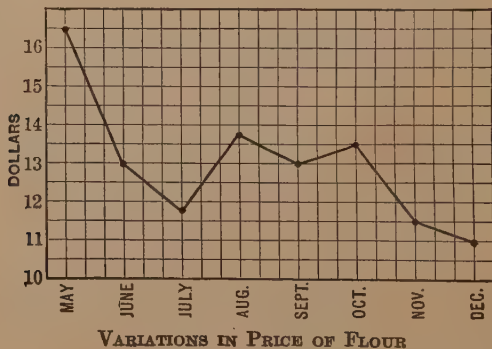
From the graph determine the approximate population in the year 1911 ; in the year 1915 ; in the year 1920. In what year was the population approximately 2000 ? 2500 ?

Broken-line graphs

When the data given represent connected facts recorded at successive intervals they are often represented by **broken-line graphs**. The broken-line graph is used for statistics more frequently than the curved-line graph.

EXERCISES

1. The following graph shows the changes in the price of a barrel of flour from May to the close of a year.



Read the price for each month. A food law was enacted in August. What was the effect of this law as shown by the graph?

2. A pupil in algebra made a graph of his grades for the term: January, 70%; February, 68%; March, 75%; April, 78%; May, 80%; June, 85%. Draw the graph.

3. Draw a graph to show that the attendance in a certain class on Monday was 25; on Tuesday, 23; on Wednesday, 26; on Thursday, 28; and on Friday, 27.

SUGGESTION. Let the horizontal axis represent the days, and the vertical axis the attendance.

4. The average weight of boys at different ages beginning at 6 years and continuing to 15 years is given below:

Age in years	6	7	8	9	10	11	12	13	14	15
Weight in pounds . .	50	53	57	62	67	72	78	85	93	105

(a) Graph the facts stated above.

(b) At about what age should a boy weigh 87 pounds?

(c) At the same rate, how much should a boy 16 years old weigh?

5. A company prepared a diagram on which was indicated the following percentage of increase in the cost of labor for eight years. Draw the diagram.

Year . .	1916	1917	1918	1919	1920	1921	1922	1923
Increase .	73%	74%	135%	150%	175%	160%	140%	120%

6. A boy working after school and on Saturday earned the following amounts. Draw a graph to show his earnings.

Monday, \$.75	Wednesday, \$.85	Friday, \$.62
Tuesday, \$1.00	Thursday, \$1.10	Saturday, \$2.12

7. Collect material for several different graphs, and bring into class the drawing of each.

8. In representing statistical material, state when you would prefer to use (1) a bar graph, (2) a circle graph, and (3) a line graph.

REVIEW EXERCISES

Draw graphs to represent each of the following sets of statistics, choosing the method which seems best fitted in each case.

1. During the fall term in a city high school the number of students who reported for the athletic teams was given as follows :

Football	45	Cross-country Running . .	23
Soccer	35	Basket Ball	28

2. At the close of the year a boy found his monthly record of income and expenses as given here. Compare his income and expenses, using a solid line to represent expense and a dotted line to represent income.

	JAN.	FEB.	MAR.	APRIL	MAY	JUNE
Income . .	\$10.50	\$9.25	\$10.00	\$8.50	\$9.75	\$10.25
Expense . .	8.25	7.00	7.20	9.25	6.50	8.75

	JULY	AUG.	SEPT.	OCT.	NOV.	DEC.
Income . .	\$35.00	\$18.00	\$8.75	\$9.20	\$10.50	\$11.25
Expense . .	4.95	15.00	7.60	7.90	8.25	13.50

(a) Show the points where the graphs cross and tell what they indicate.

(b) During July and the first two weeks of August the boy worked on a farm. How is this shown on the graph?

3. The pupils of a senior class of a high school indicated their preferences as to athletics. Of the 90 members 15 said football, 20 baseball, 17 tennis, 6 lacrosse, 10 boating, 12 walking; the remainder chose other forms. Consider the last as one group.

4. In a high school in which the course is divided into 8 terms there were enrolled 489 students as follows :

1st term — 83	3d term — 62	5th term — 57	7th term — 45
2d term — 78	4th term — 65	6th term — 51	8th term — 48

5. The following table gives the number of games won and lost by the winning team of a baseball league for eight consecutive years.

	GAMES WON	GAMES LOST		GAMES WON	GAMES LOST
1st year	53	31	5th year	51	42
2d year	57	33	6th year	63	39
3d year	60	32	7th year	65	40
4th year	55	41	8th year	60	45

6. A boy catalogued his own library of 62 books. He divided them into the following groups: school books 18, travel 8, biography 4, stories 21, adventure 6, and history 5.

7. Draw a graph to show the great increase in the number of automobiles manufactured each year. The numbers given are to the nearest thousand.

1910	10	1913	24	1916	90	1919	305
1911	11	1914	25	1917	128	1920	322
1912	22	1915	74	1918	222	1921	155

8. The mean temperature in the city of New York for two recent years to the nearest degree for each month was:

	JAN.	FEB.	MAR.	APRIL	MAY	JUNE	JULY	AUG.	SEPT.	OCT.	NOV.	DEC.
First year	24	29	41	48	58	68	73	73	67	60	44	38
Second year	34	35	48	55	60	70	76	71	71	56	45	34

Construct a graph to compare these two records.

CHAPTER IV

FORMULAS

Table of related numbers

Bankers and others who have much interest to compute use interest tables to save time. The following is a small section of such a table :

Interest on \$100 at 5%	\$5	\$10	\$15	\$20	\$25	\$30	\$35	\$40	\$45
Time in years	1	2	3	4	5	6	7	8	9

Tell from the table how much interest is due on \$100 if the time is 3 years. What is the interest on \$100 for 5 years? for 7 years? for 9 years? Extend the table to show the interest for 12 years; for 15 years.

Below is a similar table which shows the interest on a $4\frac{3}{4}\%$ Victory note. Extend this table to 10 years. By means of the extended table, find the interest on \$100 for 5 years; for 8 years.

Interest on \$100	\$4.75	\$9.50	\$14.25
Time in years	1	2	3

The arithmetic rule

For such simple problems as we have considered, a table is not necessary. We usually find interest by this rule: "To find the interest on \$100 at 5% for a certain number of years, multiply \$100 by 5% and multiply this product by the number of years." Write the arithmetic rule upon which the second interest table shown above is based.

Tables are often constructed on the basis of some arithmetic rule. After studying the following tables, write as briefly as possible the rule which tells how much George earns by working at odd times in a drug store; the rule which tells how much John earns.

GEORGE

Amount Earned	\$.76	\$1.52	\$1.90
Hours	2	4	5

JOHN

Amount Earned	\$.30	\$.50	\$.70	\$.90	\$1.10
Hours	1	2	3	4	5

The following table shows the distance an object falls at the end of a given number of seconds:

Distance in feet an object falls	16	64	144	256	400
Number of seconds	1	2	3	4	5

Notice that it falls 16 feet in 1 second, $2^2 \times 16$ feet in 2 seconds, $3^2 \times 16$ feet in 3 seconds, $4^2 \times 16$ feet in 4 seconds, $5^2 \times 16$ feet in 5 seconds. How far will it have fallen at the end of 6 seconds? at the end of 7 seconds? State the general rule.

The formula

An arithmetic rule can be abbreviated by the use of letters.

Int. on \$100 at 5%	\$5	\$10	\$15
Time in years	1	2	3

We have seen that we can write the rule for finding interest as follows: "To find the interest, multiply the principal (\$100) by the rate (5%) by the time expressed in years." All this may be written simply as $i = \frac{5}{100} \times \$100 \times t$, or, still more briefly, $i = 5t$. What is i ? What does t stand for?

The abbreviated statement $i = 5t$ is called a **formula**. It is a practical working rule written in its simplest mathematical form.

EXERCISES

1. State the rule for this interest table:

Interest on \$100 at $4\frac{3}{4}\%$	\$4.75	\$9.50	\$14.25
Time in years	1	2	3

2. Abbreviate your rule for exercise 1 so as to obtain a formula.

3. Another interest table is:

Interest on \$100 at $6\frac{1}{2}\%$	\$6 $\frac{1}{2}$	\$13	\$19 $\frac{1}{2}$
Time in years	1	2	3

Write the formula for finding the interest in this case.

4. (a) If i represents interest, p , principal, r , rate, and t , time in years, tell the meaning of the formula:

$$i = prt$$

- (b) Show that the first three exercises are special cases of this formula.

5. State the rules and write the formulas for the tables of wages, page 55.

6. If a represents *amount*, p , *principal*, r , *rate*, and t , *time* in years, interpret the formula:

$$a = p(1 + rt)$$

7. Find a when $p = \$1000$, $r = 6\%$, and $t = 2$ years.

8. Find a when $p = \$1000$, $r = 6\%$, and $t = 3$ years.

9. Find a when $p = \$500$, $r = 5\%$, and $t = 4$ years.

10. Write as a formula: "The cost of sending a bundle by parcel post into the sixth zone is 9 cents for 1 pound and 8 cents for each additional pound or fraction thereof."

11. Write this rule as a formula: "The cost in cents of sending a package into the third zone equals its weight (in pounds) multiplied by 2 plus 4."

12. In the formula $c = 8w + 1$, what is the cost when the package weighs 5 pounds?

13. In the formula above, what is the cost when the package weighs 6 pounds? 2 pounds? 10 pounds?

14. Write the formula for finding the perimeter of a square when you know a side.

15. What is the perimeter when the side of a square is 5 inches? 6 feet?

16. What is p in $p = 4s$ when $s = 5$ inches? 6 feet? 8 yards? $4\frac{3}{4}$ inches?

17. What is the area of a square when a side is s ? Write your answer as a formula.

18. What is A in the formula $A = s^2$ when $s = 2$ inches? 3 feet? $2\frac{1}{2}$ feet?

19. What is the perimeter of this rectangle? Write your answer as a formula.

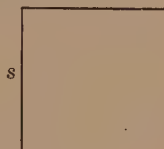
20. What is the perimeter of a rectangle if the base is 10 inches and the altitude is 5 inches?

21. What is p in the formula $p = 2a + 2b$ when a is 8 and b is 3? What is p when a is 6 inches and b is 1 inch?

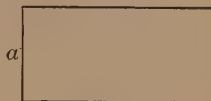
22. State the rule for finding the volume of a cube. Write it as a formula.

23. What is V in $V = e^3$ when $e = 2$ inches? 3 inches? 5 inches? 6 feet?

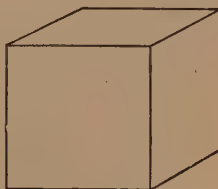
24. The adjoining figure is a parallelogram. The line a drawn perpendicular to the base is called the *altitude*. "The area of a par-



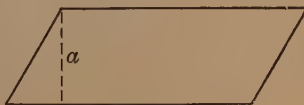
SQUARE



RECTANGLE



CUBE

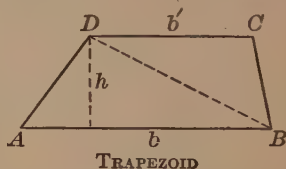


PARALLELOGRAM

allelogram equals the product of the base by the altitude." Write this as a formula.

25. Copy the figure for exercise 24, page 57, and draw the diagonal. Notice that a triangle is one half of a parallelogram. Hence we have: "The area of a triangle equals half the product of the base by the altitude." Write this as a formula.

26. The figure at the right is a trapezoid.* Notice that the diagonal DB divides it into two triangles. The area of triangle ABD equals $\frac{1}{2}bh$. The area of triangle BCD equals $\frac{1}{2}b'h$. Therefore the area of the trapezoid $ABCD$ equals $\frac{1}{2}bh + \frac{1}{2}b'h$. This may be written in the more convenient form $\frac{1}{2}h(b + b')$. Write this formula in words.



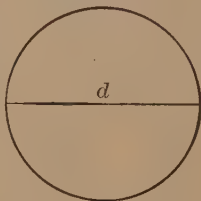
27. Write as a formula: "The volume of a rectangular solid equals the product of its length by its width by its height."

28. What is the value of V in $V = lwh$ if $h = 6$, $w = 4$, and $l = 10$? Translate this problem by a complete sentence.

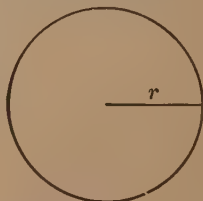
29. Write a formula for the sum of the edges of the cube shown in exercise 22.

30. Write each of the following in the form of a formula:

1. The circumference of a circle is equal to π times the length of the diameter. (The approximate number value for π is $\frac{22}{7}$, or $3\frac{1}{7}$.)



DIAMETER OF CIRCLE



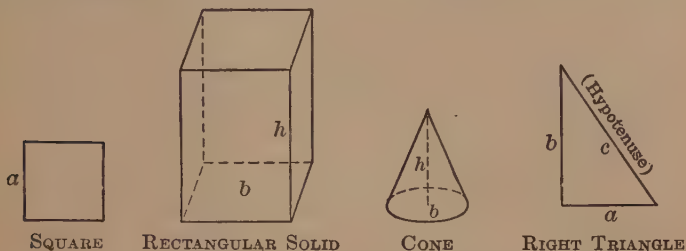
RADIUS OF CIRCLE

2. The area of a circle is equal to π times the square of the length of the radius.

* The bases are b and b' . b' is read "b prime."

3. The area of a square is equal to the square of the length of one side.

4. The volume of a rectangular solid is equal to the area of one of the bases multiplied by the altitude. The bases are rectangles.



5. The volume of a cone is equal to one third the product of the area of the base multiplied by the altitude.

6. The square of the length of the hypotenuse (longest side) of a right triangle is equal to the sum of the squares of the other two sides.

Importance of the formula

Because the formula is the briefest possible statement of a rule with the aid of symbols, it is a convenient tool not only in mathematics, but also in the arts and sciences.

Find in a trade journal or in a book on physics or general science a formula that you can explain to your classmates.

How to draw the graph of a formula

1. Frank receives 30 cents an hour for work in a drug store. Extend the table at the left to 10 hours.

Cents	30	60	90
Hours	1	2	3

(a) If c represents cents and h the hours he worked, write a formula to show how many cents Frank receives in any number of hours.

(b) Use the formula to find how much Frank receives in 4 hours; in $2\frac{1}{2}$ hours; in 6 hours.

2. Draw on squared paper a graph of the formula $c = 30h$. What is the scale of your graph?

First locate a point which represents the first pair of numbers, 1 on the time scale (hours) and 30 on the scale for cents.

Next locate a point for the pair 2 and 60. Indicate the point on the squared paper.

Locate the next point, 3 and 90.

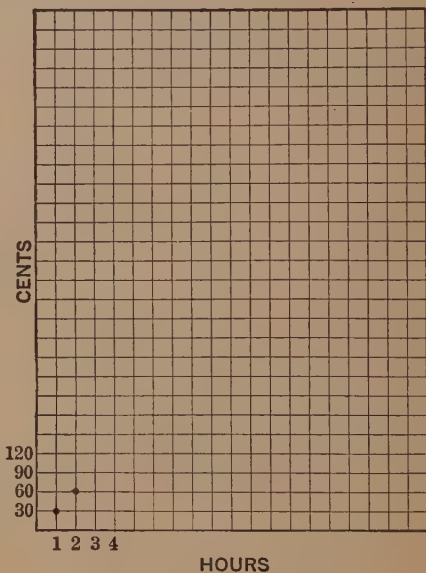
If your table as extended is correct, one pair is 5 and 150. Locate this point on the squared paper.

Locate three other points for three pairs in your table.

What important fact do you note about these points?

Connect these points by a line.

Use your graph to answer these questions: How much does Frank earn in 4 hours? in $2\frac{1}{2}$ hours? in 6 hours?



SCALE FOR GRAPH OF $c = 30h$

Graph of interest formula

In exercise 4, page 56, we found that the formula $i = prt$ represents the simple interest on p dollars at r per cent for t years. We shall now show the graph for finding the interest on \$100 at 6% for different numbers of years, noting the important steps as we work.

Building a table

If $t = 2$ years, $i = 12$

If $t = 3$ years, $i = 18$

If $t = 5$ years, $i = 30$

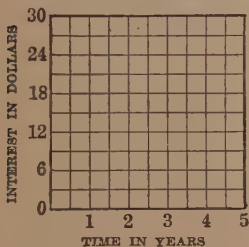
See the table at the right.

INTEREST IN DOLLARS	TIME IN YEARS
12	2
18	3
30	5

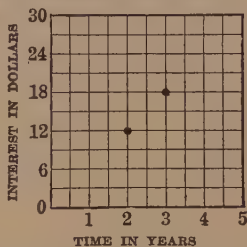
Choosing the scale

Let each two units on the horizontal axis represent 1 year.

Let each two units on the vertical axis represent \$6.



SCALE FOR INTEREST GRAPH



PLOTTING THE POINTS FOR GRAPH OF INTEREST FORMULA

Plotting the points

The first pair of numbers is (2, 12). Count to the right on the horizontal axis to the point 2 and up on the vertical axis to the point 12. Mark a dot for the point that corresponds to this pair. Take another pair (3, 18). Count to the right on the horizontal axis to the point 3 and up on the vertical axis to the point 18, and mark a dot to locate this point. Take a third pair and proceed as above. Plot as many other points corresponding to pairs of numbers in your table as you may need to make you feel sure that you can draw the line.

Drawing the line

If you plot a number of points, you will see that they all lie in a straight line. What is the least number of points that must be plotted? Why would it be well to locate a third point?

Checking the line

In checking the line it is desirable not to take two points too close together. Why?

Another way to check the line is to solve an interest problem by using the line and then solve the same problem by the use of the arithmetic rule for interest.

EXERCISES

1. Using the graph which the class obtained in the preceding work, find the interest at 6% on \$100 for $2\frac{1}{2}$ years; for 3 years; for 9 months; for 5 years.

2. Check the answers for exercise 1, using the arithmetic rule.

3. What is the meaning of the formula $c = 30h$? (See pages 59 and 60.) Here is a table for the formula $c = 30h + 10$. How does the formula $c = 30h + 10$ differ from $c = 30h$? Draw a graph, using the pairs shown in this table.

c	40	70	100
h	1	2	3

4. The interest on Victory notes was $\$4\frac{3}{4}$ on each \$100 for each year. The formula is $i = 4\frac{3}{4}t$. Draw the graph of this formula. Make up two problems for your classmates to solve by the graph.

5. In an automobile race a man drove his car at the rate of 90 miles an hour. Use d for distance and write a formula showing the relation between d and t .

Using the horizontal scale for t and the vertical scale for d , draw a graph from $t = 0$ to $t = 10$.

If you are studying general science in your school, you may be familiar with the centigrade scale for measuring temperature. The ordinary thermometer is on the Fahrenheit scale. The scale units are not the same length. This is one reason why the reading on the Fahrenheit scale is not the same as on the centigrade scale.

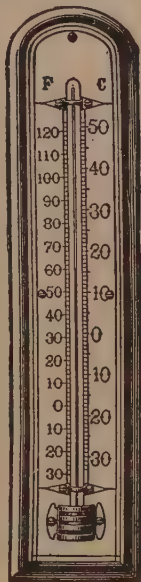
6. If you know the centigrade reading you can get the Fahrenheit reading by the formula: $F = \frac{9}{5} C + 32^\circ$. Tell in words what this formula means. Graph the formula from $C = 0^\circ$ to $C = 60^\circ$.

7. Find by the graph the corresponding Fahrenheit readings for the following centigrade readings: 15° ; 20° ; 30° ; 40° ; 45° .

Straight-line graphs

The graphs of formulas are *straight lines* whenever the unknown numbers are of the first degree only; that is, when the formulas contain no *squares* or *cubes* of the unknown numbers nor the product of two or more unknown numbers. Thus, the graphs of $c = 30h + 10$ and $p = 4s$ are straight lines. The graphs of formulas like $A = s^2$ and $d = vt$ are not straight lines.

A formula is a special kind of equation. When the graph of a formula is a *straight line*, the equation is called a **linear equation**.



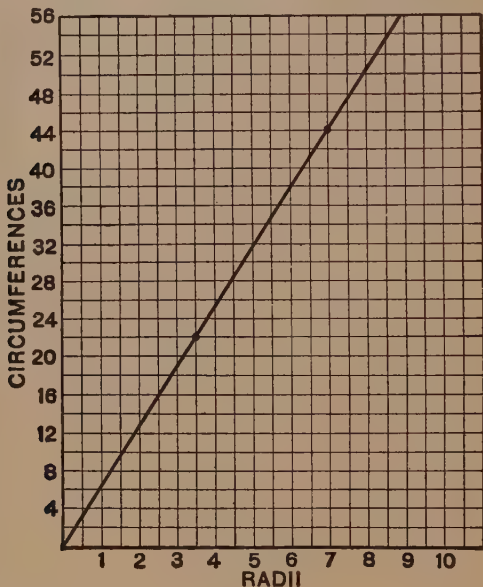
EXERCISES

1. Let c equal the number of cents George earns at shoveling snow. Let h equal the number of hours he works, at 20 cents an hour. Assume that his employer always pays his carfare of 10 cents. Then the formula $c = 20h + 10$ represents George's pay. Draw the graph. Since it is a straight line, the equation $c = 20h + 10$ is a **linear equation**.

2. Use the graph of exercise 1 to tell how long George must work to earn 50 cents; 70 cents; 90 cents.

3. Use the graph to tell how much George earns in 1 hour; in $1\frac{1}{2}$ hours; in 3 hours.

4. The graph of $c = 2\pi r$ (circumference equals twice the radius by $3\frac{1}{7}$) is found below. Since the formula is of the first degree (page 63), the graph is a straight line.

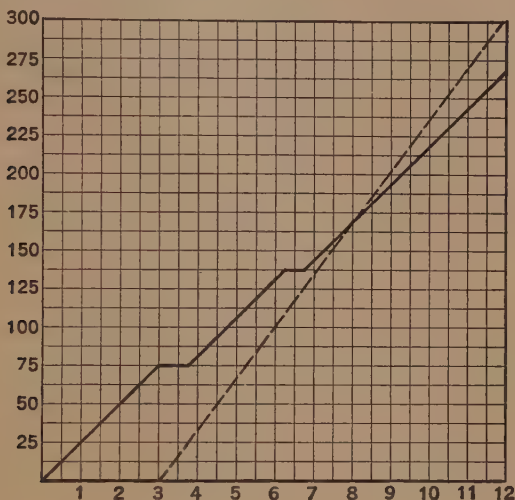


5. Copy the graph of $c = 2\pi r$. Let 2 horizontal spaces represent a unit radius and 1 vertical space two circumference units. Locate one point by letting $r = 0$, $c = 0$. Locate another point by letting $r = 7$, $c = 44$. Connect these points by a straight line and see whether the point found when $r = 3\frac{1}{2}$ lies on this line.

From the graph determine the circumference when $r = 3$, 6, and 9. What connection is there between the rate of increase of r and that of c ?

6. The distance an automobile can travel in h hours at an average rate of 25 miles an hour is found by the formula $d = 25h$. On page 65 is the graph for the following prob-

lem: A car was driven for 3 hours. The driver then stopped $\frac{3}{4}$ of an hour for lunch. After lunch he drove for $2\frac{1}{2}$ hours, then stopped $\frac{1}{2}$ hour for a soft tire. After that he resumed driving the car.



Construct this graph and that for a car which starts from the same place 3 hours after the first car and overtakes the first in 5 hours. From the graph determine the rate of the second car. How far apart will the cars be $1\frac{1}{2}$ hours after the second car starts? 6 hours after the second car starts?

Substitution in a formula.

You could not find the volume of a box by the formula $V = lwh$ unless you knew the value of the height (h), the length (l), and the width (w). In order to solve any formula you must know the values of all its letters except one. You can then *substitute* the known numbers for the letters and do the necessary computation to find the unknown number. This is called **evaluating the formula**.

EXERCISES

1. Given $F = \frac{2}{3}C + 32^\circ$. Find F if $C = 0^\circ$; 90° ; 60° .
2. Given $C = \frac{5}{9}(F - 32^\circ)$. Find C if $F = 50^\circ$; 68° ; 95° .
3. Given $d = rt$. Find d if $r = 85$ miles an hour and $t = 12$ hours.
4. Given $r = \frac{d}{t}$. Find r if $d = 20$ miles and $t = \frac{1}{2}$ hour;
if $d = \frac{2}{3}$ mile and $t = 3\frac{1}{3}$ minutes.
5. Given $V = lwh$. Find V if $l = 12$ feet, $w = 10$ feet, and $h = 14$ feet.
6. Given $A = \pi r^2$. Find A if $r = 5.1$ feet; 5 feet; 10 yards.
7. The charge (c) for a telegram from one city to another, 40¢ for 10 words and 3¢ for each additional word, may be found by the formula $c = 40 + 3(n - 10)$, in which n stands for the number of words. Find the cost of a 16-word message.
8. In the formula $i = prt$, i denotes the interest on a principal of p dollars at simple interest at r per cent for t years. Find i when $p = \$250$, $r = 5\%$, and $t = 2$ years.
9. The formula for the space (s) passed over by a body that moves with uniform velocity (v) during a given time (t) is $s = vt$. Find s when $v = 180$ feet a second and $t = 3$ seconds.
10. What is the formula for the circumference of a circle in terms of the radius? (See page 64.) Find c when $r = 3.5$.
Find r when $c = 110$; 330.

Practice in substitution in a formula

We have used the rule: "The circumference of a circle is $\frac{22}{7}$ times the diameter" and the corresponding formula: $c = \frac{22}{7}d$. It is evident that the relation between the cir-

cumference and the diameter can be expressed in a second way, namely: "The diameter is $\frac{7}{22}$ of the circumference." The new formula is $d = \frac{7}{22} c$. We have changed the subject of our sentence (the arithmetic rule) and the formula is said to be **solved** for the letter d .

Translate the following formulas for the interest laws into complete sentences:

$$(1) i = prt \quad (2) p = \frac{i}{rt} \quad (3) r = \frac{i}{pt} \quad (4) t = \frac{i}{pr}$$

The first is solved for i , and the second for p . For what letter is the third solved? the fourth? It is necessary to remember only one of the interest laws, because we can get all the others from any one. We solve a formula for some other letter by applying the equation axioms since, as we have learned, a formula is an equation.

1. If we divide both sides of the formula (1) $i = prt$ by rt we get (2) $\frac{i}{rt} = p$, or $p = \frac{i}{rt}$, which is (1) solved for p .

2. If we subtract $2b$ from each side of the formula $p = 2a + 2b$, we get $p - 2b = 2a$, or $2a = p - 2b$, which is the first formula solved for $2a$.

Which axiom is used in this case?

3. Compare the formulas:

(1) $i = prt$ and

(3) $\frac{i}{pt} = r$, or $r = \frac{i}{pt}$.

How do we get formula (3) from formula (1)?

$r = \frac{i}{pt}$ is formula (1) solved for r .

EXERCISES

1. Solve $i = prt$ for t .

2. (a) Solve $V = lwh$ for h ; for w ; for l .

(b) Translate each of your results into an arithmetic rule.

(c) Make up a problem applying each formula in (a).

3. Read the following formulas and tell what each represents and for what letter each is solved :

$$(1) A = \pi r^2$$

$$(3) s = \frac{p}{4}$$

$$(2) A = \frac{h}{2}(b + b')$$

$$(4) h = \frac{2A}{b}$$

Tell how the second form in each of the following formulas is derived from the first :

$$4. \text{ If } d = 2r, r = \frac{d}{2}.$$

$$6. \text{ If } i = prt, r = \frac{i}{pt}.$$

$$5. \text{ If } p = 2a + 2b, \\ b = \frac{p - 2a}{2}.$$

$$7. \text{ If } V = lwh, h = \frac{V}{lw}.$$

8. Which of the formulas in exercises 1 to 7 is the formula for interest? for the volume of a rectangular solid? for the area of a trapezoid? for a side of a square? for the altitude of a triangle? for the area of a circle? for the perimeter of a rectangle? What does each letter represent?

9. Which of the formulas below should be used in solving the problem: "In what time will \$300 earn \$45 at 5%?" What is the answer? What does each of the other formulas below represent?

$$(1) i = prt$$

$$(3) p = \frac{i}{rt}$$

$$(2) r = \frac{i}{pt}$$

$$(4) t = \frac{i}{pr}$$

Computations with measured numbers

Measurement is generally an approximation.

Owing to the lack of absolute accuracy in instruments used for measuring and to our inability to read the instruments exactly, measured numbers are only approximately accurate.

When we measure a line with a scale that is accurate to

tenths of an inch, we can estimate its length to hundredths of an inch. We say that a line is 4.25 inches long if it is approximately halfway between 4.2 inches and 4.3 inches long, but we cannot tell whether it is 4.246 inches or 4.254 inches long. If it is apparently just 4 inches long by this scale, we write it 4.00 inches.

A **significant figure** is one which adds significance as regards accuracy. The figures 1 to 9 and 0's placed between them are always significant; but 0's placed at the left of 1 to 9 are not significant. When placed at the right of 1 to 9, 0's are sometimes significant and sometimes not significant. They are not significant if they do not represent a definite value, but merely serve to place the decimal point.

Thus, 23,995 has 5 significant figures; 4.03 has 3 significant figures; the exact sum of \$8000 has 4 significant figures; .8000 used for $\frac{4}{5}$ has only 1 significant figure (since the three zeros in this case do not add to the accuracy of the number). If we wish to express 36.86 to the nearest tenth, we write it 36.9, which has 3 significant figures. If we wish to express it to the nearest integer, we write it 37, which has 2 significant figures.

When numerical results are called for correct to the nearest tenth, hundredth, etc., the computation is carried to the next decimal place. If this is less than 5, it is dropped. If it is 5 or more, the preceding figure is increased by 1.

The number of significant figures indicates the degree of accuracy of a measured number.

Any number used as an approximation is called a **round number**. Thus, the circumference of the equator, 24,901.96 miles, in round numbers is 25,000 miles. A round-number estimate of a result serves as a rough check and helps to avoid errors.

The accuracy of results

In computations with approximate numbers, only as many significant figures should be retained in the result as there

are in the factor which has the smallest number of significant figures. In other words, the accuracy of the result is determined by the accuracy of the least accurate part of the given data.

If the base of a rectangle is 12.375 inches and the altitude is 11.25 inches, the area, 139.2 square inches, is as accurate as it can be determined from the data (with four significant figures). The number 139.21875 square inches, obtained by multiplying 11.25 by 12.375, would indicate a degree of accuracy to hundred-thousandths of a square inch, which is greater than the given data justify.

Formulas of plane figures

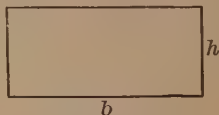
EXERCISES (Examples 1 to 7, Oral; 8 to 16, Written)

The area of a rectangle is equal to the product of its base and its altitude.

FORMULA: $A = bh$

1. Find A when $b = 6$ inches, $h = 4$ inches.

2. Find h when $A = 10$ square inches, $b = 2$ inches.



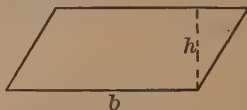
RECTANGLE

The area of a parallelogram is equal to the product of its base and its altitude.

FORMULA: $A = bh$

3. Find A when $b = 10$ feet, $h = 3$ feet.

4. Find b when $A = 12$ square feet, $h = 2$ feet.



PARALLELOGRAM

The area of a square is equal to the square of its side.

FORMULA: $A = s^2$

5. Find A when $s = 4$ inches.

6. Find the area of a rectangular building lot, the dimensions of which are 50 feet by 100 feet.



SQUARE

7. How many square yards are there in the floor of a gymnasium which is in the form of a square, 40 yards on a side?

The area of a triangle is equal to one half the product of its base and its altitude.

$$\text{FORMULA: } A = \frac{1}{2}bh$$

8. Find A when $b = 10$ inches, $h = 8$ inches.

9. Find h when $A = 36$ square inches, $b = 12$ inches.

10. Find A when $b = 6.25$ inches and $h = 7.64$ inches.

The area of a trapezoid is equal to one half the sum of its bases multiplied by its altitude.

$$\text{FORMULA: } A = \frac{1}{2}h(b + b')$$

11. Find A when $h = 6$ inches, $b = 12$ inches, $b' = 8$ inches.

12. Find A when $h = 3.2$ feet, $b = 4.8$ feet, $b' = 3.6$ feet.

In a right triangle, the square of the length of the hypotenuse (c) is equal to the sum of the squares of the lengths of the other two sides (a and b).

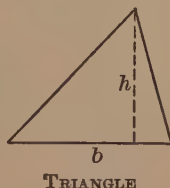
$$\text{FORMULA: } c^2 = a^2 + b^2$$

13. Find c if $a = 3$, $b = 4$.

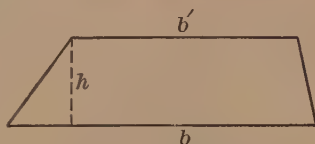
The area of a regular polygon (one which is equilateral and equiangular) is equal to one half the product of its perimeter by its apothem.

$$\text{FORMULA: } A = \frac{1}{2}ap$$

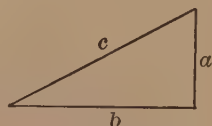
14. Find A if $a = 5.2$ inches, $p = 36$ inches.



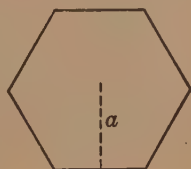
TRIANGLE



TRAPEZOID



RIGHT TRIANGLE



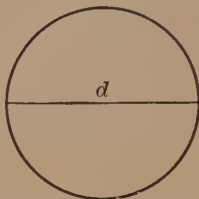
REGULAR POLYGON

The circumference of a circle is equal to the length of the diameter multiplied by π . (The approximate value of π is $3\frac{1}{7}$.)

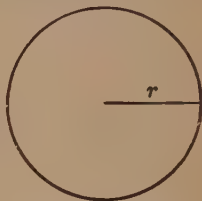
FORMULA: $c = \pi d$

15. Find c when $d = 28$ inches.

16. The diameter of a circle is equal to twice the radius ($d = 2r$). Write the above formula for the



DIAMETER OF CIRCLE



RADIUS OF CIRCLE

circumference of a circle, $c = \pi d$, obtaining a formula for c in terms of π and r . Find c when $r = 15.2$ inches.

Formulas of solids

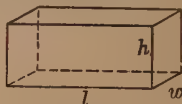
EXERCISES (Examples 1 to 7, Oral; 8 to 13, Written)

The volume of a rectangular solid is equal to the product of its three dimensions.

FORMULA: $V = lwh$

1. Find V when $l = 7$, $w = 2$, $h = 5$.

2. If $V = 48$, $h = 2$, $l = 3$, find w .

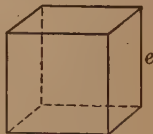


RECTANGULAR
SOLID

The volume of a cube is equal to the cube of its edge.

FORMULA: $V = e^3$

3. Find V when $e = 3$ inches.



CUBE

The surface of a cube is equal to six times the square of its edge.

FORMULA: $S = 6e^2$

4. Find S when $e = 5$ inches.

The volume of a pyramid is equal to one third the product of the area of its base and altitude.

FORMULA: $V = \frac{1}{3} Bh$

5. Find V when $h = 10$ inches and the base (B) is a square 6 inches on a side.



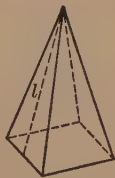
ALTITUDE
OF PYRAMID

The lateral area of a regular pyramid is equal to one half the product of its slant height by the perimeter of its base.

$$\text{FORMULA: } S = \frac{1}{2} lp$$

6. Find S when $l = 11$ inches and the base is a square 5 inches on a side.

7. Find the lateral area of a regular pyramid whose base is an equilateral triangle 6 inches on each side, and whose slant height is 12 inches.



SLANT HEIGHT
OF REGULAR
PYRAMID

The lateral surface of a cylinder of revolution (a cylinder formed by the revolution of a rectangle about one side as an axis) is equal to the product of the circumference of its base by its altitude.

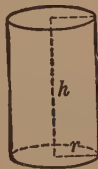
$$\text{FORMULA: } S = 2\pi rh$$

8. Find S when $r = 6\frac{1}{2}$ inches, $h = 14$ inches.

The volume of a cylinder of revolution is equal to the product of the area of the base by its altitude.

$$\text{FORMULA: } V = \pi r^2 h$$

9. Find V when $r = 4$ inches, $h = 8$ inches.



CYLINDER OF
REVOLUTION

The lateral surface of a cone of revolution (a cone formed by the revolution of a right triangle about one arm as an axis) is equal to half the product of the slant height by the circumference of the base.

$$\text{FORMULA: } S = \pi rl$$

10. Find S when $r = 6\frac{1}{2}$ inches, $l = 12$ inches.



CONE OF
REVOLUTION

The volume of a cone of revolution is equal to one third the product of its base by its altitude.

$$\text{FORMULA: } V = \frac{1}{3} \pi r^2 h$$

11. Find V when $r = 3$ inches, $h = 18$ inches.

The area of the surface of a sphere is equal to four times the square of the radius times π .

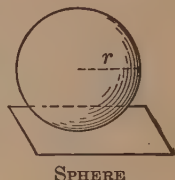
$$\text{FORMULA: } S = 4 \pi r^2$$

12. Find the surface of a sphere whose radius is 7 inches.

The volume of a sphere is equal to four thirds of the cube of the radius times π .

$$\text{FORMULA: } V = \frac{4}{3} \pi r^3$$

13. Find the volume of a sphere whose radius is $3\frac{1}{2}$ inches.



Science formulas

EXERCISES

The formula for uniform motion is :

$$s = vt$$

1. Find t when $s = 100$ meters and $v = 9$ meters a second.

The formula for uniformly accelerated motion is :

$$s = \frac{1}{2} at^2$$

2. Find s in meters when $a = 9.8$ meters per second and $t = 10$ seconds.

The formula for falling bodies (representing the space passed over by s , the constant acceleration due to gravity by g , and the time by t) is :

$$s = \frac{1}{2} gt^2$$

3. Find s (1) when $g = 980$ centimeters, $t = 3$ seconds ; (2) when $g = 32$ feet, $t = 3\frac{1}{2}$ seconds. In each case state the unit in which s is found.

Momentum is equal to mass multiplied by velocity.

$$M = mv$$

4. Find M when $m = 10$ grams, $v = 50,000$ centimeters a second.

The formula for a law of motion when f represents force, m , mass, v , velocity, and t , time, is:

$$f = \frac{mv}{t}$$

5. Find f when $m = 100$ grams, $v = 30$ centimeters a second, and $t = 10$ seconds.

The formula for the relation of weight, mass, and gravity is:

$$w = mg$$

6. Find the weight, w , when mass, m , is 12 centigrams and g is 980 centimeters.

The formula for converting F degrees on the Fahrenheit scale to C degrees on the centigrade scale is:

$$C = \frac{5}{9}(F - 32^\circ)$$

7. Find C when $F = 68^\circ$.

The formula for converting C degrees on the centigrade thermometer scale to the equivalent temperature on the Fahrenheit scale is:

$$F = \frac{9}{5}C + 32^\circ$$

8. Find F when $C = 35^\circ$.

REVIEW EXERCISES

1. Find the lateral surface of a cylinder of revolution if its altitude is 5.5 inches and the diameter of its base is 7 inches.

2. Find the volume of a pyramid, correct to one decimal place, if the base is a square 5.75 inches on a side and the altitude is 8.25 inches.

3. Find to the nearest foot the distance a body will fall in $5\frac{1}{3}$ seconds.

4. Find to the nearest degree the centigrade reading when the Fahrenheit reading is 72° .

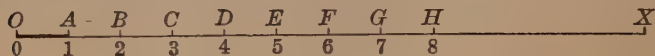
5. What is the area of a triangle, to the nearest tenth of a square foot, if its base is 4.25 feet and its altitude 6.35 feet?

CHAPTER V

POSITIVE AND NEGATIVE NUMBERS

Graphic representation of arithmetic numbers

Positive numbers such as we have in arithmetic may be pictured by lengths of lines.



In the figure above, OX is a scale the unit of which is OA . If it represents part of a yard stick, OA may represent an inch. The scale is made by taking OA as a unit and laying off equal lengths from O toward X . We can extend this scale indefinitely. OB represents two units, OC three units, etc. The lengths of the lines correspond to the numbers 1, 2, 3, etc.

Fractions may also be represented on the scale. Thus, if we place a point halfway between O and A , we can write $\frac{1}{2}$ under this point for a line which represents one half of our unit.

Locate the points which correspond to $2\frac{1}{2}$, $3\frac{1}{4}$, $4\frac{1}{5}$, and 5.25. Show where the lines representing these numbers begin and end.

Negative numbers and their representation

The preceding scale fails to represent some numbers met with in algebra and in the common affairs of life.

On page 77 is a picture of a thermometer. It shows a temperature of 5° below zero. The scale on the thermometer

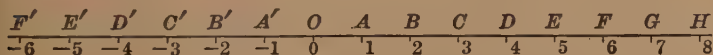
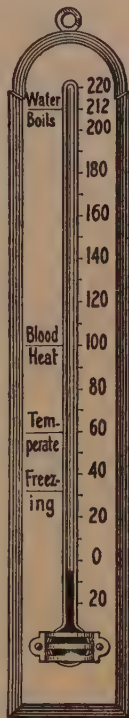
from zero upward is like the scale in the preceding section. The difference is that this scale extends in both directions from zero so that it may picture numbers of *opposite* nature. Thus, the thermometer may represent 10° above zero or 10° below zero.

For the sake of brevity, newspapers, in giving weather reports, distinguish between numbers of opposite nature by using signs. Thus, " 10° above zero" is written $+10^{\circ}$, and " 10° below zero" is written -10° . In algebra, we call a number like " 10° below zero" a **negative number**, and we write it -10° . What would -8° mean on a thermometer scale? -18° ? -6° ? -1° ? $+1^{\circ}$? $+18^{\circ}$?

We therefore have a new meaning for the signs $+$ and $-$. They not only show addition and subtraction (signs of operation in arithmetic), but they tell in what direction from zero a number lies.

An extended number scale

If now we take the scale OX of page 76 and extend it to the left by laying off equal units, we shall have a number scale that includes not only the positive numbers of arithmetic, but also the negative numbers, -1 , -2 , -3 , etc. OA' (read " OA prime") represents one unit, just as OA does. The difference is simply that the two lines ($+1$ and -1) are on *opposite* sides of zero.



Draw a similar scale on the blackboard and place points so as to show lines representing -4 , -3 , -15 , $-2\frac{1}{2}$, $-3\frac{1}{3}$, $-4\frac{1}{5}$, -2.2 , and -3.75 .

It is seen, then, that while in arithmetic the signs $+$ and $-$ are used merely to indicate operations to be performed, in algebra they have an extended meaning and use, that of *opposition*. In this sense, they are called **quality or direction signs**.

Thus, if *gains* are considered positive, indicated by $+$, *losses* are negative, indicated by $-$; if *credits* are $+$, *debits* are $-$; if distances *north* or *east* or *upstream* are $+$, distances *south* or *west* or *downstream* are $-$, respectively.

In order to distinguish between the use of the plus and minus signs as signs of quality and as signs of operations it is well to remember that the sign of operation is *always expressed*, whereas the sign of quality may frequently be omitted. Thus, 5, written without any sign preceding it, means $+5$, but there is only one way to write -5 , namely, -5 . If we wish to indicate both sign of operation and sign of quality we inclose the latter in parenthesis, thus:

$$\begin{array}{ll} 25 + (+5) & 25 - (+5) \\ 25 + (-5) & 25 - (-5) \end{array}$$

For the sake of brevity we generally omit the plus quality sign and write the first and third expressions $25 + 5$ and $25 - 5$.

In algebra positive and negative numbers, whether integers or fractions, are called **algebraic numbers**.

In arithmetic all numbers are positive.

The value of a number without regard to its sign is called, its **absolute value**.

Thus, the absolute value of both $+4$ and -4 is 4.

EXERCISES

Express algebraically:

1. (a) A man has \$10 and loses \$2.
- (b) Peter has 12¢, then earns 5¢ more, but afterwards loses 2¢.

(c) Mr. Smith bought a house for \$5000, and in selling it lost \$500.

2. An aviator flying at an elevation of 2000 feet descends 100 feet, then rises 200 feet.

3. Helen is rowing against the current of a river which is flowing at the rate of 1 mile an hour. She can row 4 miles an hour in still water. How can you find her speed per hour? If she were rowing with the current, what would be her rate per hour?

4. If a player gains 20 points in a game, his score is $+ 20$. What is the score of a player who loses 20 points?

5. Longitude west of Greenwich is often called negative. Where is the place with $- 5^{\circ}$ longitude?

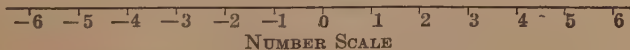
6. If the year $+ 1924$ means 1924 A.D. or after the birth of Christ, what does the year $- 60$ mean?

7. A toy balloon pulls upward with a force of 7 ounces. What happens if you tie a 4-ounce weight to it? a 7-ounce weight? an 8-ounce weight?

NOTE. Keep in mind the fact that when no sign is written the number is positive. Thus, 4 and $+ 4$ are the same. To write the sign $+$ before the first term of an expression merely emphasizes the fact that it is positive.

ADDITION AND SUBTRACTION

Addition in algebra by use of the number scale



Algebraic addition may be easily performed on the number scale.

ILLUSTRATIVE EXAMPLES

1. Add $+ 3$ and $+ 2$.

SOLUTION

Begin at $+ 3$ and count 2 units in the positive direction. The result is 5 spaces to the right of zero or in the positive direction. In other words, $3 + 2 = 5$.

2. Add $+ 3$ and $- 2$.

SOLUTION

Begin at $+ 3$ and count 2 units in the negative direction. The result is one space to the right of zero or in the positive direction. Therefore $3 - 2 = 1$.

3. Add $- 3$ and $+ 2$.

SOLUTION

Begin at $- 3$ and count 2 units in the positive direction. The result is one space to the left of zero or in the negative direction. Therefore $- 3 + 2 = - 1$.

4. Add $- 3$ and $- 2$.

SOLUTION

Begin at $- 3$ and count 2 units in the negative direction. The result is 5 spaces to the left of zero or in the negative direction. Therefore $- 3 - 2 = - 5$.

As seen in the illustrative examples above, addition of positive and negative numbers may be performed by counting along the scale of algebraic numbers.

The result of adding two or more algebraic numbers is called their **algebraic sum**.

This differs from their *arithmetic sum*, which is the sum of their absolute values.

Unless otherwise specified, "sum" in this book means "algebraic sum."

Sum of two or more numbers

EXERCISES

Give the algebraic sums :

1.	2.	3.	4.	5.	6.	7.
$+ 5$	$- 5$	$+ 5$	$+ 5$	$+ 5$	$- 5$	$- 5$
<u>$+ 5$</u>	<u>$- 5$</u>	<u>$- 5$</u>	<u>$- 4$</u>	<u>$- 9$</u>	<u>$+ 8$</u>	<u>$+ 2$</u>

To add two algebraic numbers :

RULE. If they have **like signs**, add the absolute values and prefix the common sign; if they have **unlike signs**, find the difference of the absolute values and prefix the sign of the numerically greater.

By successive applications of the above rule several numbers may be added.

Add:

$$\begin{array}{r} 8. \quad 8 \quad 10. + 8 \quad 12. - 8.1 \quad 14. - 6\frac{1}{2} \quad 16. - 6.3 \\ \underline{2} \quad \underline{+ 2} \quad \underline{+ 12.2} \quad \underline{- 8\frac{3}{4}} \quad \underline{+ 4.7} \end{array}$$

$$\begin{array}{r} 9. \quad 6 \quad 11. + 7.6 \quad 13. - 5.3 \quad 15. + 8.2 \quad 17. - 9\frac{1}{2} \\ - 3 \quad - 3.1 \quad - 3.8 \quad - 9.4 \quad 3\frac{3}{4} \\ \underline{7} \quad \underline{+ 2.9} \quad \underline{- 8} \quad \underline{+ 1.8} \quad \underline{2\frac{1}{4}} \end{array}$$

$$\begin{array}{r} 18. \quad 6.3 \quad 20. - 7.9 \\ - 9.8 \quad .8 \\ \underline{5.2} \quad \underline{- 4.5} \end{array}$$

$$22. 10 + (-4) + (-6)$$

$$23. 10 - 4 - 6 - 7$$

$$24. -40 + 6 + 8 + 7 + 6$$

$$\begin{array}{r} 19. - 8 \quad 21. \quad 8\frac{3}{5} \\ 5.7 \quad - 9\frac{2}{3} \\ \underline{9.3} \quad \underline{7\frac{2}{5}} \end{array}$$

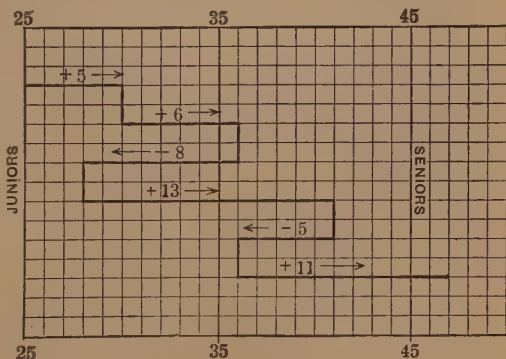
$$25. -12 + 8 + 2 + (-6)$$

$$26. -12 + 8 + 2 - 6 + 2$$

$$27. 8 - 2 + 3 + 6 - 8 - 9$$

28. Julius Cæsar was born in the year -100 (100 B.C.), and was 56 years old when assassinated. In what year was he assassinated?

29. In a football game the ball was advanced 5 yards from the Juniors' 25-yard line toward the Seniors' goal, then -8 yards (that is, it went back 8 yards), and so on, as shown in the diagram. What was the position of the ball after 3 plays? after 4 plays? after 5 plays? after 6 plays?



30. Plot the following and find the last position of the ball:

On 15-yard line; gained 4 yards; gained 5 yards; lost 2 yards; gained 30 yards; lost 6 yards; lost 2 yards; gained 12 yards.

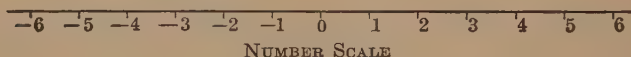
31. How far from port is a vessel if it sails 50 miles, -10 miles (is driven back 10 miles), 40 miles, -30 miles, and 89 miles?

Subtraction in algebra by use of the number scale

In addition, two numbers are given and their algebraic sum is to be found. In subtraction, the algebraic sum and one of the numbers are given, and the other number is to be found.

Subtraction is the *inverse* of addition.

The **difference** is the algebraic number which added to the subtrahend gives the minuend.



ILLUSTRATIVE EXAMPLES

1. Subtract 3 from 5. (What number added to 3 gives 5?)

SOLUTION

Begin at 3 and count in the positive direction till 5 is reached. This requires 2 units. Hence $5 - (+3) = 2$.

2. Subtract $+4$ from -3 . (What number added to $+4$ gives -3 ?)

SOLUTION

Begin at $+4$ and count in the negative direction till -3 is reached. This requires 7 units. Hence $(-3) - (+4) = -7$.

3. Subtract -4 from $+3$. (What number added to -4 gives $+3$?)

SOLUTION

Begin at -4 and count in the positive direction till $+3$ is reached. This requires 7 units. Hence $(+3) - (-4) = +7$.

4. Subtract -4 from -3 . (What number added to -4 gives -3 ?)

SOLUTION

Begin at -4 and count in the positive direction till -3 is reached. This requires 1 unit. Hence $(-3) - (-4) = +1$.

Difference of two numbers

EXERCISES

Subtract the lower number from the upper one:

1. $\begin{array}{r} 6 \\ 3 \\ \hline \end{array}$	3. $\begin{array}{r} 6 \\ 4 \\ \hline \end{array}$	5. $\begin{array}{r} 6 \\ 5 \\ \hline \end{array}$	7. $\begin{array}{r} 6 \\ 6 \\ \hline \end{array}$	9. $\begin{array}{r} 6 \\ 7 \\ \hline \end{array}$	11. $\begin{array}{r} 6 \\ 8 \\ \hline \end{array}$
2. $\begin{array}{r} -3 \\ 0 \\ \hline \end{array}$	4. $\begin{array}{r} -3 \\ 1 \\ \hline \end{array}$	6. $\begin{array}{r} -3 \\ 2 \\ \hline \end{array}$	8. $\begin{array}{r} -4 \\ 3 \\ \hline \end{array}$	10. $\begin{array}{r} -5 \\ 4 \\ \hline \end{array}$	12. $\begin{array}{r} -6 \\ 5 \\ \hline \end{array}$

Observe that *subtracting a positive number is equivalent to adding a numerically equal negative number.*

Subtract the lower number from the upper one:

13. $\begin{array}{r} 4 \\ 0 \\ \hline \end{array}$	15. $\begin{array}{r} 4 \\ -1 \\ \hline \end{array}$	17. $\begin{array}{r} 4 \\ -2 \\ \hline \end{array}$	19. $\begin{array}{r} 4 \\ -3 \\ \hline \end{array}$	21. $\begin{array}{r} 5 \\ -6 \\ \hline \end{array}$	23. $\begin{array}{r} 8.4 \\ -3.7 \\ \hline \end{array}$
14. $\begin{array}{r} -5 \\ 0 \\ \hline \end{array}$	16. $\begin{array}{r} -5 \\ -1 \\ \hline \end{array}$	18. $\begin{array}{r} -5 \\ -2 \\ \hline \end{array}$	20. $\begin{array}{r} -5 \\ -6 \\ \hline \end{array}$	22. $\begin{array}{r} -1 \\ -3 \\ \hline \end{array}$	24. $\begin{array}{r} -46.0 \\ -22.3 \\ \hline \end{array}$

Observe that *subtracting a negative number is equivalent to adding a numerically equal positive number.*

Subtract the lower number from the upper one:

25. $\begin{array}{r} 10 \\ -2 \\ \hline \end{array}$	28. $\begin{array}{r} 12 \\ 5 \\ \hline \end{array}$	31. $\begin{array}{r} 20.9 \\ -6.2 \\ \hline \end{array}$	34. $\begin{array}{r} 16\frac{1}{2} \\ -4\frac{1}{4} \\ \hline \end{array}$	37. $\begin{array}{r} 40 \\ -8 \\ \hline \end{array}$
26. $\begin{array}{r} 0 \\ -2 \\ \hline \end{array}$	29. $\begin{array}{r} -3.5 \\ -6.2 \\ \hline \end{array}$	32. $\begin{array}{r} -7.5 \\ 4.9 \\ \hline \end{array}$	35. $\begin{array}{r} 10.3 \\ -5.8 \\ \hline \end{array}$	38. $\begin{array}{r} -5 \\ 10 \\ \hline \end{array}$
27. $\begin{array}{r} 4 \\ 4 \\ \hline \end{array}$	30. $\begin{array}{r} 4 \\ -4 \\ \hline \end{array}$	33. $\begin{array}{r} -4 \\ 4 \\ \hline \end{array}$	36. $\begin{array}{r} -9.1 \\ 3.9 \\ \hline \end{array}$	39. $\begin{array}{r} -18\frac{2}{3} \\ 6\frac{1}{3} \\ \hline \end{array}$

40. Subtract 12 from -1 . 42. From 0 subtract -3 .
 41. Subtract -4 from 14. 43. From -3 subtract 0.
 44. From 0 subtract -7 ; from the result subtract -4 ; then add -2 ; add -3 ; add 7; subtract 11; and add -6 .
 45. Which is greater and how much, 3 or -5 ? -2 or -5 ? 6 or $8 - 3$? $-2 + (-8)$ or $-2 - (-8)$?

A weather map for January 16 gave the following minimum and maximum temperatures (Fahrenheit):

	CHICAGO	DULUTH	HELENA	MONTREAL	NEW ORLEANS	NEW YORK
Minimum	24°	-6°	-12°	-12°	64°	20°
Maximum	30°	2°	-4°	18°	76°	42°

46. The range of temperature in Chicago was 6° . Find the range of temperature in each of the other cities.
 47. The freezing point is 32° F. How far below the freezing point did the temperature fall in Montreal?
 48. How much colder at the minimum and at the maximum was it in Duluth than in Chicago? in Montreal than in New York? in Helena than in New Orleans? —
 49. An elevator runs from the basement, -22 feet from the first floor, to the tenth story, 105 feet above the first floor. Express its total movement from the basement to the tenth floor; from the tenth floor to the basement.

MULTIPLICATION

Primarily, *multiplication* is the process of taking one number as many times as there are units in another. The principles of multiplication are therefore related to those of addition.

Thus, $3 \times 5 = 5 + 5 + 5 = 15$

In this section and the next, the sign $\times$ is to be read "times."

In arithmetic multiplication is extended to cases that cannot be brought under the original definition; as, $3\frac{2}{3} \times 4$.

So in algebra there are cases, like -3×4 , that do not come under the original definition.

What we are concerned with is the method of finding the product consistent with the laws of operation used in arithmetic and the interpretation of the results obtained.

Product of two factors

ILLUSTRATIVE EXAMPLES

1. What is $(+3)$ times $(+4)$?

SOLUTION

From the definition of multiplication in arithmetic this means

$$(+4) + (+4) + (+4) = +12$$

That is, $(+3) \times (+4) = +12$

2. What is $(+3)$ times (-4) ?

SOLUTION

As in the previous example, this means

$$(-4) + (-4) + (-4) = -12$$

That is, $(+3) \times (-4) = -12$

3. What is (-3) times $(+4)$?

SOLUTION

This means that $(+4)$ is subtracted 3 times.

$$-(+4) - (+4) - (+4) = -12$$

That is, $(-3) \times (+4) = -12$

4. What is (-3) times (-4) ?

SOLUTION

This means that (-4) is subtracted 3 times.

$$-(-4) - (-4) - (-4) = 12$$

Since $-(-4) = 4$ (page 83), we have $+4 + 4 + 4 = 12$

That is, $(-3) \times (-4) = 12$

Summary 1. $(+3) \times (+4) = +12$

2. $(+3) \times (-4) = -12$

3. $(-3) \times (+4) = -12$

4. $(-3) \times (-4) = +12$

The preceding conclusions may be generalized as follows :

$$1. +a \times +b = +ab$$

$$2. +a \times -b = -ab$$

$$3. -a \times +b = -ab$$

$$4. -a \times -b = +ab$$

Hence, for multiplication :

Law of signs. *The sign of the product of two factors is + when the factors have like signs and - when they have unlike signs.*

EXERCISES

1. Multiply each of the following by $+2$; then by -2 :

3, 5, -6 , 10, -8 , -9 , -12

Multiply :

(a)	(b)	(c)	(d)	(e)
2. -8.2	9	6.3	$4\frac{1}{2}$	-2
<u>6</u>	<u>3</u>	<u>-5.4</u>	<u>$-7\frac{1}{2}$</u>	<u>10</u>

3. 7	8	-7	-11	-4
<u>5</u>	<u>-2</u>	<u>10</u>	<u>-12</u>	<u>8</u>

4. 10	$-4\frac{1}{2}$	-8.2	$-2\frac{2}{3}$	$-2\frac{1}{4}$
<u>-5</u>	<u>$2\frac{1}{2}$</u>	<u>-9.7</u>	<u>$-5\frac{1}{3}$</u>	<u>$-1\frac{2}{5}$</u>

Product of several factors

By the law of signs :

$$-a \times -b = +ab$$

$$-a \times -b \times -c = +ab \times -c = -abc$$

$$-a \times -b \times -c \times -d = -abc \times -d = +abcd, \text{ etc.}$$

The product of an even number of negative factors is positive; of an odd number of negative factors, negative.

Positive factors do not affect the sign of the product.

EXERCISES

Find the products indicated :

- | | |
|-----------------------|-----------------------|
| 1. $(-1)(-1)(-1)$ | 5. $(+9)(-2)(-2)$ |
| 2. $(-2)(-2)(-2)(-2)$ | 6. $(+8)(+7)(-2)(-1)$ |
| 3. $(+2)(-3)(-3)(+2)$ | 7. $(+6)(-3)(-4)(-1)$ |
| 4. $(+5)(-3)(+6)$ | 8. $(-1)(+7)(-2)(+7)$ |

DIVISION

Sign of the quotient

In multiplication two numbers are given and their product is to be found. In division the product of two numbers is given and the other number is to be found.

Division is thus the inverse of multiplication.

Thus, $3 \times 4 = 12$ illustrates multiplication ;
but $12 \div 4 = 3$ illustrates division, the *inverse* process.

Since division is the inverse of multiplication, the following are direct consequences of the law of signs for multiplication given on page 86 :

- | | |
|-------------------------|-------------------------------|
| 1. $+a \times +b = +ab$ | $\therefore +ab \div +a = +b$ |
| 2. $+a \times -b = -ab$ | $\therefore -ab \div +a = -b$ |
| 3. $-a \times +b = -ab$ | $\therefore -ab \div -a = +b$ |
| 4. $-a \times -b = +ab$ | $\therefore +ab \div -a = -b$ |

Hence, for division :

Law of signs. *The sign of the quotient is + when the dividend and divisor have like signs and - when they have unlike signs.*

EXERCISES

- Divide each of the following numbers by 2 :
6, - 6, 10, -10, 14, -12, -18, 22, - 8
- Divide each of the foregoing numbers by - 2.

Perform the indicated divisions:

3. $7 \overline{) - 14}$

10. $- 21 \div 3$

17. $\frac{- 63.7}{- 7}$

4. $4 \div (- 4)$

11. $\frac{28}{- 7}$

18. $- 1 \overline{) 9}$

5. $9 \div (- 3)$

12. $\frac{49.2}{- .6}$

19. $- 6 \div .3$

6. $\frac{36}{4}$

13. $- 3 \overline{) - 12}$

20. $- 8 \div 2$

7. $\frac{- 99}{11}$

14. $- 1 \div (- 1)$

21. $\frac{- 4.9}{- .7}$

8. $- 3 \overline{) 15}$

15. $45 \div (- 5)$

16. $\frac{- 42}{6}$

22. $\frac{73.6}{8}$

9. $22 \div (- 2)$

REVIEW

1. What are positive numbers? negative numbers?

2. What two meanings has the minus sign in algebra? If distance north is positive, what is the meaning of $- 150$ miles? of $+ 75$ miles?

3. Distinguish between arithmetic numbers and algebraic numbers.

4. Instead of subtracting a number (positive or negative), what may be done to secure the same result? Illustrate by subtracting $- 7$ from $+ 12$. What is the absolute value of each of these numbers?

5. State the law of signs for multiplication; for division.

6. What is the product of an even number of negative factors? of an odd number of negative factors?

7. $8 + 6 - 4 - 3 + 9 - 16 = ?$

8. $- 9 + 10 - 16 - 4 + 5 = ?$

9. $7.2 - 8.3 + 11.2 - 14.4 + 5.8 + 7.3 = ?$

10. $4.7 - 8.1 - 6 + 5.2 - 3.9 + 7.8 = ?$

11. $- 6 - 4 + 10.5 - 8.3 - 2 + 7.1 - 9 = ?$

12. $7 - 1.8 + 7.6 - 6.5 + 4.6 - 8.1 + 7.2 = ?$

In the next 16 exercises subtract the lower number from the upper one:

13.	20.5 <u>17.3</u>	17.	20.9 <u>-18.5</u>	21.	-15.1 <u>13.3</u>	25.	-9.8 <u>15.7</u>
14.	-28.3 <u>-7.2</u>	18.	7.93 <u>15.2</u>	22.	58.3 <u>3.9</u>	26.	38.3 <u>95.1</u>
15.	$6\frac{3}{4}$ <u>$7\frac{3}{8}$</u>	19.	$-15\frac{1}{2}$ <u>$8\frac{3}{4}$</u>	23.	$-16\frac{2}{5}$ <u>$-17\frac{3}{5}$</u>	27.	$8\frac{2}{3}$ <u>$-6\frac{1}{3}$</u>
16.	$65\frac{1}{2}$ <u>$-7\frac{1}{4}$</u>	20.	$-8\frac{3}{4}$ <u>$-19\frac{1}{8}$</u>	24.	$47\frac{2}{3}$ <u>$-5\frac{1}{4}$</u>	28.	$-7\frac{7}{8}$ <u>$-6\frac{3}{4}$</u>

29. At 9.00 A.M. one day in January the thermometer registered 12° above zero. For the next four hours it rose at the rate of 2° an hour and then began to fall 3° an hour. Find the temperature at noon; at 2.00 P.M.; at 5.00 P.M.

30. If dates after Christ (A.D.) are considered positive, and dates before Christ (B.C.) are considered negative, find the difference in years between 20 B.C. and 140 A.D.; between 400 B.C. and 300 A.D.

31. An investor makes \$300 on one transaction, loses \$500 on a second, and makes \$400 on a third. What is the result of the three transactions?

32. $3 \times (-3) \times (-3) = ?$

33. $7 \times (-4) \times (-2) = ?$

34. $(-6) \times (-4) \times (-9) = ?$

35. $18 \times (-6) \div (-2) = ?$

36. $24 \times (-2) \div (2) = ?$

37. $9 \times (-2) \div (-3) = ?$

38. $(-6) \times (-8) \div (-4) = ?$

39. $(-10) \times (-5) \div (-2) = ?$

40. $(-12) \times 8 \div [(-3) \times 2] = ?$

41. $15 \times (-6) \div [5 \times (-2)] = ?$

ADDITION

In Chapter V we distinguished positive and negative numbers and learned how to add, subtract, multiply, and divide such numbers when represented by figures. We shall now extend these operations to positive and negative numbers represented by letters.

To add monomials

1. How many are 2 plus 5? How many times a number are 2 times the number plus 5 times that number?

2. If n stands for a number, how many times n are 2 times n plus 5 times n ? $2n + 5n = ?$

3. $3 a's + 6 a's = ?$ $3 a + 6 a = ?$

4. $5c's + 7c's + 4c's = ?$ $5c + 7c + 4c = ?$

5. $3 + 4 + 6 = ?$

7. $7 + 4 - 3 = ?$

6. $3 d's + 4 d's + 6 d's = ?$ 8. $7 d + 4 d - 3 d = ?$

9. What kind of terms can we add? Explain how we add these terms.

10. Add $4a$ and $3a$.

SOLUTION

SOLUTION $4a$ $3a$ $\hline 7a$	EXPLANATION. Just as 3 a 's and 4 a 's are 7 a 's, so 3 a $+ 4a = 7a$; that is, when the monomials are similar the sum may be obtained by adding the numerical coefficients and annexing to their sum the common literal factor.
---	--

Add:

11. $\begin{array}{r} 3 \\ 6 \end{array}$	15. $\begin{array}{r} a \\ 5a \end{array}$	19. $\begin{array}{r} 12mb \\ - 2mb \end{array}$	23. $\begin{array}{r} 2.3ab^2 \\ - 3.5ab^2 \end{array}$
12. $\begin{array}{r} 2x \\ 3x \end{array}$	16. $\begin{array}{r} 7(a+b) \\ 1(a+b) \end{array}$	20. $\begin{array}{r} 7m \\ m \end{array}$	24. $\begin{array}{r} 3(a-b)^2 \\ (a-b)^2 \end{array}$
13. $\begin{array}{r} 3x \\ 6x \end{array}$	17. $\begin{array}{r} -a \\ 4a \end{array}$	21. $\begin{array}{r} -4c \\ -3c \end{array}$	25. $\begin{array}{r} 5(2a+b)^3 \\ - 7(2a+b)^3 \end{array}$
14. $\begin{array}{r} 4v \\ - 2v \\ - 7v \end{array}$	18. $\begin{array}{r} -y \\ 4y \\ - 9y \end{array}$	22. $\begin{array}{r} 40x^2 \\ - 10x^2 \\ - 60x^2 \end{array}$	26. $\begin{array}{r} 4.8a^2bc^3 \\ - 9.7a^2bc^3 \\ 3.1a^2bc^3 \end{array}$

Perform the additions indicated:

27. $8a + 3a - a + 7a - 6a + 5a$
 28. $5y - 3y + 8y + 10y - 6y - y - 2y$
 29. $8m - 3m + 7m - 4m - 5m + 2m$
 30. $3ab - 4ab + 7ab - 6ab + 4ab - 3ab$
 31. $6x^2y^2 - 4x^2y^2 + 2x^2y^2 - 7x^2y^2 - x^2y^2$
 32. $5(x+y) - 3(x+y) + 2(x+y) - (x+y)$
 33. $11(a+b)^2 - 2(a+b)^2 - (a+b)^2 + 10(a+b)^2$

Only similar terms can be united into a *single term*. Dissimilar terms are considered to have been added when the addition is *indicated*.

34. Add $6a$, $-5b$, $-2a$, $3b$, $-2c$, and $-a$.

SOLUTION

$$\text{Sum} = 6a - 2a - a - 5b + 3b - 2c = 3a - 2b - 2c$$

Add:

35. $2x, 4a, -x, -2a$ 38. $5r, -\frac{3}{4}t, -2r, -\frac{1}{4}t$
 36. $m, -3c, -6m, 4c$ 39. $\frac{1}{2}p, -\frac{2}{3}q, -\frac{1}{2}p, \frac{1}{6}q$
 37. $4z, -z, -w, +4w$ 40. $d, .4b, -.5b, .6$
 41. $x^2, 2xy, y^2, -3x^2, -4xy, 4y^2$
 42. $2r^2, rs, 3s^2, -4r^2, +5rs, 6s^2$
 43. $a^3, 3a^2b, -ab^2, b^3, -a^3, -2a^2b, ab^2, 2b^3$

To add polynomials

EXERCISES

1. Add
- $x + 2y - 2z$
- ,
- $x + y$
- , and
- $x - 4y + 2z$
- .

SOLUTION	=	CHECK ($x = 2, y = 3, z = 2$)
$x + 2y - 2z$	=	$2 + 6 - 4 = 4$
$x + y$	=	$2 + 3 = 5$
$x - 4y + 2z$	=	$2 - 12 + 4 = -6$
$3x - y$	=	$6 - 3 = 3$

EXPLANATION. For convenience, write similar terms in the same column.

Find the algebraic sum of each column, and write the results in succession with their proper signs.

CHECK. Whenever possible, the student should prove or check his results. In the check illustrated above we have substituted 2 for x , 3 for y , and 2 for z in each of the polynomials and in the result. The result, therefore, checks. A check tends to establish the correctness of the result; but cases will arise when the check may indicate that the work is correct, even though errors have been made. Any number except 0 may be used in this substitution; but in cases where there are exponents it is undesirable to substitute the number 1, for, since any power of 1 is 1, the value 1 does not check the exponents. It is best also not to substitute the same values for all the letters.

Add, and check the results:

2. $2a + 4b$ $- 6a - 2b$ $a + 3b$ 	3. $4r + 3s$ $- r - s$ $3r + 2s$ 	4. $x^2 - 2xy - y^2$ $x^2 + y^2$ $3xy + y^2$
--	---	---

5. Simplify
- $11a^2b - 7ab^2 + 2ac^2 - 9a^2b - 6ab^2 + 4ac^2$
- .

SUGGESTION. Arrange the terms so that similar terms shall stand in the same column, and proceed as above.

Simplify:

6. $2a + 2b + 3c + 4b - 4a + 6a - 2c$
7. $7w + 4x - 4y - x - 2w + 3y - 3x + 4w$
8. $7l - 6m + 3n - 8l + 4m + 11n + m - 2l$

$$9. 15r + 6s - 11t + r - 9s + t - 2s + 5r - 2t - 10r$$

$$10. 7x - 11y + 4z - 7z + 11x - 4y + 7y - 11z - 4x + y - x - z$$

$$11. a + 3b + 5c - 6a + d + 4b - 2c - 2b + 5a - d + a - b$$

$$12. 4x^2 - 3xy + 5y^2 + 10xy - 17y^2 - 11x^2 - 5xy + 12x^2 - 2xy$$

$$13. 2xy - 5y^2 + x^2y^2 - 7xy + 3y^2 - 4x^2y^2 + 5xy + 4y^2 + x^2y^2$$

Add, and check the results :

$$14. 4r - 5t, 2r - s + 3t, 2s - 3r, \text{ and } s + 2t$$

$$15. 5x - 3y - 2z, 4y - 2x, 3z - 2x, \text{ and } 4x + 2z$$

$$16. 6an - 3a^2n^2 + 4, a^2n^2 - 4an - 2, \text{ and } a^2n^2 - 6$$

$$17. 7a - 3b + 5c, 2b + d - 3c, \text{ and } 5c - 6a + 2d$$

$$18. x + y + z, x - y + z, y - z - x, z - x - y, \text{ and } x - z$$

$$19. 2a - 3b, 2b - 3c, 5c - 4a, 10a - 5b, \text{ and } 7b - 3c$$

$$20. a - \frac{1}{2}b + c, \frac{3}{2}a + \frac{1}{3}b, \frac{1}{2}a - \frac{1}{3}c, \text{ and } \frac{2}{3}b - \frac{2}{3}c$$

$$21. (2a + b)v + (a + c)y, -3(2a + b)v, \text{ and } (a + c)y$$

$$22. a + b - c, a - b - c, a - b + c, -a + b, \text{ and } -a - b + c$$

$$23. 2ab - 3b^2, 2a^2 + 5b^2, 7a^2 + ab - b^2, 5a^2 - 8ab, 6b^2 - 15a^2, \text{ and } 3a^2 + 3ab$$

$$24. 4x^3 - 2x^2 - 7x + 1, x^3 + 3x^2 + 5x - 6, 4x^2 - 8x^3 + 2 - 6x, 2x^3 - 2x^2 + 8x + 4, 2x^3 - 3x^2 - 2x + 1$$

$$25. 5x - 3y - 2z, 4y - 2x + 6z, 3a - 2x - 4y, 4b - 2z - 5y, a - 5b, 5y - 6x, 8x + 2y - 5a - 2b, \text{ and } 6x - y - 2z + 4b$$

$$26. .2x^3 - 4x^2 + x + 2, .4x^2 - 4x + .4 - x^3, 3.5x - .6 + 3x^2 + 2x^3, \text{ and } 1 - .5x + 1.2x^2 + .8x^3$$

Add :

27. $6\sqrt{x} - 5\sqrt{xy} - 3\sqrt{y}$, $6\sqrt{xy} - \sqrt{x}$, and $3\sqrt{x} + \sqrt{y}$
28. $2(a+b) + (c-d)$, $2(c-d) - 3(a+b)$, and $(a+b) - (c-d)$
29. $\frac{1}{2}x + 2y - z$, $2x - \frac{1}{2}y + \frac{1}{4}z$, and $\frac{1}{2}z - \frac{1}{2}x - \frac{1}{2}y$
30. $\sqrt{a} + \sqrt{ab}$, $2\sqrt{a} - \sqrt{b} + 2\sqrt{ab}$, and $2\sqrt{b} - 3\sqrt{ab} - \sqrt{a}$
31. $x + 3(a+1) - y$, $-(a+1) - 2x + 4y$, and $3x - 4(a+1)$
32. $(c+2)x + (a+d)y$, $(a+d)y - (c+2)x$, and $(c+2)x - (a+d)y$
33. $4x^3 - 2x^2 - 7x + 1$, $x^3 + 3x^2 + 5x - 6$, $4x^2 - 8x^3 + 2 - 6x$, $2x^3 - 2x^2 + 8x + 4$, and $2x^3 - 3x^2 - 2x + 1$
34. $2c - 7d + 6n$, $11m - 3c - 5n$, $7n - 2d - 8c$, $8d - 3m + 10c$, $4d - 3n - 8m$, $m - 6n$, and $2m - 3d$
35. $a^5 + 5a^4 + 5a + 1$, $a^4 - 2a^5 + a^3 - 2$, $a^3 - 3a^2 - 4a^4 - a^5$, and $2a^5 + a^4 - 2a^3 + 2a^2 - 3a + 1$
36. $5x^6 - x^3 + 7x - 9$, $4x^5 - 3x^6 + 6x^2 + 12$, $x^6 - 5x^5 - x - 7$, $4 - x^2 - x^6$, $4x^4 - 10x^2 + 3x^6 + 4$, and $x^6 + x^3 - 3x^2 - 4x - 5$
37. $3(a+b) + 6(b+c)$, $5(a+b) - 10(b+c)$, $2(a+b) + (b+c)$, $3(b+c) - (a+b)$, $2(b+c) - 10(a+b)$, and $3(a+b) - 3(b+c)$
38. $(a+b)x - (c-d)y$, $(a+b)x + (c-d)y$, $2(c-d)y - (a+b)x$, $(c-d)y - (a+b)x$, $(a+b)x - (c-d)y$, and $(a+b)x - (c-d)y$

SUBTRACTION

To subtract monomials

EXERCISES (Examples 1 to 18, Oral; 19 to 29, Written)

1. How many are 8 less 3? How many times a number are 8 times the number less 3 times the number?
2. Letting n stand for a number, how many times n are 8 times n less 3 times n ? $8n - 3n = ?$
3. $7a's - 3a's = ?$ $7a - 3a = ?$

4. $10\ c's - 4\ c's = ?$ $10\ c - 4\ c = ?$

5. $11\ d's - 4\ d's - 3\ d's = ?$ $11\ d - 4\ d - 3\ d = ?$

6. What kind of terms can we subtract? Explain how we subtract these terms.

7. From $10\ a$ subtract $4\ a$.

SOLUTION **EXPLANATION.** Just as $10\ a$'s less $4\ a$'s are $6\ a$'s, so $10\ a$
 $10\ a$ $- 4\ a = 6\ a$; that is, $6\ a$ must be added to $4\ a$ to give the
 $\underline{4\ a}$ sum, $10\ a$. When terms are similar their difference may be
 $6\ a$ obtained by subtracting the numerical coefficients and an-
 nexing the common literal factor.

CHECK. $4\ a + 6\ a = 10\ a$.

Subtract, and check :

$$\begin{array}{r} 8. \ 9 \\ \underline{4} \end{array}$$

$$\begin{array}{r} 10. \ 7 \\ \underline{- 3} \end{array}$$

$$\begin{array}{r} 12. \ 18\ m^2 \\ \underline{13\ m^2} \end{array}$$

$$\begin{array}{r} 14. \ 16\ ax^2 \\ \underline{9\ ax^2} \end{array}$$

$$\begin{array}{r} 9. \ 9\ x \\ \underline{4\ x} \end{array}$$

$$\begin{array}{r} 11. \ 7\ ab \\ \underline{- 3\ ab} \end{array}$$

$$\begin{array}{r} 13. \ 20\ xy \\ \underline{- 16\ xy} \end{array}$$

$$\begin{array}{r} 15. \ 14\ r^2s^3 \\ \underline{- 7\ r^2s^3} \end{array}$$

$$\begin{array}{r} 16. \ 8\ x^2y^2z \\ \underline{6\ x^2y^2z} \end{array}$$

$$\begin{array}{r} 17. \ 21(a + b) \\ \underline{- 11(a + b)} \end{array}$$

$$\begin{array}{r} 18. \ 9(x - 4) \\ \underline{- 4(x - 4)} \end{array}$$

Simplify, adding or subtracting in order as signs indicate :

19. $5\ n + 3\ n - 7\ n$

24. $8\ r - 6\ r + 5\ r - 2\ r$

20. $8\ a - 5\ a - 3\ a$

25. $7\ y + 8\ y - 6\ y + 7\ y$

21. $2\ s + 8\ s - 5\ s$

26. $5\ z + 7\ z - 2\ z - 4\ z$

22. $3\ b - 2\ b + 7\ b$

27. $9\ v - 3\ v + 2\ v - 5\ v$

23. $8\ x + 7\ x - 3\ x$

28. $7\ n - 2\ n - 3\ n + 4\ n$

29. $2\ y + 3\ y - y + 7\ y - 3\ y + 9\ y + 2\ y - 6\ y$

To subtract polynomials

EXERCISES

1. From $4\ a^2 - ab + 2\ b^2$ take $a^2 - 2\ ab - 2\ b^2$.

SOLUTION		CHECK ($a = 2, b = 3$)
$4\ a^2 - ab + 2\ b^2$	$=$	$16 - 6 + 18 = 28$
$a^2 - 2\ ab - 2\ b^2$	$=$	$4 - 12 - 18 = -26$
$\underline{3\ a^2 + ab + 4\ b^2}$	$=$	$\underline{12 + 6 + 36 = 54}$

EXPLANATION. We arrange similar terms under each other and subtract these terms separately.

CHECK. The numerical process at the right, in which 2 is substituted for a and 3 for b , checks the process at the left.

Another common way to check a result in subtraction is to add the result to the subtrahend. The sum should equal the minuend. Thus:

$$\begin{array}{r} 3a^2 + ab + 4b^2 \\ a^2 - 2ab - 2b^2 \\ \hline 4a^2 - ab + 2b^2 \end{array}$$

Subtract, and check the results:

$$\begin{array}{r} 2. \quad 41 - 2t \\ \quad 41 - t \\ \hline \end{array}$$

$$\begin{array}{r} 5. \quad n + 5n^2 - 2n^3 \\ \quad n + n^2 + n^3 \\ \hline \end{array}$$

$$\begin{array}{r} 3. \quad 9x - 7y \\ \quad - 2x + 3y \\ \hline \end{array}$$

$$\begin{array}{r} 6. \quad 2r - 3s - t \\ \quad r + s - t \\ \hline \end{array}$$

$$\begin{array}{r} 4. \quad 5r - 8s \\ \quad - 2r + 5s \\ \hline \end{array}$$

$$\begin{array}{r} 7. \quad 8a^2 - 2ab - 3b^2 \\ \quad 5a^2 - ab + 2b^2 \\ \hline \end{array}$$

8. From $5x - 3y + z$ take $2x - y + 8z$.

9. From $3a^2b + b^3 - a^3$ take $4a^2b - 8a^3 + 2b^3$.

10. From $13a^2 + 5b^2 - 4c^2$ take $8a^2 + 9b^2 + 10c^2$.

11. From $15x - 3y + 2z$ subtract $3x + 8y - 9z$.

12. From $a^2 - ab - b^2$ subtract $ab - 2a^2 - 2b^2$.

13. From $m^2 - mn + n^2$ subtract $2m^2 - 3mn + 2n^2$.

14. From $4x^2 + 3xy + y^2$ subtract $2x^2 - 5xy + 2y^2$.

15. From $3ab + a^2 + b^2$ subtract $a^2 + 4ab + b^2$.

16. From $6x^2 + 4xy - 3y^2$ subtract $4y^2 - 3xy + 6x^2$.

17. From the sum of $3a^2 - 2ab - b^2$ and $3ab - 2a^2$ subtract $a^2 - ab - b^2$.

18. From $3x - y + z$ subtract the sum of $x - 4y + z$ and $2x + 3y - 2z$.

19. Subtract the sum of $m^2n - 2mn^2$ and $2m^2n - m - n^3$ + $2mn^2$ from $m^3 - n^3$.

If $x = a^2 + b^2$, $y = 2ab$, $z = a^2 - b^2$, and $v = a^2 - 2ab + b^2$:

20. $x + y + z + v = ?$ 22. $x - y + z - v = ?$

21. $x - y - z + v = ?$ 23. $y - x - v + z = ?$

EQUATIONS

ILLUSTRATIVE EXAMPLES

1. Solve the equation $2x + 20 = 80 - 4x$.

SOLUTION

(1)	$2x + 20 = 80 - 4x$	
(2)	$2x + 4x + 20 = 80 - 4x + 4x$	(1) + $4x$
(3)	$6x + 20 = 80$	Terms in (2) united
(4)	$6x + 20 - 20 = 80 - 20$	(3) - 20
(5)	$6x = 60$	Terms in (4) united
(6)	$x = 10$	(5) $\div 6$

CHECK. Always test the answer by finding whether the value obtained is such as to make the members of the *original equation* equal.

Thus, substituting 10 for x in the given equation, we have

$$20 + 20 \mid 80 - 40$$

$$40 = 40 \quad \text{Hence 10 is the true value of } x.$$

NOTE. The process of removing a quantity from one side of an equation and placing it on the other side by adding it to, or subtracting it from, both sides of the equation is called *transposition*. A quantity is transposed from one side of an equation to the other side by changing its sign.

For example, if $x + 5 = 9$,
by transposing $+5$ we have $x = 9 - 5$, or 4.

Since, however, by using the axioms, we can "add" a quantity to both sides or "subtract" a quantity from both sides, the term *transpose* is not necessary, although it is used by many teachers of algebra.

2. Solve the equation $2x + 13 = 5x - 20$.

SOLUTION

(1)	$2x + 13 = 5x - 20$	
(2)	$2x + 33 = 5x$	(1) + 20
(3)	$33 = 3x$	(2) - $2x$
(4)	$11 = x$	(3) $\div 3$

NOTE. It is sometimes better to solve the equation by placing the unknown quantity on the right side of the equation.

CHECK.

$$\begin{array}{r|l} 2 \cdot 11 + 13 & 5 \cdot 11 - 20 \\ 22 + 13 & 55 - 20 \\ \hline 35 & = 35 \end{array}$$

Order of processes in solving equations

In the solutions of the equations on page 99, you will avoid confusion by following the order of processes suggested. Take the equation:

$$(1) \quad 2(2x + 1) - 11 = 2x - 5$$

If there are any parentheses, perform the indicated multiplications:

$$(2) \quad 4x + 2 - 11 = 2x - 5$$

Unite similar terms in each member, if necessary.

$$(3) \quad 4x - 9 = 2x - 5$$

Add to or subtract from each member the smaller unknown term, according as its sign is $-$ or $+$.

$$(4) \quad 2x - 9 = -5 \quad (3) - 2x$$

Add to or subtract from each member any known term that stands beside an unknown term.

$$(5) \quad 2x = 4 \quad (4) + 9$$

Divide each member by the coefficient of the unknown quantity.

$$(6) \quad x = 2 \quad (5) \div 2$$

Check the equation by substituting the value of the unknown quantity in the given equation.

$$\begin{array}{r|l} \text{CHECK.} & 2(2 \times 2 + 1) - 11 \quad | \quad 2 \times 2 - 5 \\ & 10 - 11 \quad | \quad 4 - 5 \\ & -1 = -1 \end{array}$$

EXERCISES

Solve and check the following:

1. $7n + 12 = 5n + 16$
2. $4r - 11 = 19 - 2r$
3. $13x + 4 = 5x + 12$
4. $3s + 14 + 7s = 78 + 2s$
5. $9x + 23 + 2x = 4x + 37$
6. $5x - 7 + 15x = 13x + 14$
7. $2(3n - 5) = 4n + 12$
8. $2p - 9 = 7(2p - 3)$
9. $22 - 6x = 40 - 8x$
10. Solve $46 + 3x - 60 = 5x - 10 - 4x$.

SOLUTION

$$(1) \quad 46 + 3x - 60 = 5x - 10 - 4x$$

$$(2) \quad 3x - 14 = x - 10$$

Like terms in (1) united

$$(3) \quad 2x = 4$$

$$(2) \quad + 14 - x$$

$$(4) \quad x = 2$$

$$(3) \div 2$$

Note that in (3) above the solution has been shortened by applying the addition and subtraction axioms in one step.

Solve and check:

11. $2x - 4 + 6x = 22 - 15 + 21$
12. $30 + 5x - 32 = 7x + 12 - 9x$
13. $6x + 5x - 70 = 5x + 54 - 70$
14. $5x + 16 - 6x = 16 + 25 - 6x$
15. $9x + 15 - 2x = 32 + 4x - 11$
16. $10x - 39 + 12x - 9x + 42 - 4x = 42 - 4x$
17. $16x + 12 - 75 + 2x - 12 - 110 = 8x - 50 - 25$
18. $3x - 18 + 27 + 10x - 11 = 25 + 4x - 7x + 12$
 $+ 3x$
19. $18x + 16 = 8 + 12x + 8 - 13 + 25x - 9 + 100$
 $- 25x$

Problems

Review the *directions for solving problems* given on page 22.

1. The Borough of Manhattan contains 22,000 elevators. If 2000 more are for freight than for passengers, how many freight elevators are there?

2. The total height of a certain brick chimney in St. Louis is 172 feet. Its height above ground is 2 feet more than 16 times its depth below ground. How high is the part above ground?

3. There are 3141 islands in the Philippine Islands. Of these 195 more are named than unnamed. Find the number of each.

4. The Canadian Falls in the Niagara River are 158 feet high. This is 8 feet more than $\frac{10}{11}$ of the height of the American Falls. Find the height of the American Falls.

SOLUTION

Let f = the number of feet in the height of American Falls

$\frac{10}{11}f + 8$ = the number of feet in the height of Canadian Falls

$$(1) \quad \frac{10}{11}f + 8 = 158$$

$$(2) \quad \frac{10}{11}f = 150 \quad (1) - 8$$

$$(3) \quad f = 165 \quad (2) \div \frac{10}{11}$$

The height of the American Falls is 165 feet.

Complete the solution by showing each step in the *check*.

5. James and Katherine together had \$190 in a savings bank. The amount in James's account was \$30 more than 3 times the amount in Katherine's account. How much had each?

6. The Pennsylvania Railroad station in New York is 780 feet long, and this is $404\frac{1}{3}$ feet more than $\frac{1}{2}$ the length of the Capitol at Washington. Find the length of the Capitol.

7. A basket-ball team won 16 games, or 4 less than $\frac{2}{3}$ of the games they played. Find the number of games they played.

8. A rectangle having a perimeter of 46 feet is 5 feet longer than it is wide. Find its dimensions.

9. A lawn is 7 rods longer than it is wide. If the distance around it is 62 rods, what are its dimensions?

10. The distance around a desk top is 170 inches. If the desk top is 15 inches longer than it is wide, how wide is it?

11. How old is a man whose age 16 years hence will be 4 years less than twice his present age?

12. Adrian is 8 years younger than his sister. In 4 years the sum of their ages will be 26 years. How old is each?

13. A prime dark sea-otter skin cost \$400 more than a brown one. If the first cost 3 times as much as the second, how much did each cost?

14. A wagon loaded with coal weighed 4200 pounds. The coal weighed 1800 pounds more than the wagon. How much did the wagon weigh? the coal?

15. A mining company sold copper ore at \$5.28 a ton. The profit per ton was \$.22 less than the cost. What was the profit on each ton?

16. The length of a tunnel was $22\frac{1}{2}$ times its width. If the length had been 50 feet less, it would have been 20 times the width. Find the length of the tunnel; the width.

17. The St. Lawrence River at a point where it is spanned by a bridge is 1800 feet wide. This is 180 feet less than $\frac{3}{8}$ of the length of the bridge. How long is the bridge?

18. There were 43 members of a high-school radio club. If the number of those who made their own receiving sets was 4 more than twice the number of those who bought their sets, find how many made their own sets.

19. George is 15 years older than James. In 5 years George will be twice as old as James. Find the present age of each.

20. John and Frank have \$72.60. How many dollars has each if John has \$12.20 more than Frank?

21. The largest thermometer in the world has a glass tube 16 feet long. Find the length of the thermometer if the tube is .8 of the entire length.

22. If $\frac{1}{5}$ of a number is added to the number, the sum is 12. What is the number?

SUGGESTION. $n + \frac{n}{5} = 12$; that is, $\frac{6}{5}n = 12$.

23. If $\frac{1}{3}$ of a number is subtracted from twice the number, the difference is 35. What is the number?

SUGGESTION. $2n - \frac{1}{3}n = 35$; that is, $\frac{5}{3}n = 35$.

24. The total cost of a state house was \$13,000,000. If the furnishings cost $2\frac{1}{4}$ times as much as the construction, what was the cost of each?

25. A young man saved \$300 of his year's income. If this was \$60 less than 30% of his income for the year, find his annual income.

PARENTHESES

You have seen how parentheses, (), are used to group numbers that are to be regarded as a single number. Other signs of grouping, used in the same way, are brackets, [], braces, { }, and the horizontal bar, —.

When numbers are included by any of the signs of grouping, they are commonly said to be *in parenthesis*, *in a parenthesis*, or *in parentheses*. The signs in most common use are () and []. The bar is used in frac-

tions; as, $\frac{3}{a+b}$.

Removal of parentheses preceded by + or -

ILLUSTRATIVE EXAMPLES

1. Remove parentheses and simplify $3a + (b + c - 2a)$.

SOLUTION

The given expression indicates that $(b + c - 2a)$ is to be *added* to $3a$. This may be done by writing the terms of $(b + c - 2a)$ after $3a$ in succession, each with its proper sign, and uniting terms.

$$\therefore 3a + (b + c - 2a) = 3a + b + c - 2a = a + b + c$$

2. Remove parentheses and simplify $4a - (2a - 2b)$.

SOLUTION

The given expression indicates that $(+2a - 2b)$ is to be *subtracted* from $4a$. Proceeding as in subtraction, that is, changing the sign of each term of the subtrahend and adding, we have

$$4a - (2a - 2b) = 4a - 2a + 2b = 2a + 2b$$

PRINCIPLES. 1. A parenthesis preceded by a plus sign may be removed from an expression without changing the signs of the terms in parenthesis.

2. A parenthesis preceded by a minus sign may be removed from an expression if the signs of all the terms in parenthesis are changed.

EXERCISES (Examples 1 to 20, Oral; 21 to 36, Written)

Simplify by removing the parentheses:

- | | |
|---|--------------------------|
| 1. $4 + (6 - 3)$ | 12. $5a - (7a + 2)$ |
| 2. $5 + (7 - 1)$ | 13. $6x - (2x - 3y)$ |
| 3. $a + (b - 2a)$ | 14. $5x - (7x + 7y)$ |
| 4. $a + (b - c)$ | 15. $2x - (x + y)$ |
| 5. $2a + (a - b)$ | 16. $x - (y - z)$ |
| 6. $6x + (7x - 2)$ | 17. $x - (-y + z)$ |
| 7. $2 + (3 - 1)$ | 18. $m - n - (-a)$ |
| 8. $4 + (6 - 5)$ | 19. $m - (n - 2a)$ |
| 9. $2a + (7a - b)$ | 20. $5x - (2x + y)$ |
| 10. $5a - (4a + b)$ | 21. $a - m + (n - m)$ |
| 11. $3 + (6 - 2)$ | 22. $5a - 2b - (a - 2b)$ |
| 23. $a - (b - c + a) - (c - b)$ | |
| 24. $2xy + 3y^2 - (x^2 + xy - y^2)$ | |
| 25. $m + (3m - n) - (2n - m + n)$ | |
| 26. $4x - y - (3x - y) - (2x + 4y)$ | |
| 27. $(2x^2 - 4xy + y^2) - (x^2 + 2xy - 3y^2)$ | |
| 28. $(2a + 3b - 4c) + (a - 3b - 4c)$ | |

When an expression contains parentheses within parentheses, they may be removed *in succession*, beginning with either the outermost or the innermost, preferably the latter.

29. Simplify $6x - [3a + (9b - 2a) + 4x - 10b]$.

SOLUTION

$$\begin{aligned}
 & 6x - [3a + (9b - 2a) + 4x - 10b] \\
 = & 6x - [3a + 9b - 2a + 4x - 10b] && \text{Principle 1} \\
 = & 6x - [a - b + 4x] && \text{Terms united} \\
 = & 6x - a + b - 4x && \text{Principle 2} \\
 = & 2x - a + b && \text{Terms united}
 \end{aligned}$$

Simplify:

30. $b - [x + 4a + (x - 7)]$

31. $y - [5 - 3b - (7y - 4a)]$

32. $ab - [ab + ac - a - (2a - ac) + 2a - 2ac]$

33. $a - [(7y - 4a - 1) - (a + y)]$

34. $4m - [p + 3n + (m + n) + 3 - (6p - 3n - 5m)]$

35. $7.6m^2 - [4.2m^2 - (m^2 - 3) + 7]$

36. $3.8a^2 - [5.2a^2 - (a^2 - 3) + 9]$

Grouping terms by means of parentheses

It follows from pages 102 and 103 that:

PRINCIPLES. 1. Any number of terms of an expression may be inclosed in a parenthesis preceded by a plus sign without changing the signs of the terms to be inclosed.

2. Any number of terms of an expression may be inclosed in a parenthesis preceded by a minus sign, if the signs of the terms to be inclosed are changed.

ILLUSTRATIVE EXAMPLES

1. In $a^2 + 2ab + b^2$, group the terms involving b .

SOLUTION

$$a^2 + 2ab + b^2 = a^2 + (2ab + b^2)$$

2. In $a^2 - x^2 - 2xy - y^2$, group as a subtrahend the terms involving x and y .

SOLUTION

$$a^2 - x^2 - 2xy - y^2 = a^2 - (x^2 + 2xy + y^2)$$

EXERCISES

1. In $ax^2 + ab + 2x^2 + 2b$, group the terms involving x^2 , and also the terms involving b , as quantities to be added.

2. In $a^3 + 3a^2b + 3ab^2 + b^3$, group the first and fourth terms, and also the second and third terms, as quantities to be added.

In each of the following expressions group the last three terms together preceded by a minus sign:

$$3. a^2 - b^2 - 2bc - c^2 \qquad 5. xy - xz - y^2 + 2yz - z^2$$

$$4. a^2 - b^2 + 2bc - c^2 \qquad 6. ac + bc - a^2 - 2ab - b^2$$

$$7. a^2 + 2ab + b^2 - c^2 + 2cd - d^2$$

$$8. a^2 - 2ab + b^2 - c^2 - 2cd - d^2$$

9. Explain why $x^3 - x^2 + x - 1$ does not equal $(x^3 - 1) - (x^2 + x)$.

10. Explain why $x^2 - y^2 + 2yz - z^2$ does not equal $x^2 - (y^2 + 2yz - z^2)$.

To unite literal coefficients by using parentheses

EXERCISES

Add:

$$\begin{array}{llll} 1. & \frac{ax}{(a+b)x} & 3. & \frac{bm - cm}{(b-c)m} \\ 5. & \frac{-cx - dx}{-(c+d)x} & 7. & \frac{(t+r)x + (t+2r)x}{(2t+3r)x} \end{array}$$

$$\begin{array}{llll} 2. & \frac{ax}{nx} & 4. & \frac{cy - dy}{-dy} \\ 6. & \frac{-mp - np}{-np} & 8. & \frac{(a+b)x + (2a+c)x}{(2a+c)x} \end{array}$$

Subtract the lower expression from the upper one:

$$\begin{array}{lll} 9. & \frac{mx}{nx} & 11. & \frac{dy + az}{ey - bz} \\ 13. & \frac{ax - by}{2x - cy} \end{array}$$

$$\begin{array}{lll} 10. & \frac{a^2x + aby}{b^2x + aby} & 12. & \frac{mx - ny}{nx - my} \\ 14. & \frac{(2r - s)y}{(r + 2s)y} \end{array}$$

15. Unite the coefficients of x and y in $ax - ay - bx - by$

SOLUTION

The entire coefficient of x is $(a - b)$; the entire coefficient of y is $(-a - b)$, or $-(a + b)$.

$$\therefore ax - ay - bx - by = (a - b)x - (a + b)y$$

Unite in alphabetical order the coefficients of x and y in each of the following, giving each parenthesis the sign of the first coefficient to be inclosed therein :

16. $ax - by - bx - cy + dx - ey$

17. $5ax + 3ay - 2dx + ny - 5x - y$

18. $cx - 2bx + 7ay + 3ax - lx - ty$

19. $bx + cy - 2ax + by - cx - dy$

20. $x + ax - y + ay$

22. $bx - cy - 2ay + by$

21. $x - ay - ax - y$

23. $rx - ay - sx + 2cy$

Group the same powers of x in each of the following :

24. $ax^3 + bx^2 - cx + ex^3 - dx^2 - fx$

25. $x^3 + 3x^2 + 3x - ax^2 - 3ax^3 + bx$

26. $x^2 - abx - x^3 - bx^2 - cx - mnx^3 + dx$

27. $ax^4 - x^4 - ax^2 + x^2 + ax - x - abx^3 + x^3$

REVIEW EXERCISES

Find the numerical value of each of the following if $a = 1$, $b = 2$, $c = 3$, $m = 7$, $n = 8$, $x = 4$, $y = 5$, $z = 6$:

1. $6a + 7b - 4c - 2a - 4b + 5c$

2. $2x^2 - xy + y^2 - 2xy - 2y^2$

3. $2a^2 + 3ab - b^2 - (a^2 - ab - 2b^2)$

4. $2m - (m - n) - (y + 2z) + (3a + b)$

5. $[2a - b - (a + 2b) - 3y - (y - 2z)]$

6. $(3x - 2y - 4z) - [4x - 2y - (2x - y)]$

7. $m^2 - 2mn - (3m^2 + 2mn - n^2)$

8. $m^2 - 3mn - n^2 + (4m^2 - 3mn) - (m^2 + n^2)$

MULTIPLICATION

Product of two monomials

In algebra, as in arithmetic, the product of two numbers contains all the factors of both numbers. It makes no difference how these factors are grouped.

Since $a^2 = aa$ and $a^3 = aaa$,

$$a^2 \cdot a^3 = (aa)(aaa) = aaaaa = a^5.$$

That is, $a^2 \cdot a^3 = a^{2+3} = a^5$. (Exponents added)

Similarly, $3 a^2 \cdot 5 a^3 = (3 \cdot 5)(a^2 \cdot a^3) = 15 a^5$.

(Coefficients multiplied)

Again, $3 a^2b \cdot 5 a^3b^2 = (3 \cdot 5)(a^2 \cdot a^3)(b^1 \cdot b^2) = 15 a^5b^3$.

Hence, for multiplication:

Law of exponents. *The exponent of a number in the product is equal to the sum of its exponents in the multiplicand and the multiplier.*

Law of coefficients. *The coefficient of the product is equal to the product of the coefficients of the multiplicand and the multiplier.*

Law of signs. (See page 86.) *The sign of the product of two factors is + when the factors have like signs, and - when they have unlike signs.*

EXERCISES

Tell the products quickly:

- | | | | |
|---|---|---|---|
| 1. $\begin{array}{r} 7a \\ 3a \\ \hline \end{array}$ | 5. $\begin{array}{r} 3x \\ -4y \\ \hline \end{array}$ | 9. $\begin{array}{r} -3.5m \\ -2m^3n \\ \hline \end{array}$ | 13. $\begin{array}{r} -8ab^2x^3 \\ 2.5b^4cx^4 \\ \hline \end{array}$ |
| 2. $\begin{array}{r} 3y \\ -4y \\ \hline \end{array}$ | 6. $\begin{array}{r} 4x^2 \\ -7x^2 \\ \hline \end{array}$ | 10. $\begin{array}{r} -8aw \\ -3aw \\ \hline \end{array}$ | 14. $\begin{array}{r} -12a^3bc \\ 3a^2b^2d^2 \\ \hline \end{array}$ |
| 3. $\begin{array}{r} ab^2 \\ -a^3b \\ \hline \end{array}$ | 7. $\begin{array}{r} -3xy^3 \\ 9xz \\ \hline \end{array}$ | 11. $\begin{array}{r} 2ax \\ 2by \\ \hline \end{array}$ | 15. $\begin{array}{r} -4x^4y^3z \\ -3x^2yz^2 \\ \hline \end{array}$ |
| 4. $\begin{array}{r} x^4y \\ -xy \\ \hline \end{array}$ | 8. $\begin{array}{r} -7pq^2 \\ -p^5q \\ \hline \end{array}$ | 12. $\begin{array}{r} -8c^2d \\ .4d^3z \\ \hline \end{array}$ | 16. $\begin{array}{r} -6a^3b^2c^4 \\ -3a^4bc^2 \\ \hline \end{array}$ |

Multiply :

- | | | | |
|---|---|---|---|
| 17. $\begin{array}{r} 4 a^2 \\ - 1 \\ \hline \end{array}$ | 22. $\begin{array}{r} 10 m^4 n^3 \\ - 3 n^2 m^3 \\ \hline \end{array}$ | 27. $\begin{array}{r} - p^2 q \\ ap^2 q^3 \\ \hline \end{array}$ | 32. $\begin{array}{r} - 2 a^2 m^3 n^4 \\ 8 b^5 n^6 m^7 \\ \hline \end{array}$ |
| * 18. $\begin{array}{r} 2 a^{m+1} \\ 3 a^2 \\ \hline \end{array}$ | 23. $\begin{array}{r} - 2 a^r b^{2z} \\ 7 a^{3r} b^{4z} \\ \hline \end{array}$ | 28. $\begin{array}{r} - x^n y^n \\ xy \\ \hline \end{array}$ | 33. $\begin{array}{r} 4 x^{n-1} \\ - 2 x^{n+1} \\ \hline \end{array}$ |
| 19. $\begin{array}{r} 5 y \\ 3 y^{n-2} \\ \hline \end{array}$ | 24. $\begin{array}{r} a^2 b^3 x^3 y^{n-2} \\ a^n b^{n-3} y^2 \\ \hline \end{array}$ | 29. $\begin{array}{r} - x^{1-n} \\ - x^n \\ \hline \end{array}$ | 34. $\begin{array}{r} a^{n-1} b^{n-2} c^3 \\ a^{n+1} b^2 c^{n-1} \\ \hline \end{array}$ |
| 20. $\begin{array}{r} - 5 x^n \\ x \\ \hline \end{array}$ | 25. $\begin{array}{r} - xyz \\ x^{b-1} z^{n+4} \\ \hline \end{array}$ | 30. $\begin{array}{r} ab^2 c^3 \\ d^m b^{2n} c \\ \hline \end{array}$ | 35. $\begin{array}{r} 2^{5-n} z^{b-a} \\ 2^{n-3} z^{2a-b} \\ \hline \end{array}$ |
| 21. $\begin{array}{r} - a^m \\ - a^n \\ \hline \end{array}$ | 26. $\begin{array}{r} - 4 a^r b^s \\ - 3 a^{r-2} b^s \\ \hline \end{array}$ | 31. $\begin{array}{r} y^{n-m} \\ y^{m-n+1} \\ \hline \end{array}$ | 36. $\begin{array}{r} - x^{n-1} y^{n-2} \\ - xy \\ \hline \end{array}$ |

To multiply a polynomial by a monomial

Multiplying as in arithmetic, we have :

$\begin{array}{r} 321 \\ 3 \\ \hline 963 \end{array}$	$\begin{array}{r} 300 + 20 + 1 \\ 3 \\ \hline 900 + 60 + 3 \end{array}$	$\begin{array}{r} 3h + 2t + u \\ 3 \\ \hline 9h + 6t + 3u \end{array}$
---	---	--

Also $x + y + z$ multiplied by a equals $ax + ay + az$.

The product of a polynomial by a monomial is equal to the sum of the partial products obtained by multiplying each term of the polynomial by the monomial.

EXERCISES (Examples 1 to 8, Oral ; 9 to 13, Written)

Multiply :

- | | | |
|---|--|---|
| 1. $\begin{array}{r} x + 2 \\ 4 \\ \hline \end{array}$ | 3. $\begin{array}{r} ax^2 + y \\ ax \\ \hline \end{array}$ | 5. $\begin{array}{r} 5 m^2 s + 2 t \\ 3 st^2 \\ \hline \end{array}$ |
| 2. $\begin{array}{r} m^2 + n^3 \\ mn \\ \hline \end{array}$ | 4. $\begin{array}{r} x^2 + 2 xy + y^2 \\ - xy \\ \hline \end{array}$ | 6. $\begin{array}{r} xy + yz + xz \\ xyz \\ \hline \end{array}$ |

* Literal exponents follow the same laws as numerical exponents. For example, $a^{m+1} \times a^2 = a^{m+1+2} = a^{m+3}$.

$$\begin{array}{r} 7. \quad 1 + 2x + 6x^2 + 4x^3 \\ \hline - x^3 \end{array}$$

$$\begin{array}{r} 8. \quad 4x^3 + 6x^2 + 2x + 1 \\ \hline - 3x^2 \end{array}$$

In exercise 7, the multiplicand is **arranged** according to the **ascending powers** of x ; in exercise 8, according to the **descending powers** of x .

Arrange according to the ascending or descending powers of some letter and perform the multiplications indicated:

9. $ab(6a^2 + a^4 + 1 + 4a^3 + 4a)$

10. $.5a(3a + 2.5a^2 + 1)$

11. $\frac{2}{3}x(3x + 1) - \frac{1}{3}x(6x - 2)$

12. $1.3a^2b(b - 4) + 2.5ab^2(a - 3)$

13. Find the value of $6\left(\frac{a^2}{15} - \frac{b^2}{18} + \frac{ab}{10}\right)$ if $a = 5$, $b = 3$.

To multiply a polynomial by a polynomial

EXERCISES

1. Multiply $x + 5$ by $x + 2$ and check the result.

SOLUTION

CHECK ($x = 2$)

$$x + 5$$

$$= 7$$

$$x + 2$$

$$= 4$$

$$x \text{ times } (x + 5) = x^2 + 5x$$

$$2 \text{ times } (x + 5) = 2x + 10$$

$$x(x + 5) + 2(x + 5) = x^2 + 7x + 10$$

$$= 28$$

CHECK. The product must equal the multiplicand multiplied by the multiplier, regardless of what value x may represent. To check the result, therefore, we may assign to x any value we choose and observe whether, for that value, the *product obtained* = the *multiplicand* $\times$ the *multiplier*. When $x = 2$, multiplicand = 7, multiplier = 4, and $x^2 + 7x + 10 = 28$. Since $7 \times 4 = 28$, it may be assumed that $x^2 + 7x + 10$ is the correct product.

RULE. Multiply the multiplicand by each term of the multiplier and find the sum of the partial products.

2. Multiply $x + 4$ by $x + 6$; check the result when $x = 2$.

3. Multiply $x + 1$ by $x - 2$; check the result when $x = 3$.

4. Multiply $2x + 3$ by $4x - 1$; check the result when $x = 3$.

5. Multiply $x^2 - x + 1$ by $x + 1$; check the result when $x = 2$.

6. Multiply $2a + b - c$ by $3a + b$; check the result when $a = 2$, $b = 3$, and $c = 2$.

Multiply, and check each result:

$$\begin{array}{r} 7. \ x + y \\ \underline{x + y} \end{array}$$

$$\begin{array}{r} 8. \ x - 4 \\ \underline{x + 9} \end{array}$$

$$\begin{array}{r} 9. \ 2x + 1 \\ \underline{3x - 5} \end{array}$$

$$\begin{array}{r} 10. \ 5y - 3z \\ \underline{4y + z} \end{array}$$

$$11. \ 2x + 3 \text{ by } x + 2$$

$$15. \ 3l - 5t \text{ by } 2l + 6t$$

$$12. \ 4x + 1 \text{ by } 3x - 4$$

$$16. \ 4y - 6b \text{ by } 2y + b$$

$$13. \ 5n + 1 \text{ by } 4n - 5$$

$$17. \ 2b + 5c \text{ by } 5b - 2c$$

$$14. \ h - 2k \text{ by } 3h + k$$

$$18. \ ax + by \text{ by } ax - by$$

An indicated product is said to be **expanded** when the multiplication is performed.

Expand, and check each result:

$$19. \ (c^2 + d^2)(c^2 - d^2)$$

$$24. \ (2b + 5c)(3b + 8c)$$

$$20. \ (3a + b)(3a + b)$$

$$25. \ (\frac{1}{2}a + \frac{1}{3}b)(\frac{1}{2}a + \frac{1}{3}b)$$

$$21. \ (2n^2 - l)(n^2 + 2l)$$

$$26. \ (\frac{2}{3}x - \frac{1}{4}y)(\frac{3}{4}x + \frac{1}{2}y)$$

$$22. \ (3x - .5y)(2x - .1y)$$

$$27. \ (2xy - 3y)(4xy + 7y)$$

$$23. \ (2a^n + 3b)(3a^n + b)$$

$$28. \ (4ax + 3by)(4ax + 3by)$$

Arrange both multiplicand and multiplier according to the ascending or the descending powers of the letter involved; multiply, and check each result.

$$29. \ x + x^3 + 1 + x^2 \text{ by } x - 1$$

$$30. \ x^3 + 10 - 7x - 4x^2 \text{ by } x - 2$$

$$31. \ 14 - 9x - 6x^2 + x^3 \text{ by } x + 1$$

$$32. \ a^3 - 30 - 11a + 4a^2 \text{ by } a - 1$$

$$33. \ 4a^2 - 2a^3 - 8a + a^4 - 3 \text{ by } 2 + a$$

$$34. \ x^2 + y^2 + 2xy \text{ by } x^2 - 2xy + y^2$$

$$35. \ b^2 + 5b - 4 \text{ by } -4 + 2b^2 - 3b$$

Multiply :

36. $a^5 + a^4 + 4a^3 - a^2 + a$ by $a + 1$
37. $4y^2 - 10 + 2y$ by $2y^2 - 3y + 5$
38. $31x^3 - 27x^2 + 9x - 3$ by $3x + 1$
39. $5x - 5x^2 + 10$ by $12 - 30x + 2x^2$
40. $4x - 3x^2 + 2x^3$ by $3x - 10x^2 + 10$
41. $2a^2 - 3b^2 - ab$ by $3a^2 - 4ab - 5b^2$
42. $a + b + c + d$ by $a - b - c + d$
43. $4x^3 - 3x^2y + 5xy^2 - 6y^3$ by $5x + 6y$
44. $3n^2 + 3m^2 + mn$ by $m^3 - 2mn^2 + m^2n$
45. $a^2 + b^2 + c^2 - ab - ac - bc$ by $a + b + c$
46. $m^8 - m^6n^3 + m^4n^6 - m^2n^9 + n^{12}$ by $m^2 + n^3$
47. $x^{3a} - 3x^{2a} + 3x^a - 1$ by $x^{2a} - 2x^a + 1$
48. $ax^{2n} + ay^{2n}$ by $ax^{2n} - ay^{2n}$

Expand :

49. $(3a^2 + 5b^3)^2$
50. $(4c + d)(4c - d)$
51. $(\frac{1}{2}x + y)(\frac{1}{2}x + y)$
52. $(3x^3 - \frac{1}{2}y)(3x^3 - \frac{1}{2}y)$
53. $(\frac{1}{2}a + \frac{1}{3}b)(\frac{1}{2}a - \frac{1}{3}b)$
54. $(\frac{2}{3}x + \frac{1}{3}y)(\frac{3}{4}x + \frac{1}{4}y)$
55. $(2x + 2y)^3$
56. $(a + b + c)^2$
57. $(a - b)(a + b)(a^2 + b^2)$
58. $(a^m - b^n)(a^m + b^n)(a^{2m} + b^{2n})$
59. $(ax^{n-1} + y^{n-1})(3ax^{n-1} + 2y^{n-1})$
60. $(x^m - nx^{n-1}y + \frac{1}{2}nx^{n-2}y^2)(x + y)$
61. $(x^{2n} + x^ny^n + y^{2n})(x^{2n} - x^ny^n + y^{2n})$
62. $(a^{6n} + a^{4n}b^{2c} + a^{2n}b^{4c} + b^{6c})(a^{2n} - b^{2c})$

ALGEBRAIC REPRESENTATION

1. Express in the shortest way the sum of five x 's; the product of five x 's.

2. When the multiplicand is x and the multiplier y , express the product in three ways.

3. Express the sum of 2 and 3; of x , $-y$, and $-z$.
4. What number is 10 less than 25? n less than 25?
5. Express the number that exceeds 4 by 9; a by b .
6. How much does 20 exceed $10 + 4$? z exceed $10 + y$?
7. What number must be added to 6 so that the sum shall be 10? to m so that the sum shall be 4?
8. Mary read 10 pages in a book, stopping at the top of page a . On what page did she begin to read?
9. A man bought a chairs at d dollars each and paid with a 10-dollar bill. Express the number of dollars in change due him; the number of cents.
10. Albert has x dollars and Charles, y dollars. If Albert gives Charles m dollars, how much will each then have?
11. If 40 is separated into two parts, one of which is x , represent the other part.
12. If n is any integer, $2n$ will always represent an even number. What are the next two consecutive even numbers? What are the two odd numbers nearest to $2n$?
13. If n is any integer, $2n + 1$ will be an odd number. What are the next two consecutive odd numbers?
14. If n represents the numerator of a fraction and d the denominator, write the fraction. Write the fraction with its numerator decreased by 2 and its denominator increased by 3.
15. A boy who had d dollars bought p pads at c cents each. How many cents had he left?
16. The number 25 may be written $20 + 5$. Write the number whose first digit is t and second digit u .
17. The number 376 may be written $300 + 70 + 6$. Write the number whose first digit is h , second digit t , and third digit u .
18. Add $8t + 6u$, $7t + 3u$, and $9t + 8u$. Compare the result with the sum of $86 + 73 + 98$.

19. Indicate the product of $x + y - d$ by xy .

20. Show that the product of the sum of two numbers $(a + b)$ by their difference $(a - b)$ equals the difference of the squares of the numbers.

21. If a man earns a dollars a month and his expenses are b dollars a month, how much will he save in a year?

22. Indicate the sum of x and z multiplied by m times the sum of x and y .

23. State in an equation that x equals d times $(y - c)$.

24. How many seconds are c hours $+ d$ minutes?

25. Express in cents the interest on y dollars for x years, if the interest on 1 dollar for 1 month is z cents.

26. How far can a man ride in a hours at the rate of b miles an hour? How far will he have ridden in a hours if he stops c hours of the time to rest?

27. How many square rods are there in a square field one of whose sides is $2b$ rods long? $(x - y)$ rods long?

28. What is the number of square rods in a rectangular field whose length is $(a + b)$ rods and width $(a - b)$ rods?

29. A fence is built across a rectangular field so as to make the part on one side of the fence a square. If the field is a rods long and b rods wide, what is the area of each part?

30. Represent $(a - b)$ times the number whose tens' digit is t and units' digit u .

DIVISION

(Review page 87.)

To divide a monomial by a monomial

Because $a^2 \cdot a^3 = a^{2+3} = a^5$

$\therefore a^5 \div a^3 = a^{5-3} = a^2$ (Exponents subtracted)

Similarly, because $3a^2 \cdot 5a^3 = (3 \cdot 5)a^{2+3} = 15a^5$

$\therefore 15a^5 \div 5a^3 = (15 \div 5)a^{5-3} = 3a^2$ (Coefficients divided)

The quotient may be obtained, just as in arithmetic, by dividing equal factors in dividend and divisor.

$$(1) \quad \frac{15 a^5}{5 a^3} = \frac{\overset{1}{3} \cdot \overset{1}{\cancel{5}} \cdot \overset{1}{\cancel{a}} \cdot \overset{1}{\cancel{a}} \cdot \cancel{a} \cdot a \cdot a}{\underset{1}{\cancel{5}} \cdot \underset{1}{\cancel{a}} \cdot \underset{1}{\cancel{a}} \cdot \underset{1}{\cancel{a}}} = 3 a^2$$

$$(2) \quad \frac{21 a^5 b^3}{-3 a^3 b^2} = \frac{\overset{1}{\cancel{3}} \cdot \overset{1}{7} \cdot \overset{1}{\cancel{a}} \cdot \overset{1}{\cancel{a}} \cdot \overset{1}{\cancel{a}} \cdot a \cdot a \cdot \overset{1}{\cancel{b}} \cdot \overset{1}{\cancel{b}} \cdot b}{-\overset{1}{\cancel{3}} \cdot \overset{1}{\cancel{a}} \cdot \overset{1}{\cancel{a}} \cdot \overset{1}{\cancel{a}} \cdot \overset{1}{\cancel{b}} \cdot \overset{1}{\cancel{b}}} = -7 a^2 b$$

$$\text{or } (2) \quad \frac{21 a^5 b^3}{-3 a^3 b^2} = \frac{21}{-3} a^{5-3} b^{3-2} = -7 a^2 b^1 = -7 a^2 b$$

Hence, for division :

Law of exponents. *The exponent of a number in the quotient is equal to its exponent in the dividend minus its exponent in the divisor.*

Since a number divided by itself equals 1, $a^5 \div a^5 = 1$; but $a^5 \div a^5 = a^{5-5} = a^0$; hence $a^0 = 1$. That is, any number whose exponent is 0 is equal to 1.

Law of coefficients. *The coefficient of the quotient is equal to the coefficient of the dividend divided by the coefficient of the divisor.*

Law of signs. *When the signs of the dividend and the divisor are alike, the sign of the quotient is +. When the signs of the dividend and the divisor are unlike, the sign of the quotient is -.*

EXERCISES

Tell the quotients quickly :

1. $a) \underline{5 a^3}$

6. $7 c^2 d^4) \underline{35 c^4 d^5}$

11. $2 a^3) \underline{a^5 x}$

2. $x^2) \underline{2 x^3}$

7. $3^4 \div 3^4$

12. $4^0 m^3) \underline{4^5 m^4 n^2}$

3. $\frac{12 a^3 b^3}{4 a b^2}$

8. $\frac{-18 x^4 y^4}{3 x^2 y}$

13. $\frac{-21 a b^3 c^2}{-7 b^3}$

4. $\frac{28 a^4 b^2 c^2}{-4 a b c}$

9. $\frac{16 x^3 y^3 z^3}{-4 x y^3 z}$

14. $\frac{24 x^{2a} y^{2b} z^{3a}}{8 x^{2a} z^{2a}}$

5. $\frac{-20 a^4 b^5 y^2}{-4 a^2 b^2 y^2}$

10. $\frac{-36 a^4 y^2 z^3}{9 a^4 z^2}$

15. $\frac{a^{4x} b^{2y} c^{3z}}{a^{2x} b^y c^{5z}}$

To divide a polynomial by a monomial

Dividing as in arithmetic, we have

$$(1) \begin{array}{r} 2 \overline{)86} \\ 43 \end{array} \quad \begin{array}{r} 2 \overline{)80+6} \\ 40+3 \end{array} \quad \begin{array}{r} 2 \overline{)8 \text{ tens} + 6 \text{ units}} \\ 4 \text{ tens} + 3 \text{ units} \end{array} \quad \begin{array}{r} 2 \overline{)8t+6u} \\ 4t+3u \end{array}$$

(2) Since $(a + b)x = ax + bx$,
if $ax + bx$ is regarded as the dividend and x as the divisor,
 $(ax + bx) \div x = a + b$; that is,

The quotient of a polynomial divided by a monomial is equal to the sum of the partial quotients obtained by dividing each term of the polynomial by the monomial.

EXERCISES

1. Divide $9x^2y^2 + 15xy^2z^2$ by xy^2 ; by $3xy$.

SOLUTIONS

$$\begin{array}{r} xy^2 \overline{)9x^2y^2 + 15xy^2z^2} \\ 9x \quad + 15z^2 \end{array}$$

$$\begin{array}{r} 3xy \overline{)9x^2y^2 + 15xy^2z^2} \\ 3xy \quad + 5yz^2 \end{array}$$

Find the quotients:

2. $\begin{array}{r} 4cd \overline{)4c^2d - 20cd^2} \end{array}$

6. $\begin{array}{r} 24a^6b^2 + 32a^5b^3 + 40a^4b^4 \\ - 8a^4b^2 \end{array}$

3. $\begin{array}{r} xz^2 + 3xz - x^2z^2 \\ xz \end{array}$

7. $\begin{array}{r} 7ab \overline{)14a^4b^3 + 49a^2b} \end{array}$

4. $\begin{array}{r} 5x^2y - 10x^2y^2 + 15xy^2 \\ - 5xy \end{array}$

8. $\begin{array}{r} 35x^2y^3z^4 + 45x^4y^3z^2 \\ - 5x^2y^2z \end{array}$

5. $\begin{array}{r} 4a^2b^3 - 12a^3b^2 + 16a^4b \\ 4a^2b \end{array}$

9. $\begin{array}{r} 36a^3b^4c^6 + 60a^2b^5c^7 \\ - 12a^2b^4c^6 \end{array}$

10. $\begin{array}{r} 24r^3s^2 + 30r^2s^2 - 42r^2s^3 \\ 6r^2s^2 \end{array}$

11. $\begin{array}{r} 9x^2yz + 36xy^2z^3 - 45xyz^5 \\ - 9xyz \end{array}$

12. $(8a^7b^3 + 28a^6b^4 - 16a^5b^5 - 4a^4b^6) \div 4a^4b^3$

13. $(3x^3yz^2 + 15x^5y^2z^3 + 6x^4yz^3 - 18x^6y^3z) \div 3x^3yz$

Divide:

14. $a^2y - 2ay^2$ by ay ; by $-ay$

15. $9x^2y^2 + 15xy^2$ by $3xy^2$; by $-3xy^2$

16. $-xz^3 - 3xz + x^2z^2$ by $-xz$; by xz

17. $3x^3 - 6x^5 + 9x^7 - 12x^9$ by $3x^2$; by $-3x^2$

18. $30r^3s^3 + 15r^2s^2 - 45rs^4 + 75r$ by $15r$; by $-15r$

19. $-t^5u - t^4uv + tu^4v - t^3u^3v^2 + tu^2v^3$ by tu ; by $-tu$

To divide a polynomial by a polynomial

ILLUSTRATIVE EXAMPLES

1. Divide $3x^2 - 35 + 8x$ by $x + 5$; check the result.

	SOLUTION	CHECK ($x = 2$)
	$\begin{array}{r l} 3x^2 + 8x - 35 & x + 5 \\ 3x \text{ times } (x + 5) & 3x^2 + 15x \\ \hline & -7x - 35 \\ -7 \text{ times } (x + 5) & -7x - 35 \\ \hline & 0 \end{array}$	$\begin{array}{r} -7 \div 7 \\ = -1 \end{array}$

EXPLANATION. For convenience, the divisor is written at the right of the dividend and the quotient below the divisor. Both dividend and divisor are arranged according to the descending powers of x .

Since the dividend is the product of the quotient and the divisor, it is the sum of all the partial products formed by multiplying each term of the quotient by each term of the divisor. Hence if $3x^2$, the first term of the dividend, as arranged, is divided by x , the first term of the divisor, the result, $3x$, is the first term of the quotient.

As in arithmetic, the whole of the undivided part of the dividend is not brought down for each division, but only as much of it as may be needed each time.

Subtracting $3x$ times $(x + 5)$ from the dividend leaves $-7x - 35$, the part of the dividend still to be divided.

Proceeding as before, we find $-7x \div x = -7$, the next term of the quotient. -7 times $(x + 5)$ equals $-7x - 35$. Subtracting, we have no remainder. Hence the quotient is $3x - 7$.

CHECK. When $x = 2$ the dividend equals -7 and the divisor, 7 . The quotient then should equal $-7 \div 7$, or -1 . On substituting 2 for x , we find that the quotient is equal to -1 . Presumably, then, the result is correct. Another way of checking the result is to multiply the quotient by the divisor. The product should be the dividend.

2. Divide $x^3 + 6x^2 + 12x - 1$ by $x + 2$.

SOLUTION

$$\begin{array}{r|l}
 x^3 + 6x^2 + 12x - 1 & x + 2 \\
 \hline
 x^3 + 2x^2 & x^2 + 4x + 4 + \frac{-9}{x+2} \\
 \hline
 4x^2 + 12x & \\
 4x^2 + 8x & \\
 \hline
 4x - 1 & \\
 4x + 8 & \\
 \hline
 -9 &
 \end{array}$$

CHECK ($x = 2$)

$$55 \div 4 = 13\frac{3}{4}$$

$$\text{Quotient} = 4 + 8 + 4 - \frac{9}{4} = 16 - 2\frac{1}{4} = 13\frac{3}{4}$$

NOTE. In the check above, 2 is substituted for x . The result may be checked by substituting any other value for x , except such as gives for the result $0 \div 0$ or any number divided by 0, because we are unable to determine the numerical value of such results. It is better, however, not to substitute 1 for x , as this does not check the exponents.

RULE. Arrange both dividend and divisor according to the ascending or the descending powers of a common letter.

Divide the first term of the dividend by the first term of the divisor, and write the result for the first term of the quotient.

Multiply the whole divisor by this term of the quotient, and subtract the product from the dividend. The remainder will be a new dividend.

Divide the new dividend as before, and continue to divide in this way until the division is exact or until the divisor is of a lower degree than the new dividend.

If there is a remainder after the last division, write it over the divisor in the form of a fraction, and add the fraction to the part of the quotient previously obtained.

EXERCISES

Divide, and check each result :

1. $x^2 + 4x + 4$ by $x + 2$
2. $x^2 - 4x + 4$ by $x - 2$
3. $x^2 + x - 20$ by $x + 4$
4. $x^2 + 7x + 12$ by $x + 3$
5. $x^2 - 3x - 18$ by $x - 6$
6. $l^4 - 6l^2 + 16$ by $l^2 + 2$
7. $9 + 6x + x^2$ by $3 + x$
8. $x^2 + 8x + 18$ by $x + 3$
9. $m^2 - 18 - 3m$ by $m - 6$
10. $x^2 + 2xy + y^2$ by $x + y$

Divide, and check each result :

11. $3a^3 + 10a^2 - 5a + 12$ by $a + 4$

12. $5x^4 + x^3 - 3x^2 - x - 2$ by $x^2 - 1$

13. $25x^5 - x^3 - 8x - 2x^2$ by $5x^2 - 4x$

14. $a^8 + a^6 + a^4 + 3a - 1$ by $a + 1$

15. $b^9 + 6b^6 + b^4 + 9b^3 + 4b + 8$ by $b^3 + 4$

16. $6z^4 + 4 + 10z^2 + 4z^6$ by $4z^2 + 2$

17. $x^4 - x^3y + 2x^2y^2 - xy^3 + y^4$ by $x^2 + y^2$

18. $8m^3 + 8m^2n + 4mn^2 + n^3$ by $2m + n$

19. $2a^4 - 5a^3b + 6a^2b^2 - 4ab^3 + b^4$ by $a^2 - ab + b^2$

20. $140 + 46a + 27a^2 + a^4 + 2a^5$ by $7 + 3a + 2a^2$

21. $4y^4 - 9y^2 - 1 + 6y$ by $3y + 2y^2 - 1$

22. $x^4 - 3x^3 - 36x^2 - 71x - 21$ by $x^2 - 3 - 8x$

23. $6a^2 + 13ab + 6b^2$ by $3a + 2b$

24. $3m^2 - 4am^3 + a^2m^4$ by $am - 1$

25. $6a^3 - 18a + 12$ by $a - 1$

26. $4a^3 + 7a^2 + 2a + 8$ by $a + 2$

27. $ax^3 - a^2x^2 - bx^2 + b^2$ by $ax - b$

28. $20x^2y - 25x^3 - 18y^3 + 27xy^2$ by $6y - 5x$

29. $a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4$ by $a^2 - 2ax + x^2$

30. Divide $c^3 - 8$ by $c + 2$.

SOLUTION

$$\begin{array}{r|l}
 c^3 & -8 \\
 c^3 + 2c^2 & \\
 \hline
 -2c^2 & -8 \\
 -2c^2 - 4c & \\
 \hline
 & 4c - 8 \\
 & 4c + 8 \\
 & \hline
 & -16
 \end{array}
 \quad
 \begin{array}{l}
 c + 2 \\
 \hline
 c^2 - 2c + 4 + \frac{-16}{c + 2}
 \end{array}$$

CHECK ($c = 3$)
 $19 \div 5 = 3\frac{4}{5}$

Divide, and check results :

31. $x^5 + 32$ by $x + 2$

33. $m^5 - n^5$ by $m + n$

32. $x^6 - y^6$ by $x^2 + y^2$

34. $m^5 + n^5$ by $m + n$

35. $x^7 + 2x^6 - 2x^4 + 2x^2 - 1$ by $x + 1$

36. $7x^3 + 2x^4 - 27x^2 + 16 - 8x$ by $x^2 + 5x - 4$

37. $28x^4 + 6x^3 + 6x^2 - 6x - 2$ by $2 + 2x + 4x^2$

38. Divide $81 + 9a^{2n} + a^{4n}$ by $a^{2n} - 3a^n + 9$; check the result by multiplication.

SOLUTION

$$\begin{array}{r}
 a^{4n} \qquad \qquad + 9a^{2n} \qquad \qquad + 81 \quad | \quad a^{2n} - 3a^n + 9 \\
 a^{4n} - 3a^{3n} + 9a^{2n} \quad \quad \quad \quad \quad \quad \quad | \quad a^{2n} + 3a^n + 9 \\
 \hline
 3a^{3n} \qquad \qquad \qquad \qquad \qquad \qquad + 81 \\
 3a^{3n} - 9a^{2n} + 27a^n \\
 \hline
 9a^{2n} - 27a^n + 81 \\
 9a^{2n} - 27a^n + 81 \\
 \hline
 \end{array}$$

CHECK

$$\begin{array}{r}
 a^{2n} - 3a^n + 9 \\
 a^{2n} + 3a^n + 9 \\
 \hline
 a^{4n} - 3a^{3n} + 9a^{2n} \\
 3a^{3n} - 9a^{2n} + 27a^n \\
 9a^{2n} - 27a^n + 81 \\
 \hline
 a^{4n} \qquad \qquad + 9a^{2n} \qquad \qquad + 81
 \end{array}$$

Divide, and check by multiplication :

39. $a^{4n} + 16 + 4a^{2n}$ by $2a^n + a^{2n} + 4$

40. $x^{5a} - 61x^a - 60$ by $x^{2a} - 2x^a - 3$

41. $a^{5n} - 41a^n - 120$ by $a^{2n} + 4a^n + 5$

42. $25x^{5n} - x^{3n} - 8x^n - 2x^{2n}$ by $5x^{2n} - 4x^n$

43. $a^{8n} + a^{6n} + a^{4n} + a^{2n} + 3a^n - 1$ by $a^n + 1$

44. $4y^{2n+4} - 9y^{2n+2} + 6y^{n+2} - y^2$ by $2y^{n+2} - 3y^{n+1} + y$

ALGEBRAIC REPRESENTATION

1. Express 5 dollars in terms of cents ; 5 cents in terms of dollars ; m dollars as cents ; m cents as dollars.
2. Find the value of x that will make 6 x equal to 48.
3. By what number must 25 be multiplied to produce 300? How many times 10 equals x ? How many times r equals s ?
4. Represent (the third power of a minus the fifth power of x) divided by (m plus n^2).
5. Express the multiplicand when lmn is the product and lm the multiplier.
6. Find an expression for 5 per cent of x ; y per cent of z .
7. It takes a men c days to do a piece of work. How long will it take 1 man to do it? 2 men? x men?
8. If D is the dividend, r the remainder, d the divisor, and q the quotient, show that $D - r = dq$. Find an expression for $\frac{D}{d}$ in terms of q , r , and d .
9. A train ran s miles in r hours and m miles in the succeeding h hours. Find its average rate per hour during each period and during the whole time.
10. If tennis balls cost r cents a dozen last year and the price has advanced 60 cents a dozen, how much will half a dozen cost at the present rate?
11. If a man has D dollars and spends $\frac{1}{3}$ of that sum, how many dollars has he left?
12. If f represents the number of feet in the length of a line, express $\frac{1}{4}$ of it increased by 5 inches.
13. If an automobile goes m miles an hour, what is its rate in feet per minute? per second?
14. A man walks m miles in h hours. How far can he walk in 3 hours? in a hours?

15. How many pounds of sugar at b cents a pound would cost as much as e dozen eggs at c cents a dozen?
16. How many yards are there in i inches? in f feet?
17. A boy will be y years old 2 years hence. How old was he 4 years ago?
18. A book cost d dollars. How many dollars will have to be paid for a second book if it costs c cents more than the first?
19. A rectangle is b inches long and c inches wide. Find its area and its perimeter.

Problems

Solve the following problems and check the solutions:

1. I bought 55 stamps for 95 cents. If part of them were 2-cent stamps and part 1-cent stamps, how many of each did I buy?

SOLUTION

Let s = the number of 2-cent stamps

Then $55 - s$ = the number of 1-cent stamps

- | | | |
|-----|-----------------------|---------------------|
| (1) | $2s + 1(55 - s) = 95$ | Equality axiom |
| (2) | $2s + 55 - s = 95$ | Parenthesis removed |
| (3) | $s + 55 = 95$ | Terms in (2) united |
| (4) | $s = 40$ | (3) - 55 |
| (5) | $55 - s = 15$ | Substitution |

$\therefore$ I bought 40 2-cent stamps and 15 1-cent stamps.

CHECK. The results obtained may satisfy the *equation of the problem* and still be incorrect, because the equation may be incorrect. If, however, the results satisfy the *conditions of the problem*, the solution is presumably correct.

First condition: The whole number of stamps bought is 55

$$40 + 15 = 55$$

Second condition: The total cost of the stamps = 95¢

The cost of 40 stamps @ 2¢ and 15 stamps @ 1¢ = 95¢

2. A rectangle is twice as long as it is wide. Its perimeter is 240 feet. Find its dimensions.
3. My lawn is 45 feet deeper than it is wide. The distance around it is 270 feet. Find its dimensions.

4. The square of a number is 15 less than the product of this number and the next higher one. Find the number.

5. The difference of the squares of two consecutive numbers is 33. Find the numbers.

6. The students of a school numbering 210 raised \$175 with which to buy pictures. The seniors gave \$1.50 each; the rest, \$.50 each. Find the number of seniors.

(7.) A certain paper mill produces each week 350 tons of paper from sawdust. Of this 50 tons more is used for newspapers than for wrapping paper. How many tons are used for each?

8. The roadway of the Connecticut Avenue concrete bridge in Washington, D. C., together with two sidewalks, is 52 feet wide. How wide is the roadway, if it is 8 feet narrower than twice the combined width of the sidewalks?

SUGGESTION. Let $2f$ equal the number of feet in the combined width of the sidewalks.

9. One year the box factories of New England used 6,000,000 feet of boards. The amount of white pine used was 1,200,000 feet less than 3 times the amount of the other timber used. How much white pine was used?

(10.) Edward and Stanley together caught 14 fish. Edward caught 4 less than 2 times the number Stanley caught. How many fish did each catch?

11. Of the 160,000 inhabitants of Hawaii one year, twice as many were Japanese as Chinese. The rest of the inhabitants, or $\frac{1}{4}$ of the total, were Americans and Europeans. Find the number of Chinese.

12. It cost a man 60¢ to send a telegram at "30 - 2," that is, 30¢ for the first 10 words and 2¢ for each additional word. How many words did the message contain?

SUGGESTION. Let w represent the number of words in the message. Then $w - 10$ will represent the number of words in excess of 10 words.

$$\therefore 30 + 2(w - 10) = 60$$

13. How many words can be sent by telegraph for 65¢ at the day rate, "35 - $2\frac{1}{2}$ "?

14. A long-distance telephone message cost \$1.25. The rate was 50¢ for the first 3 minutes and 15¢ for each additional minute. How long did the conversation last?

15. Charles was twice as old as his sister Alice 4 years ago. Now Alice is $\frac{2}{3}$ as old as Charles. Find the present age of each.

16. A 20-hour train between New York and Chicago was late 91 times during its first year's run. It was late at Chicago 10 more times than 50% as many times as it was late at New York. How many times was it late at New York?

SUGGESTION. Let n represent the number of times the train was late at New York. Then $91 - n$ represents the number of times the train was late at Chicago.

17. John and his father were working for a builder at different rates per hour. John's father received 40 cents more per hour than John, and in 20 hours they earned together \$28. How much did each earn per hour?

REVIEW

1. In the following expression point out the positive numbers; the negative numbers. Unite similar terms.

$$3ax + 7by - 9bx + 10by - 4ax - 3bx + 4ax - 2ax - 12by$$

2. If we multiply any quantity by zero, what is the result?

3. What is the sign of the product of an even number of negative factors? of an odd number of negative factors?

4. State the law of signs for multiplication; for division.

5. In what respect do $(a - b)$ and $(b - a)$ differ?

6. How may a parenthesis preceded by a minus sign be removed from an algebraic expression without changing the value of the expression?

7. Add $3a + 5b - 11c$, $b - 2a + c$, $2c + 3a - b$, $7c - b + 6a$, $5b - 4a - 2c$, $b - a$, and $c + b - a$.

8. Subtract the sum of $x - 2y + 3z - 5w$ and $7x + w - 2z$ from $10x - y + z - 8w$.

9. If $x = r^2 + rs - s^2$, $y = 2r^2 + 4rs + 2s^2$, and $z = r^2 - 3rs - s^2$, find the value of $x + y - z$.

Simplify:

10. $17x - [3y + 4z - (z + 5a + x - 3a - 2y)]$

11. $a + 2b - [4c - 2(a + 2b) - b]$

When $x = 2$, $y = -3$, and $z = 5$, find the value of:

12. $xz - (x + y + z)$

13. $3(x - y) + 2(y - x) - zy$

14. $x^2 - 3x(y + z) + y^2 - z$

15. $(x - y)(y + z) - z^2(y - z)$

16. Unite similar terms within parentheses:

$$ax^2 - cy + ax - 2ax^2 + 2cy^2 - ax - cy^2 + ax^2 + cy$$

17. Unite the coefficients of x and y :

$$7ax - 3by - 22a^2x + 5ay - 17bx + cy - 4x + 13y$$

Solve, and check results:

18. $3b + 5 = -2$

19. $8a - 5 + 2a = 4a - 14 + 3$

20. $6 - 3x = 14 - 5x$

21. $3x + 7(x - 2) - 13 = 12 - 3x$

22. $20 = 7 - 5(3 - x) + 9(x + 2)$

23. $x^3 - 1 = x(x^2 - x) + x + 3 + x^2$

24. $(x - 2)(x + 3) - (x + 5)(x - 7) = 1$

25. $(2x - 3)(x + 2) = 2(x + 5)(x - 1) - 10$

CHAPTER VII

SPECIAL PRODUCTS AND FACTORING

Products and factors

Having learned that $5 \times 7 = 35$, you know that the factors of 35 are 5 and 7. Also having learned that $2 \times 3 \times 3 = 18$, you know that the factors of 18 are 2, 3, and 3.

In algebra, also, you need to know how to multiply some numbers rapidly and how to do the reverse process called **factoring**.

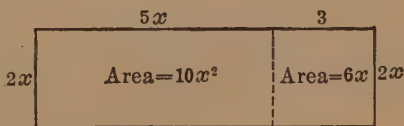
Thus, when you know that the product of x and $x + y = x^2 + xy$, you know also that the factors of $x^2 + xy$ are x and $x + y$.

A number that has no integral factors except itself and 1 is called a **prime number**.

Algebra is a language which tells much in a few words and symbols. Instead of writing: "Find the product of $5x + 3$ and $2x$," we may write, "Find the value of $2x(5x + 3)$." The parenthesis around $5x + 3$ shows us that $5x + 3$ is one number. The fact that there is no sign between $2x$ and $5x + 3$ means that we are to find the product of the two numbers.

Practice in finding products

The area of the figure at the right, which is $(5x + 3)$ units long and $2x$ units wide, may be expressed by the algebraic number $2x(5x + 3)$. Why? We can find this product from the figure if we look at the two parts of the rectangle made by the dotted line. By adding the areas of the two parts, we see that $10x^2 + 6x$ is the total area. This is the product of $2x$ and $5x + 3$.



EXERCISES

1. Let a be some convenient length. Draw a rectangle whose dimensions are $4a + 2$ and $3a$. Find, by inspecting the figure, the product of $4a + 2$ and $3a$.

2. Draw a figure showing the product of $3x$ and $5x + 7$.

3. Draw a figure showing the product of $6n(2n + 3)$.

4. What is the rule for multiplying numbers like $2x(3x + 2)$ and $2b(5b + 2)$?

5. Since $3a(a - 4) = 3a^2 - 12a$, what are the factors of $3a^2 - 12a$?

6. Since $4a(2b + 3c + 5) = 8ab + 12ac + 20a$, what are the factors of $8ab + 12ac + 20a$?

To factor an expression whose terms have a monomial factor

EXERCISES

1. Find the factors of $3xy - 6x^2y + 9xy^2$.

SOLUTION

$$3xy - 6x^2y + 9xy^2 \\ = 3xy(1 - 2x + 3y)$$

EXPLANATION. By examining the terms of the expression, we see that the monomial $3xy$ is a factor of every term and that there is no other common factor.

Dividing by this common factor gives the other factor.

Hence the factors of $3xy - 6x^2y + 9xy^2$ are $3xy$ and $1 - 2x + 3y$.

CHECK. The product of the factors should equal the given expression. Thus: $3xy(1 - 2x + 3y) = 3xy - 6x^2y + 9xy^2$

Always find the prime factors. Look first for a monomial factor.

Factor, and check each result:

(2.) $5x^3 - 5x^2$

3. $8x^2 + 2x^4$

4. $3x^3 - 6x^2y$

5. $4a^2 - 6ab$

6. $5m^2 - 3mn$

7. $3x^3y^2 - 3x^2y^3$

8. $4a^3b - 6a^2b^2$

9. $5m^4n - 10m^3n^2$

10. $3x^4 - 9x^3 - 6x^2$

11. $3a^4 - 2a^3b + a^2$

12. $x^{12} + x^{11} + x^{10} - x^9$

13. $ac - bc - cy - abc$

14. $3x^3y^3 - 3x^2y^2 + 12xy$

15. $3m^5 - 12m^3n^2 + 6mn^4$

16. Factor $a(a - b) + b(a - b)$.

SOLUTION

In this expression we see that the binomial $(a - b)$ is a factor of each term. Removing the binomial factor as we do the monomial factor, we find the factors to be $(a - b)(a + b)$.

17. Factor $x(2a - 3b) + y(3b - 2a)$.

SOLUTION

$$\begin{aligned} x(2a - 3b) + y(3b - 2a) &= x(2a - 3b) + 3by - 2ay \\ &= x(2a - 3b) - 2ay + 3by = x(2a - 3b) - y(2a - 3b) \\ &= (2a - 3b)(x - y) \end{aligned}$$

Factor :

18. $a(x - y) + 5(x - y)$

20. $c^2(y + z)^2 - d(y + z)^2$

19. $4(a + b) - x(a + b)$

21. $5(x - y) - 2a(y - x)$

22. $2a(3x - 5y) + b(5y - 3x)$

To factor a polynomial whose terms may be grouped to show a common polynomial factor

EXERCISES

1. Factor $ax + ay + bx + by$.

SOLUTION

$$\begin{aligned} ax + ay + bx + by &= a(x + y) + b(x + y) \\ &= (a + b)(x + y) \end{aligned}$$

2. Factor $ax - ay - bx + by$.

SOLUTION

$$\begin{aligned} ax - ay - bx + by &= a(x - y) - b(x - y) \\ &= (a - b)(x - y) \end{aligned}$$

Factor, and check each result :

3. $am - an + mx - nx$

8. $x^2 + xy - ax - ay$

4. $bc - bd + cx - dx$

9. $c^2 - 4c + ac - 4a$

5. $pq - px - rq + rx$

10. $2x - y + 4x^2 - 2xy$

6. $ay - by - ab + b^2$

11. $1 - m + n - mn$

7. $x^2 - xy - 5x + 5y$

12. $2p + q + 6p^2 + 3pq$

13. $b^2 - bc + ab - ac$

16. $y^3 + y^2 - 3y - 3$

14. $ar - rs - ab + bs$

17. $x^5 + x^3 + x^2y + y$

15. $x^3 + x^2 + x + 1$

18. $2 - 2n - n^2 + n^3$

19. $x^2 - x + ax - a$

20. $3x^3 - 15x + 10y - 2x^2y$

21. $12a^3 - 8ab - 3a^4 + 2a^2b$

22. $3m^2n - 9mn^2 + am - 3an$

23. $15ab^2 - 9b^2c - 35ab + 21bc$

24. $16ax + 12ay - 6by - 8bx$

25. $ax^2 - ax - axy + ay + x - 1$

26. $xy + x - 3y^2 - 3y - 4y - 4$

27. $ax - a - bx + b - cx + c$

28. $mx - nx - x - my + ny + y$

29. $bx^2 - b - xy - y + yx^2 - bx$

30. $m^2 + mn + mn + n^2 + m + n$

To square a binomial

EXERCISES

Find the value of each of the following by actual multiplication and write the result as shown in example 1:

1. $(x + 6)^2 = x^2 + 12x + 36$

2. $(a + 4)^2$

6. $(y - 2)(y - 2)$

3. $(m + 3)^2$

7. $(r - \frac{1}{2})(r - \frac{1}{2})$

4. $(x - 5)^2$

8. $(a + b)(a + b)$

5. $(x - 8)(x - 8)$

9. $(a + 3b)^2$

10. Write the sum of a and b . Write the square of the sum of a and b (to represent the square of the sum of two numbers).

From exercises like 1 to 10 we have the following rule:

RULE. *The square of the sum of two numbers equals the square of the first, plus twice the product of the first by the second, plus the square of the second.*

11. Write the difference of two numbers, a and b . Write the square of $a - b$ (to represent the square of the difference of two numbers).

12. Compare the square of $a + b$ with the square of $a - b$.

13. Write a rule for the square of the difference of two numbers based on the rule for the square of the sum of two numbers. In what respect do the two rules differ?

Find the following squares at sight. Check by multiplying if you have any doubt about your result:

- | | | |
|------------------|------------------|--------------------|
| 14. $(y + 3)^2$ | 17. $(k + 10)^2$ | 20. $(2y - 5)^2$ |
| 15. $(a - 2b)^2$ | 18. $(k - 10)^2$ | 21. $(3a + 2bc)^2$ |
| 16. $(a - 4t)^2$ | 19. $(a + b)^2$ | 22. $(8x - 5ay)^2$ |

Square the following binomials by inspection:

- | | | |
|---------------|---------------|---------------|
| 23. $x + m$ | 30. $2a - 6x$ | 37. $2a + 3b$ |
| 24. $m - n$ | 31. $3m - 8n$ | 38. $2x - 7y$ |
| 25. $x - 6$ | 32. $4x + y$ | 39. $8x - 5y$ |
| 26. $p - 8$ | 33. $p + 3q$ | 40. $3x - 2y$ |
| 27. $r + 7$ | 34. $4a - 3$ | 41. $5m - 3n$ |
| 28. $x^2 + 9$ | 35. $5x + 4$ | 42. $3p - 5q$ |
| 29. $y^2 - 6$ | 36. $ab - 3$ | 43. $4xy - 7$ |

Expand the following:

- | | | |
|---------------------------|----------------------------|----------------------------|
| 44. $(x + \frac{1}{2})^2$ | 51. $(r + \frac{4}{9})^2$ | 58. $(2x - \frac{1}{2})^2$ |
| 45. $(y - \frac{1}{3})^2$ | 52. $(s - \frac{3}{5})^2$ | 59. $(3x + \frac{1}{3})^2$ |
| 46. $(a + \frac{4}{3})^2$ | 53. $(x - \frac{4}{11})^2$ | 60. $(2x - \frac{1}{5})^2$ |
| 47. $(m - \frac{2}{7})^2$ | 54. $(z - \frac{3}{8})^2$ | 61. $(3x - \frac{1}{7})^2$ |
| 48. $(5x + z)^2$ | 55. $(a^3 + b^3)^2$ | 62. $(5x + 2y)^2$ |
| 49. $(2a + x)^2$ | 56. $(a^5 + b^5)^2$ | 63. $(7z + 3c)^2$ |
| 50. $(ab + cd)^2$ | 57. $(a^n + b^n)^2$ | 64. $(3b + 10x)^2$ |

Expand :

- | | | |
|-------------------|-----------------------------|--------------------------|
| 65. $(2a - x)^2$ | 77. $(3x - 2)^2$ | 89. $(.4x - .15y)^2$ |
| 66. $(3m - n)^2$ | 78. $(2x - 5y)^2$ | 90. $(2a + .3b)^2$ |
| 67. $(4x - y)^2$ | 79. $(5m - 3n)^2$ | 91. $(1.5a^2 + 2.1)^2$ |
| 68. $(m - 4n)^2$ | 80. $(4p - 5q)^2$ | 92. $(.7x^a - 1.1y^b)^2$ |
| 69. $(p - 3q)^2$ | 81. $(a^n - b^n)^2$ | 93. $(1.3x^m - .4y^n)^2$ |
| 70. $(a - 7b)^2$ | 82. $(x^m - y^n)^2$ | 94. $(2ab + .8c^2)^2$ |
| 71. $(4a - 3)^2$ | 83. $(a^2 - 2b^2)^2$ | 95. $(1 + .03)^2$ |
| 72. $(5x - 4)^2$ | 84. $(y^3 - 6x)^2$ | 96. $(1 + .04)^2$ |
| 73. $(ab - 3)^2$ | 85. $(ab - 2c^2)^2$ | 97. 1.06^2 |
| 74. $(2z - 7y)^2$ | 86. $(x^{t+r} - y^{2r})^2$ | 98. 1.05^2 |
| 75. $(7x - 5y)^2$ | 87. $(mx^m - ny^n)^2$ | 99. 2.03^2 |
| 76. $(9w - 2v)^2$ | 88. $(x^{m-1} - y^{n-1})^2$ | 100. 2.05^2 |

$$35^2 = (30 + 5)^2 = 30^2 + 10 \times 30 + 5^2 \\ = 30(30 + 10) + 25 = 30(40) + 25 = 1225$$

NOTE. We may also write 4.5^2 as $(4 + .5)^2$ and use the above principle.

Square the following numbers :

- | | | | |
|---------------------|----------------------|----------|----------|
| 101. $5\frac{1}{2}$ | 104. 42 | 107. 29 | 110. 45 |
| 102. $8\frac{1}{2}$ | 105. 53 | 108. 39 | 111. 7.5 |
| 103. 41 | 106. $12\frac{1}{2}$ | 109. 115 | 112. 9.5 |

To factor a trinomial that is a square

By multiplication:

$$(a + b)(a + b) = a^2 + 2ab + b^2 \\ \therefore a^2 + 2ab + b^2 = (a + b)(a + b)$$

By multiplication:

$$(a - b)(a - b) = a^2 - 2ab + b^2 \\ \therefore a^2 - 2ab + b^2 = (a - b)(a - b)$$

These two trinomials are **squares**, for each may be separated into two *equal* factors. They are types, showing the form of all **trinomial squares**.

A trinomial is a square, therefore, if these two conditions are fulfilled:

1. Two terms, as $+a^2$ and $+b^2$, must be squares.
2. The other term must be numerically equal to twice the product of the square roots of the terms that are squares.

$25x^2 - 20xy + 4y^2$ is a square, for $25x^2 = (5x)^2$, $4y^2 = (2y)^2$, and $-20xy = -2(5x)(2y)$.

Every number has two square roots, one positive and the other negative. In factoring, usually only the *positive* square root is taken.

Thus, $\sqrt{25}$ (read "the square root of 25") equals 5 or -5 , for $5 \cdot 5 = 25$ and $(-5)(-5) = 25$.

$a^2 + 2ab + b^2 = (a + b)(a + b)$ or $(-a - b)(-a - b)$, but we usually factor trinomial squares in the first way only.

RULE. *Arrange the trinomial in order of the descending powers of one of its letters. See whether the middle term is equal to twice the product of the square roots of the first and last terms. If it is, connect the square roots of the first and last terms with the sign of the middle term and indicate that the result is to be taken twice as a factor.*

EXERCISES

Make each of the following a trinomial square by writing the missing term:

- | | | |
|--------------------|--------------------|---------------------|
| 1. $x^2 + ? + y^2$ | 4. $c^2 - 2cd + ?$ | 7. $a^2 + 14ab + ?$ |
| 2. $a^2 - ? + b^2$ | 5. $x^2 + 4xy + ?$ | 8. $p^2 - 8pq + ?$ |
| 3. $y^2 + ? + z^2$ | 6. $r^2 - 8rs + ?$ | 9. $x^2 + 6xy + ?$ |

Factor, and check each result:

- | | |
|-----------------------|-------------------------------|
| 10. $x^2 + 2xy + y^2$ | 16. $5x^2 + 30x + 45$ |
| 11. $p^2 - 2pq + q^2$ | 17. $16p^2 - 24p + 9$ |
| 12. $c^2 + 2cd + d^2$ | 18. $9x^2 - 42x + 49$ |
| 13. $x^2 - 2x + 1$ | 19. $9 + 42b^3 + 49b^6$ |
| 14. $x^2 + 4x + 4$ | 20. $36n^8 - 12n^4 + 1$ |
| 15. $x^2 + 6x + 9$ | 21. $16x^4 - 96x^2y + 144y^2$ |

Factor:

22. $4 - 4a + a^2$

23. $4a - 4a^2 + a^3$

24. $m^2 - 8m + 16$

25. $a^2 - 16a + 64$

26. $1 + 8b + 16b^2$

27. $3x^2 + 6xy + 3y^2$

28. $2m^2 - 8mn + 8n^2$

29. $1 - 6a^3 + 9a^6$

30. $16x^2 - 8xy + y^2$

31. $4 + 12m^2 + 9m^4$

32. $9 - 12ab + 4a^2b^2$

33. $s^2 - 12rs + 36r^2$

34. $4x^2 + 4axy + a^2y^2$

35. $9a^2 + 15ab + 6.25b^2$

36. $75x^2 - 210xy + 147y^2$

37. $x^{2r} + 8x^rz + 16z^2$

Ans. $(x^r + 4z)(x^r + 4z)$

38. $4a^2 + 12ab^{2n} + 9b^{4n}$

39. $.09a^{6n} + .3a^{3n}b + .25b^2$

40. $4x^2y^2 - 20xy + 25$

41. $4x^2 + 12xyz + 9y^2z^2$

42. $2x + 20a^2x + 50a^4x$

43. $8a^2b + 40ab^2 + 50b^3$

44. $4x^{2a} + 8x^ay^b + 4y^{2b}$

45. $x^{2n} - 2x^ny^n z^n + y^{2n} z^{2n}$

46. $4x^{2m} + 16x^my^n + 16y^{2n}$

47. $1 + 10mn + 25m^2n^2$

48. $ax^2 + 6axy^2 + 9ay^4$

49. $a^2b^2 - 2abcd + c^2d^2$

50. $9x^2y^2 + 12xyrs + 4r^2s^2$

51. $16x^2 - 56xy + 49y^2$

52. $2.25a^4b^2 - .9a^2b + .09$

53. $49a^2b^2 - 18.2ab^2 + 1.69b^2$

54. $45x^6 - 120x^3y + 80y^2$

55. Factor $x^2 + 6x(x - y) + 9(x - y)^2$.

SOLUTION

$$\begin{aligned}
 & x^2 + 6x(x - y) + 9(x - y)^2 \\
 &= [x + 3(x - y)][x + 3(x - y)] \\
 &= (x + 3x - 3y)(x + 3x - 3y) \\
 &= (4x - 3y)(4x - 3y)
 \end{aligned}$$

Factor:

56. $x^2 + 2x(x - y) + (x - y)^2$

57. $t^2 - 4t(t - 1) + 4(t - 1)^2$

58. $c^2 - 6c(a - c) + 9(a - c)^2$

59. $(r + s)^2 - 4(r + s) + 4$

60. $16 - 24(t - 1) + 9(t - 1)^2$

61. $14(x - y) + (x - y)^2 + 49$

62. $(a + b)^2 - 2(a + b)(b + c) + (b + c)^2$

63. $(a - x)^2 + 4(a - x)(x - b) + 4(x - b)^2$

64. $16(a - x)^2 + 32(a - x)(x + b) + 16(x + b)^2$

Factoring test

Factor the following. If up to standard, you should be able to do at least eighteen examples in six minutes with not more than two errors.

1. $x^2 + 6x + 9$

2. $m^2 + 4mn + 4n^2$

3. $r^2s^2 - 8rs + 16$

4. $16p^2 - 24p + 9$

5. $4x^4 + 4x^2y^2 + y^4$

6. $9 + 42b^3 + 49b^6$

7. $9a^2 - 60ab + 100b^2$

8. $1 - 6m^4 + 9m^8$

9. $4x^2y^2 - 20xy + 25$

10. $4x^2 + 12xyz + 9y^2z^2$

11. $49x^2 + 70x + 25$

12. $x^2 - 20xy + 100y^2$

13. $49a^4 + 112a^2 + 64$

14. $16b^2 - 16bc + 4c^2$

15. $25x^2 + 70xy + 49y^2$

16. $x^2 - \frac{2}{3}x + \frac{1}{9}$

17. $a^2 - \frac{2}{5}a + \frac{1}{25}$

18. $4a^2 + a + \frac{1}{16}$

19. $9b^2 - \frac{6}{5}b + \frac{1}{25}$

20. $16m^2 - \frac{8}{3}mn + \frac{1}{9}n^2$

To find the product of the sum and the difference of two numbers

EXERCISES

Find by multiplication the following products and write the results as the one in example 1 is written :

1. $(x + 2)(x - 2) = x^2 - 4$

4. $(r + 4)(r - 4)$

2. $(a - b)(a + b)$

5. $(x + y)(x - y)$

3. $(m + 10)(m - 10)$

6. $(x + 2y)(x - 2y)$

Multiply the next four problems by inspection :

7. $(x + 4)(x - 4)$

9. $(5m + 2b)(5m - 2b)$

8. $(c + 5)(c - 5)$

10. $(3t + 25)(3t - 25)$

The product of the sum of two numbers by their difference equals the difference of the squares of the two numbers.

Practice in finding the product of the sum and the difference of two numbers by inspection

EXERCISES

Write the products at sight. (The examples should be done with considerable speed.)

- | | |
|----------------------------------|--|
| 1. $(x + y)(x - y)$ | 14. $(12 + xy)(12 - xy)$ |
| 2. $(a + c)(a - c)$ | 15. $(3m^2 - b)(3m^2 + b)$ |
| 3. $(p + q)(p - q)$ | 16. $(2x^3 + 5y^2)(2x^3 - 5y^2)$ |
| 4. $(p + 5)(p - 5)$ | 17. $(2a^2 + 3b^2)(2a^2 - 3b^2)$ |
| 5. $(x + 1)(x - 1)$ | 18. $(-5n - b)(-5n + b)$ |
| 6. $(x^2 + 1)(x^2 - 1)$ | 19. $(\frac{1}{2}x + 3)(\frac{1}{2}x - 3)$ |
| 7. $(ab - c^2)(ab + c^2)$ | 20. $(\frac{2}{3}a - 4b)(\frac{2}{3}a + 4b)$ |
| 8. $(4y + z^2)(4y - z^2)$ | 21. $(-x - 2y)(-x + 2y)$ |
| 9. $(rs - 5m)(rs + 5m)$ | 22. $(3x^m + 7y^n)(3x^m - 7y^n)$ |
| 10. $(3m - 4n)(3m + 4n)$ | 23. $(mx^a + 2y^b)(mx^a - 2y^b)$ |
| 11. $(lx - 5m)(lx + 5m)$ | 24. $(a^nb^m + a^mb^n)(a^nb^m - a^mb^n)$ |
| 12. $(3m + 4n)(3m - 4n)$ | 25. $(x^{m-1} + y^{n+1})(x^{m-1} - y^{n+1})$ |
| 13. $(4x^3y^3 + 5)(4x^3y^3 - 5)$ | 26. $(5a^3b^2 + 2x^2)(5a^3b^2 - 2x^2)$ |

27. 32×28

SOLUTION

$$32 \times 28 = (30 + 2)(30 - 2) = 30^2 - 2^2 = 896$$

- | | | | |
|--------------------|--------------------|---------------------|--------------------|
| 28. 19×21 | 32. 89×91 | 36. 57×63 | 40. 75×65 |
| 29. 31×29 | 33. 45×55 | 37. 58×62 | 41. 27×33 |
| 30. 44×36 | 34. 34×26 | 38. 37×43 | 42. 93×87 |
| 31. 56×64 | 35. 22×38 | 39. 105×95 | 43. 66×54 |

To factor a binomial which is the difference of two squares

ILLUSTRATIVE EXAMPLES

- $a^2 - 9b^2 = (a + 3b)(a - 3b)$
- $32x^2 - 98y^2 = 2(4x - 7y)(4x + 7y)$
- $x^4 - 1 = (x^2 + 1)(x^2 - 1) = (x^2 + 1)(x + 1)(x - 1)$

RULE. (1) Find the square roots of the two terms which are squares.

(2) One factor is the sum of the square roots and the other is the difference of these square roots.

EXERCISES

Write by inspection the prime factors of the following :

- | | | | |
|-----------------------|----------------------|-------------------------------|---------------------|
| 1. $x^2 - 4$ | 5. $c^2 - d^2$ | 9. $x^4 - 49$ | 13. $a^2x^2 - 4c^2$ |
| 2. $a^2 - 9$ | 6. $a^2 - 16$ | 10. $a^4 - 81$ | 14. $25 - 9x^2$ |
| 3. $x^2 - y^2$ | 7. $x^2 - m^2$ | 11. $25x^2 - 1$ | 15. $9b^2 - c^2d^2$ |
| 4. $z^2 - 4$ | 8. $25 - c^2$ | 12. $b^2c^2 - 16$ | 16. $a^{16} - b^8$ |
| 17. $4x^2 - 25y^2$ | 26. $400x^2 - 36y^2$ | 35. $2x^8 - 2y^8$ | |
| 18. $9a^2 - 49b^2$ | 27. $144m^2 - 16n^2$ | 36. $a^2 - \frac{1}{9}$ | |
| 19. $m^4 - 16n^4$ | 28. $18c^2 - 50$ | 37. $x^2 - .01$ | |
| 20. $16a^4 - 81b^4$ | 29. $3x^4 - 3y^6$ | 38. $x^4 - \frac{1}{16}$ | |
| 21. $x^3 - xy^2$ | 30. $36a^4 - 225$ | 39. $x^{2n} - y^{2b}$ | |
| 22. $8s^2 - 2t^2$ | 31. $x^4y^4 - 256$ | Ans. $(x^n + y^b)(x^n - y^b)$ | |
| 23. $169 - a^2b^2$ | 32. $3m^5 - 3m$ | 40. $x^{4a} - y^{4b}$ | |
| 24. $2x^4 - 2y^4$ | 33. $4a^4 - 4b^4$ | 41. $4a^2 - .25$ | |
| 25. $121b^2 - a^2c^2$ | 34. $5x^8 - 5$ | 42. $4a^{2n} - b^{4m}$ | |

43. Find the value of $32^2 - 28^2$.

SOLUTION

$$32^2 - 28^2 = (32 + 28)(32 - 28) = 60 \times 4 = 240$$

Find the value of :

- | | | |
|--------------------------|--------------------------|--------------------------|
| 44. $(12.7)^2 - (9.8)^2$ | 46. $(10.5)^2 - (6.3)^2$ | 48. $(7.8)^2 - (3.2)^2$ |
| 45. $(16.4)^2 - (3.6)^2$ | 47. $85^2 - 15^2$ | 49. $(12.3)^2 - (8.3)^2$ |

Factoring and multiplication test

Cover column *B* on page 136 and give the factors of the numbers printed in column *A* in three minutes. Then cover column *A* and give the products of the numbers in column *B* in three minutes.

Column A

Column B

$$1. y^2 - 16 = (y + 4)(y - 4)$$

$$2. x^4 - y^2 = (x^2 + y)(x^2 - y)$$

$$3. a^2 - 1 = (a - 1)(a + 1)$$

$$4. 1 - y^2 = (1 + y)(1 - y)$$

$$5. b^2 - 9y^2 = (b + 3y)(b - 3y)$$

$$6. 49 - 36b^2 = (7 + 6b)(7 - 6b)$$

$$7. 1 - 25a^2b^2 = (1 - 5ab)(1 + 5ab)$$

$$8. 100x^2 - 36 = 4(5x - 3)(5x + 3)$$

$$9. 121r^2 - 16t^2 = (11r + 4t)(11r - 4t)$$

$$10. .25a^2 - .16b^2 = (.5a - .4b)(.5a + .4b)$$

To factor when either square is a square of a polynomial

EXERCISES

$$1. \text{ Factor } 25a^2 - (3a + 2b)^2.$$

SOLUTION

One factor is $5a + (3a + 2b)$ and the other is $5a - (3a + 2b)$.

$$5a + (3a + 2b) = 5a + 3a + 2b = 8a + 2b = 2(4a + b)$$

$$5a - (3a + 2b) = 5a - 3a - 2b = 2a - 2b = 2(a - b)$$

$$\therefore 25a^2 - (3a + 2b)^2 = 2(4a + b) \cdot 2(a - b) \\ = 4(4a + b)(a - b)$$

Factor:

$$2. a^2 - (b + c)^2$$

$$3. a^2 - (a + b)^2$$

$$4. b^2 - (2a + b)^2$$

$$5. 4c^2 - (b + c)^2$$

$$6. 9b^2 - (a - x)^2$$

$$7. 9a^2 - (2a - 5)^2$$

$$8. x^4 - (3x^2 - 2y)^2$$

$$9. 49a^2 - (5a - 4b)^2$$

$$10. \text{ Factor } (3a - 2b)^2 - (2a - 5b)^2.$$

SOLUTION

$$(3a - 2b)^2 - (2a - 5b)^2$$

$$= [(3a - 2b) + (2a - 5b)][(3a - 2b) - (2a - 5b)]$$

$$= (3a - 2b + 2a - 5b)(3a - 2b - 2a + 5b)$$

$$= (5a - 7b)(a + 3b)$$

Factor:

11. $(2a + 3b)^2 - (a + b)^2$ 12. $(2x + 5)^2 - (5 - 3x)^2$

13. Factor $4a^2 - 4ab + b^2 - c^2$.

SOLUTION

$$(4a^2 - 4ab + b^2) - c^2 = (2a - b)^2 - c^2 = (2a - b - c)(2a - b + c)$$

14. Factor $25x^2 - 9y^2 + 12yz - 4z^2$.

SOLUTION

$$\begin{aligned} 25x^2 - (9y^2 - 12yz + 4z^2) &= (5x)^2 - (3y - 2z)^2 \\ &= (5x - 3y + 2z)(5x + 3y - 2z) \end{aligned}$$

Factor:

15. $a^2 - x^2 - 9y^2 + 6xy$

28. $r^2 - s^2 - 25 + 10s$

16. $9a^2 + 4bc - 4c^2 - b^2$

29. $x^2 + 6x - 16y^2 + 9$

17. $4x^2 + y^2 - 4xy - a^2$

30. $x^2 + 4x - y^2 + 4$

18. $(5a + 6b)^2 - (a + b)^2$

31. $(5b - 4c)^2 - (3a - 2c)^2$

19. $(5a - 3b)^2 - (a - b)^2$

32. $(4x - 3y)^2 - (2x - 3a)^2$

20. $(3x + 4)^2 - (2 - 5x)^2$

33. $(9x + 6y)^2 - (4x - 3y)^2$

21. $(a - 2b)^2 - (a - 5)^2$

34. $(x^3 + x^2)^2 - (2x + 2)^2$

22. $(2x - 3y)^2 - (3y + z)^2$

35. $(a + b + c)^2 - (a - b - c)^2$

23. $a^2 - 2ax + x^2 - n^2$

36. $b^2 - x^2 - y^2 + 2xy$

24. $b^2 + 2by + y^2 - n^2$

37. $4c^2 - x^2 - y^2 - 2xy$

25. $1 - 4q + 4q^2 - a^2$

38. $9c^2 - x^2 - y^2 + 2xy$

26. $r^2 - 2rx + x^2 - 16t^2$

39. $bc^2 - 9a^2b - b^3 - 6ab^2$

27. $c^2 - a^2 - b^2 - 2ab$

40. $ab^2 - 4a^3 - 12a^2c - 9ac^2$

41. $a^2 - 2ab + b^2 - c^2 + 2cd - d^2$

42. $x^2 - 2xy + y^2 - m^2 + 10m - 25$

43. $x^2 - a^2 + y^2 - b^2 + 2xy - 2ab$

44. $4x^2 + 9 - 12x + 10mn - m^2 - 25n^2$

The product of two binomials that have a common term

Let $x + a$ and $x + b$ represent any two binomials having a common term, x . Multiply $x + a$

by $x + b$:

$$\begin{array}{r}
 x + a \\
 x + b \\
 \hline
 x^2 + ax \\
 bx + ab \\
 \hline
 x^2 + (a + b)x + ab
 \end{array}$$

	x	a
x	x^2	ax
b	bx	ab

PRINCIPLE. *The product of two binomials having a common term is equal to the sum of the square of the common term, the product of the sum of the unlike terms by the common term, and the product of the unlike terms.*

EXERCISES

1. Expand: $(x + 2)(x + 5)$; check the result.

SOLUTION

The square of the common term is x^2 .

The sum of 2 and 5 is 7.

The product of 2 and 5 is 10.

$$\therefore (x + 2)(x + 5) = x^2 + 7x + 10$$

$$\text{CHECK. } (x = 2) \quad 4 \cdot 7 = 4 + 14 + 10, \text{ or } 28 = 28$$

2. Expand: $(a + 1)(a - 4)$; check the result.

SOLUTION

The square of the common term is a^2 .

The sum of 1 and -4 is -3 .

The product of 1 and -4 is -4 .

$$\therefore (a + 1)(a - 4) = a^2 - 3a - 4$$

$$\text{CHECK. } (a = 4) \quad 5 \cdot 0 = 16 - 12 - 4, \text{ or } 0 = 0$$

Expand by inspection, and check each result:

3. $(x + 5)(x + 6)$ 8. $(x - 3)(x - 2)$ 13. $(n - 6)(n + 15)$
 4. $(x + 7)(x + 8)$ 9. $(x - 5)(x - 1)$ 14. $(x^2 + 5)(x^2 - 3)$
 5. $(x - 7)(x + 8)$ 10. $(x + 5)(x + 8)$ 15. $(x^3 - 7)(x^3 + 6)$
 6. $(x + 7)(x - 8)$ 11. $(p - 4)(p + 1)$ 16. $(x^n - 5)(x^n + 4)$
 7. $(x - 5)(x - 4)$ 12. $(r - 3)(r - 1)$ Ans. $x^{2n} - x^n - 20$

- | | |
|----------------------------|------------------------------------|
| 17. $(x^n - a)(x^n - 2a)$ | 23. $(ab - 6)(ab + 4)$ |
| 18. $(y - 2b)(y + 3b)$ | 24. $(x^2y^2 - a)(x^2y^2 + 2a)$ |
| 19. $(z - 4a)(z + 3a)$ | 25. $(3xy + y^2)(y^2 - xy)$ |
| 20. $(2x - .7)(2x + .5)$ | 26. $(b^2c^2 + ef)(b^2c^2 + ef)$ |
| 21. $(3y - 1)(3y + 2)$ | 27. $(.5ab + 2c^2)(.5ab + 2c^2)$ |
| 22. $(4x^2 + 1)(4x^2 - 7)$ | 28. $(.3x^2 + 2y^2)(.3x^2 - 2y^2)$ |

To factor a trinomial of the form $x^2 + bx + c$

By multiplication: $(x + a)(x + b) = x^2 + (a + b)x + ab$
 $\therefore x^2 + (a + b)x + ab = (x + a)(x + b)$

This trinomial consists of an x^2 -term, an x -term, and a term without x , that is, a **constant term**. Therefore it has the type form $x^2 + bx + c$.

Hence, if a trinomial of this form is factorable, it may be factored by *finding two numbers whose algebraic product is c (the constant term) and whose algebraic sum is b (the coefficient of x)*. The first factor is $x +$ one of these numbers; the second factor is $x +$ the other number.

$$\begin{aligned}\text{Thus,} \quad x^2 + 8x + 15 &= (x + 3)(x + 5) \\ x^2 - 8x + 15 &= (x - 3)(x - 5) \\ x^2 + 2x - 15 &= (x - 3)(x + 5) \\ x^2 - 2x - 15 &= (x + 3)(x - 5)\end{aligned}$$

EXERCISES

1. Resolve $x^2 - 13x - 48$ into two binomial factors.

SOLUTION

The first term of each factor is evidently x .

Since the product of the second terms of the two binomial factors is -48 , the second terms must have opposite signs; and since their algebraic sum, -13 , is negative, the negative term must be numerically larger than the positive term.

The two factors of -48 whose sum is negative may be 1 and -48 , 2 and -24 , 3 and -16 , 4 and -12 , or 6 and -8 . Since the algebraic sum of 3 and -16 is -13 , 3 and -16 are the factors of -48 sought.

$$\therefore x^2 - 13x - 48 = (x + 3)(x - 16)$$

2. Factor $72 - m^2 - m$.

SOLUTION

Arranging the trinomial according to the descending powers of m and making m^2 positive gives:

$$\begin{aligned} 72 - m^2 - m &= -m^2 - m + 72 \\ &= -(m^2 + m - 72) \\ &= -(m - 8)(m + 9) \\ &= (-m + 8)(m + 9) \\ &= (8 - m)(9 + m) \end{aligned}$$

Separate into factors, and check each result by assigning a numerical value to each letter:

3. $x^2 + 7x + 12$

4. $y^2 - 7y + 12$

5. $p^2 - 8p + 12$

6. $r^2 + 8r + 12$

7. $15 + 2a - a^2$

8. $b^2 + b - 12$

9. $30 - r^2 + r$

10. $c^2 - c - 72$

11. $c^2 - 5c - 14$

12. $x^2 - x - 110$

13. $x^2 - 3x + 2$

14. $a^2 + 9a + 14$

15. $y^2 - 5y + 6$

16. $b^2 + 2b - 15$

17. $x^2 + bx - 20b^2$

18. $2x^2 - 2xy - 12y^2$

19. $-12r^2 - dr + d^2$

20. $4xy - x^2 + 21y^2$

21. $x^2 + 5ax + 6a^2$

22. $x^2 - 6ax + 5a^2$

23. $y^2 - 4by - 12b^2$

24. $y^2 - 3ny - 28n^2$

25. $z^2 - anz - 2a^2n^2$

26. $-x^2 + 25x - 100$

27. $x^4 + 19cx^2 + 90c^2$

28. $x^6 + 12ax^3 + 20a^2$

29. $x^8 - 11b^2x^4 + 24b^4$

30. $n^2x^2 - 11nxy + 30y^2$

Factoring test

Try to factor eighteen of these correctly in eight minutes:

1. $x^2 + 6x + 8$

2. $a^2 + 8a + 15$

3. $b^2 - 12b + 32$

4. $r^2 - 14r + 48$

5. $r^2 - 19r + 90$

6. $m^2 - 7m - 30$

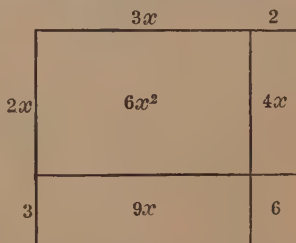
7. $x^2 - 8x - 20$

8. $y^2 + 8y - 33$

- | | |
|------------------------------|--|
| 9. $k^2 + 5k - 36$ | 16. $c^2 - 34cx - 440x^2$ |
| 10. $a^2 + 7a - 8$ | 17. $x^2 - 15x - 700$ |
| 11. $x^2 - 6xy - 16y^2$ | 18. $m^2 - 5.2m + 1.0$ |
| 12. $r^2s^2 - 11rst + 24t^2$ | 19. $z^2 + 1.5z + .5$ |
| 13. $a^2 + 16ab + 39b^2$ | 20. $x^2 + \frac{3}{4}x + \frac{1}{8}$ |
| 14. $c^2 - 16cd + 48d^2$ | 21. $x^2 + 3x + \frac{9}{4}$ |
| 15. $x^2 - 22ax - 135a^2$ | 22. $y^2 - .10y - .02$ |

Multiplication of other binomials

How many square inches are there in a sheet of paper whose dimensions are $(2x + 3)$ inches and $(3x + 2)$ inches? The figure tells us what the product is. We can see that the whole figure is made up of four parts. The areas of these parts are $6x^2$, $9x$, $4x$, and 6 . The sum of the areas is $6x^2 + 9x + 4x + 6$, or $6x^2 + 13x + 6$.



By the usual method of multiplication we can get the product of $(2x + 3)(3x + 2)$ and also of $(7x + 3y)(4x - 5y)$, as below:

(a)	(b)
$ \begin{array}{r} 2x + 3 \\ 3x + 2 \\ \hline 6x^2 + 9x \\ + 4x + 6 \\ \hline 6x^2 + 13x + 6 \end{array} $	$ \begin{array}{r} 7x + 3y \\ 4x - 5y \\ \hline 28x^2 + 12xy \\ - 35xy - 15y^2 \\ \hline 28x^2 - 23xy - 15y^2 \end{array} $

Notice in (a) that the first term of the answer is the product of the first terms of the two factors. $6x^2 = 2x(3x)$. Notice that the last term is the product of the last terms of the two factors. $6 = 3(2)$. The middle term is the product of the second term of the multiplicand by the first term of the multiplier, plus the product of the first term of the multiplicand by the second term of the multiplier. $13x = 3(3x) + 2(2x)$.

Show that the terms in the product in (b) are derived in the same way.

EXERCISES

1. Expand $(2x - 5)(3x + 4)$.

SOLUTION

$$2x - 5$$

$$3x + 4$$

$$\hline 6x^2 - 7x - 20$$

EXPLANATION. The product must have a term in x^2 , a term in x , and a numerical, or *constant*, term.

The x^2 -term is the product of $2x$ and $3x$; the x -term is the sum of the partial products $-5 \cdot 3x$ and $2x \cdot 4$, called the **cross products**; and the constant term is the product of -5 and 4 .

Expand:

2. $(7y + 5)(2y + 6)$

9. $(3a - 2)(2a + 6)$

3. $(3y + 4)(2y + 3)$

10. $(3a + 2)(2a + 6)$

4. $(8x + 5)(7x - 3)$

11. $(8y + 5)(7y + 3)$

5. $(2x + 3)(3x - 2)$

12. $(8y + 5)(7y - 3)$

6. $(2r + 4)(3r + 1)$

13. $(8y - 5)(7y + 3)$

7. $(3r + 5)(2r + 3)$

14. $(8y - 5)(7y - 3)$

8. $(2t - 5)(2t + 1)$

15. $(7x + 4)(5x - 2)$

Complete the next ten products without doing all the multiplication:

16. $(2x + 6)(3x + 2) = 6x^2 + 22x + ?$

17. $(9x - 3)(2x + 2) = 18x^2 + 12x - ?$

18. $(7x + 6)(2x + 1) = 14x^2 + 19x + ?$

19. $(2x - 6y)(3x - 2y) = 6x^2 - 22xy + ?$

20. $(y - 3)(2y - 5) = 2y^2 - 11y + ?$

21. $(x - 5a)(x + 6a) = x^2 + ax - ?$

22. $(3a - 7)(4a + 5) = 12a^2 - 13a - ?$

23. $(6n + 8)(9n - 3) = 54n^2 + 54n - ?$

24. $(5x - 2y)(2x + 3y) = 10x^2 + 11xy - ?$

25. $(4a - 2b)(3a + 3b) = 12a^2 + 6ab - ?$

Complete the next six products without multiplying, by writing the first term of each product:

26. $(5x - 2y)(3x - 3y) = ? - 21xy + 6y^2$
27. $(7a - 6)(-3a - 2) = ? + 4a + 12$
28. $(2y + 6b)(-3y + 2b) = ? - 14by + 12b^2$
29. $(6b + 1)(3b - 4) = ? - 21b - 4$
30. $(-2x + 5n)(2x + 3n) = ? + 4nx + 15n^2$
31. $(2x + 6y)(-5x + 3y) = ? - 24xy + 18y^2$

Write the middle term in each of the following products:

32. $(2x + 6)(3x + 2) = 6x^2 + ? + 12$
33. $(2x - 6y)(3x + 2y) = 6x^2 - ? - 12y^2$
34. $(2x - 6)(3x - 2) = 6x^2 - ? + 12$
35. $(5y + 3x)(3y + 8x) = 15y^2 + ? + 24x^2$
36. $(9a - 6)(3a + 4) = 27a^2 + ? - 24$

Expand at sight, using the following device, if helpful:

$$37. (4x + 5)(3x + 2)$$

Products

$$(4x + 5)(3x + 2)$$

Sums

$$(4x + 5)(3x + 2)$$

- | | |
|-------------------------|--------------------------|
| 38. $(4x + 3)(3x + 2)$ | 42. $(x + 9b)(x + 6b)$ |
| 39. $(x - 7y)(2x - 3y)$ | 43. $(2x + 5c)(3x - 2c)$ |
| 40. $(2x - 8)(3x + 5)$ | 44. $(2x + 4y)(3x - 5y)$ |
| 41. $(4x + 3)(3x - 4)$ | 45. $(3a + 2b)(5a - 6b)$ |

Expand by inspection, and check each result:

- | | |
|-------------------------|------------------------------|
| 46. $(2x + 5)(3x + 4)$ | 51. $(2a + 5b)(5a + 2b)$ |
| 47. $(3x - 2)(2x - 3)$ | 52. $(7n^2 - 2p)(2n^2 - 7p)$ |
| 48. $(3a - 4)(4a + 3)$ | 53. $(2m - 3n)(4m - 6n)$ |
| 49. $(3x - y)(x - 3y)$ | 54. $(5x^2 - y)(3x^2 + y)$ |
| 50. $(7z - a)(3z + 2a)$ | 55. $(7y^2 - 2y)(3y^2 + y)$ |

Cross-product method of factoring

Suppose that we wish to find the factors of $3x^2 + 7x + 2$. We can easily see that the first terms of the factors must be $3x$ and x . It is helpful to write the blank form of the two parentheses like this:

$$3x^2 + 7x + 2 = (\quad) (\quad)$$

We are then ready to write the terms we have just found. Thus:

$$3x^2 + 7x + 2 = (3x \quad)(x \quad)$$

Next we find the second term of each factor. From a preceding section, we know that the last terms are 2 and 1 or 1 and 2. To see whether 2 or 1 belongs to the first factor, we need to try the two combinations and test to see whether the middle term will be $7x$.

$$\text{Suppose we take } 3x^2 + 7x + 2 = \overbrace{(3x+2)(x+1)}^{3x \atop 2x}.$$

The middle term is $3x + 2x$, or $5x$, which is not correct.

If we interchange terms 1 and 2, we get

$$3x^2 + 7x + 2 = \overbrace{(3x+1)(x+2)}^{6x \atop x}.$$

The middle term is $6x + x$, or $7x$. Hence we have the correct pair of factors, $(3x + 1)$ and $(x + 2)$.

Now let us look at a more difficult problem.

To factor a trinomial of the form $ax^2 + bx + c$

ILLUSTRATIVE EXAMPLE

Factor $3x^2 + 11x - 4$.

SOLUTION

If this trinomial is the product of two binomial factors, these factors may be found by reversing the process of multiplication.

Since $3x^2$ is the product of the *first terms* of the binomial factors, the first terms, each containing x , are $3x$ and x .

Since -4 is the product of the last terms, they must have unlike signs, and the only possible *last terms* are 4 and -1 , -4 and 1 , or 2 and -2 .

Hence associating these pairs of factors of -4 with $3x$ and x in all possible ways, we have:

$$\begin{array}{cccccc} 3x+4 & 3x-1 & 3x-4 & 3x+1 & 3x+2 & 3x-2 \\ x-1 & x+4 & x+1 & x-4 & x-2 & x+2 \end{array}$$

Of these we select *by trial* the pair that gives $+11x$ (the middle term of the given trinomial) for the algebraic sum of the "cross products."

Try the first pair:

$4x - 3x$ does not equal $11x$. Therefore this pair is not correct.

Try the second pair:

$-x + 12x$ does equal $11x$. Therefore this pair is correct.

$$\therefore 3x^2 + 11x - 4 = (3x - 1)(x + 4)$$

From the preceding solution it may be observed that:

1. When the sign of the last term of the trinomial is $+$, the last terms of the factors must be both $+$ or both $-$, and like the sign of the middle term of the trinomial.

2. When the sign of the last term of the trinomial is $-$, the sign of the last term of one factor must be $+$, and the sign of the last term of the other factor must be $-$.

3. The sum of the cross products of the two terms must equal the middle term of the trinomial.

EXERCISES

Factor, and check each result:

1. $2y^2 + 3y + 1$

9. $2x^2 + x - 15$

2. $5x^2 + 9x - 2$

10. $5x^2 + 13x + 6$

3. $3x^2 - 7x - 6$

11. $3x^2 - 17x + 10$

4. $4r^2 + 8r + 3$

12. $6x^2 - 11x - 35$

5. $6x^2 - 7x + 2$

13. $15x^2 + 17x - 4$

6. $2x^2 - 5x - 12$

14. $15x^2 - 14x - 8$

7. $10t^2 + t - 3$

15. $2x^2 + 3xy - 2y^2$

8. $6n^2 - 13n + 6$

16. $3x^2 - 10xy + 3y^2$

REVIEW EXERCISES IN SPECIAL PRODUCTS

Expand :

- | | | |
|---|--|--|
| 1. $(x^n + y^n)^2$ | 9. $(1 + r^a s^b)^2$ | 17. $(x + 3)(x - 4)$ |
| 2. $(3 - 2y^2)^2$ | 10. $(.2x^3 - 5)^2$ | 18. $(2x + 1)(2x + 2)$ |
| 3. $(a^n + b^m)^2$ | 11. $(\frac{1}{2}x^2 + a^2)^2$ | 19. $(\frac{1}{2}a - 2)(\frac{1}{2}a - 4)$ |
| 4. $(1 - 3xy)^2$ | 12. $(\frac{1}{3}m^2 - \frac{3}{4}n)^2$ | 20. $(ax - 1)(ax + 3)$ |
| 5. $(2x^4 + .5)^2$ | 13. $(abc - \frac{1}{2}d)^2$ | 21. $(xy + 4)(xy - 4)$ |
| 6. $(r^3 - .2s)^2$ | 14. $(\frac{3}{4}x - \frac{2}{3}y)^2$ | 22. $(a^{2n} - b^m)(a^{2n} + b^m)$ |
| 7. $(y^m - y^n)^2$ | 15. $(3ab - c)^2$ | 23. $(3r + 2)(2r + 10)$ |
| 8. $(4a + .3b)^2$ | 16. $(2xy + z^3)^2$ | 24. $(5x - .12)(5x + .12)$ |
| 25. $(1.5x - .3y^4)^2$ | 43. $(2a + y)(2a + x)$ | |
| 26. $(3x^2y^2 + 2a^nz^2)^2$ | 44. $(2y + z)(3y - 2z)$ | |
| 27. $(\frac{1}{4}rs - \frac{1}{2}t^{a+1})^2$ | 45. $(y^2z^2 - 8)(y^2z^2 + 5)$ | |
| 28. $(bd - ce)^2$ | 46. $(5r^2 + 2s)(2r^2 - 5s)$ | |
| 29. $(m^{a+2} - m^bn^{c+1})^2$ | 47. $(3x^n + m^n)(3x^n + n^m)$ | |
| 30. $(2.5 + 12p^2q^2r^3)^2$ | 48. $[r + (s + t)][r - (s + t)]$ | |
| 31. $[(2a + 3b) + 4]^2$ | 49. $(5cdx + 1)(5cdx - 6)$ | |
| 32. $(x^{m-n} - y^{m+n})^2$ | 50. $(3bx^3 + 7)(3 + 7bx^3)$ | |
| 33. $(a^2b^2c^2 + d^2e^2)^2$ | 51. $(ad^2x^3 - 10)(ad^2x^3 - 3)$ | |
| 34. $(4.5x^2 - 3.5)^2$ | 52. $(x^{a+b} + 3y)(x^{a+b} - 2y)$ | |
| 35. $(\frac{2}{3}m^4n^3 + \frac{3}{4}p^2q^2)^2$ | 53. $(t + u + v)(t - u - v)$ | |
| 36. $(.7x - 5)(.6x + 7)$ | 54. $(a^mb^n + x^y)(a^mb^n - x^y)$ | |
| 37. $(4def - \frac{1}{4}abc)^2$ | 55. $(x^{a-1} - y)(2x^{a-1} + 3y)$ | |
| 38. $(5r^2s^3 + 1.5r^{a-1})^2$ | 56. $(7a^2x^3 + 6z^n)(7a^2x^3 - 6z^n)$ | |
| 39. $(.2a - 3)(.2a + .5)$ | 57. $(a^{m-1} + b^{n-1})(a^{m-1} + b^{n-1})$ | |
| 40. $(x + 5b)(x - 5b)$ | 58. $(4az^3 - 3y^2)(2az^3 + y)$ | |
| 41. $(3x + 4)(3x - 5)$ | 59. $(x^ay^b + x^by^a)(x^ay^b - x^by^a)$ | |
| 42. $(2a + b)(3a - 2b)$ | 60. $[(a+b) + (c+d)][(a+b) - (c+d)]$ | |

Multiplication test

Cover the numbers in column *B*. Practice until you can write the products of the factors in column *A* in six minutes, without making more than two errors.

<i>Column A</i>	<i>Column B</i>
1. $(2x + 1)(x + 2)$	$= 2x^2 + 5x + 2$
2. $(2x + 3)(x + 1)$	$= 2x^2 + 5x + 3$
3. $(5x - 1)(x - 7)$	$= 5x^2 - 36x + 7$
4. $(2a + 3)(3a + 2)$	$= 6a^2 + 13a + 6$
5. $(5y - 1)(y - 1)$	$= 5y^2 - 6y + 1$
6. $(3a + 1)(a + 1)$	$= 3a^2 + 4a + 1$
7. $(2x + 7)(x - 4)$	$= 2x^2 - x - 28$
8. $(x - 6y)(x - 4y)$	$= x^2 - 10xy + 24y^2$
9. $(a + 3)(a + 3)$	$= a^2 + 6a + 9$
10. $(a + 3y)(a + 3y)$	$= a^2 + 6ay + 9y^2$
11. $(y - 4)(y - 4)$	$= y^2 - 8y + 16$
12. $(y - 4x)(y - 4x)$	$= y^2 - 8xy + 16x^2$
13. $(x - 8)(x - 3)$	$= x^2 - 11x + 24$
14. $(x - 8y)(x - 3y)$	$= x^2 - 11xy + 24y^2$
15. $(z - 2)(z - 10)$	$= z^2 - 12z + 20$
16. $(z - 2r)(z - 10r)$	$= z^2 - 12rz + 20r^2$
17. $(5a + 1)(a + 3)$	$= 5a^2 + 16a + 3$
18. $(x - 6)(x + 4)$	$= x^2 - 2x - 24$
19. $(x - 5)(x + 1)$	$= x^2 - 4x - 5$
20. $(x - 5a)(x + a)$	$= x^2 - 4ax - 5a^2$

Factoring test

Cover up column *A* above and write the factors of each trinomial in column *B*. When you have finished, check your answers with column *A*. The standard is eight minutes with not more than two errors.

Factoring completely or finding prime factors

It is important always to get prime factors. $2x + 3$ is prime, but $6x + 9$ is not prime. (See page 125.)

ILLUSTRATIVE EXAMPLE

Factor $5x^2 - 35x + 60$.

SOLUTION

By inspection, we see that 5 is a common factor. Hence we may write the expression in this form: $5(x^2 - 7x + 12)$. Since prime factors are to be found, we must see whether the second factor ($x^2 - 7x + 12$) can be factored further. We find that it equals $(x - 3)(x - 4)$. Hence the complete solution is:

$$5x^2 - 35x + 60 = 5(x - 3)(x - 4)$$

EXERCISES

In each of the following examples look for a common factor, then find the other factors. Factor each completely:

1. $8x^2 + 56x + 96$

16. $11a^4 - 11b^4$

2. $15y^2 - 135$

17. $n^2x^2 - 11nxy + 30y^2$

3. $ab^2 - ab - 12a$

18. $3x^{10} - 33b^2x^5 + 72b^4$

4. $3x^2 + 21x + 36$

19. $5x^2 - 45x - 110$

5. $12x^2 - 12y^2$

20. $y^4 - 2y^2 + 1$

6. $45 + 6a - 3a^2$

21. $2a^4 - 4a^2 + 2$

7. $5a^2 + 10ab + 5b^2$

22. $120x^4 - 120y^4$

8. $2x^4 - 2y^4$

23. $13a^4 - 13b^4$

9. $2a^8 - 2b^8$

24. $7y^2 - 28by - 84b^2$

10. $m^4 - 1$

25. $8x^2 - 48ax + 40a^2$

11. $2m^2n + 4mn^2 + n^3$

26. $9x^2 + 45ax + 54a^2$

12. $25x^2 + 70x + 49$

27. $10b^2 + 10b - 120$

13. $3x^2 - 12$

28. $3m^4 - 12$

14. $m^3 - 9m^2$

29. $2m^4 - 128$

15. $50 - 2x^2$

30. $5a^4b^4 - 5c^4d^4$

SUMMARY OF FACTORING

I. Monomial factors of expressions whose terms have a common factor; as,

$$nx + ny + nz$$

II. Binomials that are the difference of two squares; as,

$$a^2 - b^2$$

III. Trinomials that are squares; as,

$$a^2 + 2ab + b^2 \text{ and } a^2 - 2ab + b^2$$

IV. Trinomials of the form $x^2 + bx + c$.

V. Trinomials of the form $ax^2 + bx + c$.

VI. Polynomials whose terms may be grouped to show a common polynomial factor; as,

$$ax + bx + ay + by$$

General directions for factoring polynomials

1. Remove monomial factors if there are any.
2. Try to bring the polynomial under one of the cases II-VI.
3. Resolve into prime factors.
4. Group all the prime factors together as a product.

Each factor should be eliminated from the given expression, as soon as found, in order to simplify the discovery of the remaining factors.

REVIEW

1. By means of factoring we may sometimes simplify arithmetic work. Since $an + bn = n(a + b)$ and $an - bn = n(a - b)$, show how to find quickly the value of $25 \times 17 + 25 \times 8$; of $15 \times 18 - 15 \times 13$; of $12 \times 7^2 - 9 \times 7^2$.

2. Since $a^2 - b^2 = (a + b)(a - b)$, find the value of $46^2 - 4^2$; of $93^2 - 7^2$; of $35^2 - 25^2$.

3. The area of a circular ring is found by subtracting the area of the inner circle from that of the larger circle; that is, $A = \pi R^2 - \pi r^2$. Simplify the work by factoring, and find A when $R = 5.50$ and $r = 5$; when $R = 8.30$ and $r = 5.80$; when $R = 22$ and $r = 18$; when $R = 3.40$ and $r = 2.60$.



Factor:

- | | |
|--|---------------------------------|
| 4. $8x^3 - 2x^2$ | 32. $a^{4n} - b^2$ |
| 5. $b^8 - 81$ | 33. $x^{2n+2} - 1$ |
| 6. $4x^4y^8 - 4z^8$ | 34. $8x^4y^4 + 10x^3y^3$ |
| 7. $3x^5 - 3x$ | 35. $3y^2 - 3y + \frac{3}{4}$ |
| 8. $3x^3 - 12x$ | 36. $x^2 + 3xy + 2y^2$ |
| 9. $x^4y - y^5$ | 37. $3 + 4x + x^2$ |
| 10. $y^{2a} - 1$ | 38. $b^2 - 1.96$ |
| 11. $25a^2b^4 - 16a^4b^2$ | 39. $a^4 - 256$ |
| 12. $x^{n+1} + x$ | 40. $450 - 2a^2$ |
| Ans. $x(x^n + 1)$ | 41. $1 - 6m^a + 9m^{2a}$ |
| 13. $y^{2n+1} - y$ | 42. $x^2 + 9x - 90$ |
| 14. $\frac{1}{81}x^{4n} - \frac{1}{16}y^8$ | 43. $3x^2 - 2x - 8$ |
| 15. $a^2 - .04b^2$ | 44. $15 + 6x - 9x^2$ |
| 16. $d^3 - 9d$ | 45. $17 - 16a - a^2$ |
| 17. $4a^{2n} - 6.25b^{4n+6}$ | 46. $a^2b^2 + ab - 56$ |
| 18. $4x^3 - 4x$ | 47. $6x + 5x^2 - x^3$ |
| 19. $2x^2 - 4x - 16$ | 48. $a^2x^2 - 4ax + 3$ |
| 20. $a^2x - 2a^2x^2$ | 49. $x^2 - ax - 72a^2$ |
| 21. $x^2 + 3x + 2$ | 50. $n^2 - an - 90a^2$ |
| 22. $a^2 - (b + c)^2$ | 51. $x^2 - xy - 132y^2$ |
| 23. $3.5x^2 - 3.5x - 21$ | 52. $3x^2 + 30x + 27$ |
| 24. $2a^2 + ab - 6b^2$ | 53. $6x^2 - 19x + 15$ |
| 25. $7x^2 - 20xy - 3y^2$ | 54. $4a - 3ax - ax^2$ |
| 26. $x^2 - 2.9x - .3$ | 55. $x^{2n} + 2x^ny^p + y^{2p}$ |
| 27. $5a^2b^2 + 5a^4b^4$ | 56. $b^4c - 13b^2c + 42c$ |
| 28. $x^2 + .5x + .06$ | 57. $2x^3 + 28x^2 + 66x$ |
| 29. $36(x - y)^2 - z^2$ | 58. $y^2 + 16ay - 36a^2$ |
| 30. $12a^{2n-2} - 36.75b^4$ | 59. $ax^2 + 10ax - 39a$ |
| 31. $x^{2a} - y^{2a+2}$ | 60. $7x^2 - 77xy - 84y^2$ |

Factor:

61. $y^2 - 25xy + 136x^2$
 62. $a^2z^4 + 2a^2z^2 + a^2$
 63. $8x^4 - 6x^2 - 35$
 64. $77 - 30x^2 - 37x$
 65. $5x^8 + 10x^4 - 15$
 66. $5x^2 - 26xy + 5y^2$
 67. $8a^2 - 21ab - 9b^2$
 68. $4m^2 + 12mn + 9n^2$
 69. $60a^2 + 8ax - 3x^2$
 70. $10a^2c + 33ac - 7c$
 71. $25x^2 + 60xy + 36y^2$
 72. $6ax^2 + 5axy - 6ay^2$
 73. $25x^2 - 9y^2 - 24yz - 16z^2$
 74. $21 + 3x + 7x^2 + x^3$
 75. $6 - 6y + y^2 - y^3$
 76. $y^2 + xy - yz - xz$
 77. $9a^2 + 30ab + 25b^2$
 78. $a + b + ab + 1$
 79. $9a^2 - a^2b^2 + 4b^2 - 36$
 80. $m^2 - 6mn - 16x^2y^2 + 9n^2$
 81. $x^4 + 2x^2y^2 - 4a^2b^2 + y^4$
 82. $16a^4 - 9a^2 - b^2 + 6ab$
 83. $m^2 + 8m + 7$
 84. $(y^2 + 4)^2 - 16y^2$
 85. $(a - b)^2 - (c - d)^2$
 86. $x^2 + 20xy + 100y^2$
 87. $(2x^n - 3y)^2 - (3x^n + 2y)^2$
 88. $4a^2 + 6an + 2ab + 3bn$
 89. $ab^2 - 4a^3 - 12a^2c - 9ac^2$
 90. $9b^2 + c^2 - 6bc - 9$
 91. $6.25y^4 - 1 - 4a - 4a^2$
 92. $z^2 - 8z - 9$
 93. $9 + 12d + 4d^2$
 94. $z^2 + bz - 12b^2$
 95. $(x^2 + y^2)^2 - (y^2 - z^2)^2$
 96. $c^2 - 17c + 72$
 97. $4a^2 + 32a + 39$
 98. $a^2 - 6ad - 72d^2$
 99. $16a^2 + 8ab - 15b^2$
 100. $9x^4 + 12x^2y^2 + 4y^4$
 101. $3a^4 - 30a^2b^2 + 75b^4$
 102. $28x - 15 - 12x^2$
 103. $(m - 2a)^2 - 4(a + b)^2$
 104. $12a^2 - 11ad - 15d^2$
 105. $a^2 + 2ac - b^2 + c^2$
 106. $5 \times 100^2 - 5 \times 10^2$
 107. $16a^{2x+2} - 64a^{2b^{4y}}$
 108. $3x^2 - 3x + \frac{9}{16}$
 109. $(2a + 3b)^2 - 4c(2a + 3b) + 4c^2$
 110. $a^2(2a - 3b) + b^2(3b - 2a)$
 111. $.5ab(3x + 2y) - 7bc(3x + 2y)$
 112. $23 \times 15.5 + 23 \times 11.5 - 23 \times 2$
 113. $17 \times 28.2 - 17 \times 26 + 17 \times 17.8$

CHAPTER VIII

FRACTIONS

The meaning of a fraction

In algebra, an indicated division is called a **fraction**.

The fraction $\frac{a}{b}$ means $a \div b$ and is read "*a divided by b.*"

The *dividend*, written above a line, is the **numerator**; the *divisor*, written below the line, is the **denominator**; the numerator and the denominator are called the **terms** of the fraction.

An arithmetic fraction also indicates division, but in arithmetic a fraction is considered as one or more of the *equal* parts of a unit; that is, in arithmetic, the terms of a fraction are *positive integers*; while in algebra they may be *any numbers whatever*.

The signs of a fraction

A fraction, as we have seen, is an *indicated quotient*. Hence its signs are governed by the law of division.

There are *three signs* to be considered in connection with a fraction; namely, the sign of the numerator, the sign of the denominator, and the sign written before the division line, called the *sign of the fraction*.

The sign of the fraction belongs to the fraction as a whole, and not to the numerator or to the denominator alone.

In $-\frac{x}{3z}$, the sign of the fraction is $-$, while the signs of x and of $3z$ are $+$.

An expression like $\frac{-a}{-b}$ indicates a process in division, in which the quotient is to be found by dividing a by b and prefixing the sign according to the law of signs in division.

$$\frac{-a}{-b} = +\frac{a}{b} \qquad \frac{+a}{+b} = +\frac{a}{b}$$

$$\frac{-a}{+b} = -\frac{a}{b} \qquad \frac{+a}{-b} = -\frac{a}{b}$$

By observing the above fractions and their values the following principles may be deduced :

PRINCIPLES. 1. *The signs of both the numerator and the denominator of a fraction may be changed without changing the sign of the fraction.*

2. *The sign of either the numerator or the denominator of a fraction may be changed, provided the sign of the fraction is changed.*

When either the numerator or the denominator is a polynomial, its sign is changed by changing the sign of each of its terms. Thus, the sign of $a - b$ is changed by writing it $-a + b$, or $b - a$.

EXERCISES

Reduce to fractions with positive numbers in both terms :

- | | | |
|----------------------|------------------------|------------------------------|
| 1. $\frac{-3}{-4}$ | 4. $\frac{-4c}{-b-y}$ | 7. $-\frac{-2-m}{2+n}$ |
| 2. $\frac{2}{-5}$ | 5. $-\frac{-a-b}{c+d}$ | 8. $\frac{-3(x+y)}{4(-a-b)}$ |
| 3. $\frac{-a-x}{2x}$ | 6. $-\frac{-2}{-a-y}$ | 9. $\frac{-4(a+b)}{5(-x-y)}$ |

By the law of signs for multiplication, the product of two negative factors is *positive*; of three negative factors, *negative*; of four negative factors, *positive*, and so on.

PRINCIPLES. 3. *The sign of either term of a fraction is changed by changing the signs of an odd number of its factors.*

4. *The sign of either term of a fraction is not changed by changing the signs of an even number of its factors.*

10. Show that $\frac{(a-b)(d-c)}{(c-a)(b-c)} = \frac{(a-b)(c-d)}{(a-c)(b-c)}$.

SOLUTION

Changing $(d-c)$ to $(c-d)$ changes the sign of *one* factor of the numerator and therefore changes the sign of the numerator (Prin. 3).

Similarly, changing $(c-a)$ to $(a-c)$ changes the sign of the denominator (Prin. 3).

We have changed the signs of both terms of the fraction. Therefore the sign of the fraction is not affected (Prin. 1).

11. Show that $\frac{(b-a)(d-c)}{(c-b)(a-c)} = -\frac{(a-b)(c-d)}{(b-c)(a-c)}$.

SOLUTION

Changing the signs of *two* factors of the numerator does not change the sign of the numerator (Prin. 4).

Changing the sign of *one* factor of the denominator changes the sign of the denominator (Prin. 3).

Since we have changed the sign of only *one* term of the fraction, we must change the sign of the fraction (Prin. 2).

12. Show that $\frac{-b}{b-a}$ may be properly changed to $\frac{b}{a-b}$.

13. From $\frac{-a}{b-a+c}$ derive $\frac{a}{a-b-c}$ by proper steps.

14. Prove that $\frac{3}{1-x} = -\frac{3}{x-1}$; that $-\frac{2}{4-x^2} = \frac{2}{x^2-4}$.

15. $\frac{1}{(b-a)(c-b)} = \frac{?}{(a-b)(b-c)}$.

16. $\frac{(m-n)(m+n)}{(a-c)(b-a)} = \frac{?}{(a-c)(a-b)}$.

17. $\frac{(a-b)(b-a+c)}{(y-x)(z-y)(z-x)} = \frac{?}{(x-y)(y-z)(x-z)}$.

18. $\frac{-2a}{9-a^2} = \frac{?}{(a+3)(a-3)}$. 19. $\frac{2a-3}{5+a} = \frac{?}{a+5}$

REDUCTION OF FRACTIONS

To reduce a fraction to higher or lower terms, proceed as in arithmetic. Thus:

$$\frac{3 \times 3}{4 \times 3} = \frac{9}{12} \quad \frac{a \times m}{b \times m} = \frac{am}{bm} \quad \frac{9 \div 3}{12 \div 3} = \frac{3}{4} \quad \frac{am \div m}{bm \div m} = \frac{a}{b}$$

When the numerator and the denominator of a fraction have no common factor except 1, the fraction is said to be in its **lowest terms**.

(1) $\frac{7}{8}, \frac{2}{11}, \frac{m}{n}, \frac{x^2}{y}, \frac{a+b}{a-b}$ are in their lowest terms.

(2) $\frac{6}{8}, \frac{2}{12}, \frac{m}{mn}, \frac{x^2}{xy}, \frac{a^2-b^2}{a-b}$ are *not* in their lowest terms.

To reduce a fraction to its lowest terms, use the same principle as in arithmetic.

In $\frac{36}{12}$, divide both terms by 12, getting the result $\frac{3}{1}$.

In $\frac{a^2-b^2}{a-b}$, divide both terms by $a-b$, getting $a+b$.

When both terms of a fraction are multiplied or divided by the same number, the fraction is multiplied or divided by 1. Hence its value is not changed.

Thus, $\frac{1}{2} \times \frac{2}{2} = \frac{1}{2} \times 1 \quad \frac{1}{2} \times \frac{3}{3} = \frac{1}{2} \times 1$
 $\frac{6}{12} \div \frac{2}{2} = \frac{6}{12} \div 1 \quad \frac{6}{12} \div \frac{3}{3} = \frac{6}{12} \div 1$

PRINCIPLE. *Multiplying or dividing both terms of a fraction by the same number does not change the value of the fraction.*

ILLUSTRATIVE EXAMPLES

1. Reduce to lowest terms $\frac{36 x^3 y^3 z^2}{20 x^2 y^2 z}$.

$$\begin{array}{c} \text{SOLUTION} \\ \frac{36 x^3 y^3 z^2}{20 x^2 y^2 z} = \frac{\overset{1}{\cancel{2^2}} \cdot 3^2 \cdot \overset{x}{\cancel{x^2}} \cdot \overset{y}{\cancel{y^2}} \cdot \overset{z}{\cancel{z^2}}}{\overset{1}{\cancel{2^2}} \cdot 5 \cdot \overset{1}{\cancel{x^2}} \cdot \overset{1}{\cancel{y^2}} \cdot \overset{1}{\cancel{z}}} = \frac{9xyz}{5} \end{array}$$

First find the prime factors of the numerator and of the denominator. Then divide both numerator and denominator by their common factors. In other words, *cancel* the common factors.

2. Reduce to lowest terms $\frac{12(a^2 - 9)}{8(a + 3)}$.

SOLUTION

$$\frac{12(a^2 - 9)}{8(a + 3)} = \frac{2 \cdot 2 \cdot 3 \cdot (a - 3)(\cancel{a + 3})}{2 \cdot 2 \cdot 2 \cdot (\cancel{a + 3})} = \frac{3(a - 3)}{2}$$

NOTE. The 1's resulting from cancellation of factors may be omitted, as in the example above; but when all the factors are canceled, it should be remembered that the answer is 1, not 0.

RULE. (1) *Express the numerator and the denominator in terms of their prime factors.*

(2) *Divide the numerator and the denominator by all their common factors.*

The product of all the common prime factors of two or more numbers is called their **highest common factor (H. C. F.)**

$$9ax^2y + 3bx^2y = 3 \cdot x^2 \cdot y(3a + b)$$

$$6x^4y^2 - 6x^2y^4 = 2 \cdot 3 \cdot x^2 \cdot y^2(x + y)(x - y)$$

Their *common factors* are 3, x , x , and y .

Hence their H. C. F. = $3x^2y$.

NOTE. The **degree** of a term with regard to a particular letter is the same as the exponent of the letter in the term. Thus, $2ab^2$ is of the first degree in a but of the second degree in b . The degree of a term with regard to several letters equals the sum of all the exponents of those letters in the term. Thus, $2a^2b$ is of the *third degree* in a and b and is *higher* than $5ab$, which is of the *second degree* in a and b .

The term of an expression that has the greatest number of literal factors determines the *degree of the expression*. Thus, the expression $2a^2b + 5ab$ is of the *third degree*.

EXERCISES (Examples 1 to 10, Oral; 12 to 59, Written)

1. Reduce to lowest terms: $\frac{20}{24}$; $\frac{21a^2x^2y}{30a^3xz}$.

SOLUTIONS

$$\frac{20}{24} = \frac{2 \cdot 2 \cdot 5}{2 \cdot 2 \cdot 2 \cdot 3} = \frac{5}{6}$$

$$\frac{21a^2x^2y}{30a^3xz} = \frac{3 \cdot 7a^2x^2y}{3 \cdot 10a^3xz} = \frac{7xy}{10az}$$

EXPLANATION. Since a fraction is in its lowest terms when its terms have no common factor, a fraction may be reduced to its lowest terms by removing in succession all common factors of its numerator and denominator; or by dividing the terms by their *highest common factor*.

Reduce to lowest terms :

$$2. \frac{a^2xy^2}{a^3xy}$$

$$5. \frac{16 m^2nx^2z^2}{40 am^3yz^3}$$

$$8. \frac{-7 a^2bcd^3}{42 ab^2cd^4}$$

$$3. \frac{m^3n^3}{am^2n^4}$$

$$6. \frac{210 bc^2d}{750 ab^2c}$$

$$9. \frac{tr^2 + ts^2}{3t}$$

$$4. \frac{a^2b^2x^2}{b^3xy^2}$$

$$7. \frac{-25 x^2y^5z^2}{-100 x^4y^3}$$

$$10. \frac{2x}{4x^2 - 6ax}$$

11. Reduce $\frac{bx - ax}{a^2 - b^2}$ to its lowest terms.

SOLUTION

$$\begin{aligned} \frac{bx - ax}{a^2 - b^2} &= \frac{x(b - a)}{(a + b)(a - b)} \\ &= \frac{-x(a - b)}{(a + b)(a - b)} = \frac{-x}{a + b} = -\frac{x}{a + b} \end{aligned}$$

Reduce to lowest terms :

$$12. \frac{xz}{xz + yz}$$

$$20. \frac{x - y}{(x - y)^2}$$

$$28. \frac{a - b}{a^2 - 2ab + b^2}$$

$$13. \frac{a}{a^2 + ab}$$

$$21. \frac{d + e}{d^2 - e^2}$$

$$29. \frac{x + y}{x^2 + 2xy + y^2}$$

$$14. \frac{3b^3}{3b + 3}$$

$$22. \frac{a + b}{(a + b)^2}$$

$$30. \frac{c + 2}{c^2 + 4c + 4}$$

$$15. \frac{bc + bd}{bx + by}$$

$$23. \frac{b - c}{b^2 - c^2}$$

$$31. \frac{s - 3}{s^2 - 6s + 9}$$

$$16. \frac{mn - rs}{amn - ars}$$

$$24. \frac{(x - y)^2}{x^2 - y^2}$$

$$32. \frac{b^3 - 3b^2}{b^2 - 6b + 9}$$

$$17. \frac{5a + 10b}{5c + 15d}$$

$$25. \frac{3y + 1}{9y^2 - 1}$$

$$33. \frac{x^2 + 2xy + y^2}{x^2 - y^2}$$

$$18. \frac{7a - 21b}{14a - 28b}$$

$$26. \frac{5a - 6b}{25a^2 - 36b^2}$$

$$34. \frac{b^2 - 3b + 2}{b^2 - 1}$$

$$19. \frac{5c + 10d}{7c + 14d}$$

$$27. \frac{4a + 7}{16a^2 - 49}$$

$$35. \frac{a^2 - 3a - 10}{a^2 - 6a + 5}$$

Reduce to lowest terms :

$$36. \frac{1 + 5m + 6m^2}{1 + 6m + 8m^2}$$

$$37. \frac{a^2 - 7a + 12}{a^2 - 8a + 16}$$

$$38. \frac{y^2 - 81}{y^2 + 7y - 18}$$

$$39. \frac{a^2 - 11a + 24}{a^2 - a - 6}$$

$$40. \frac{x^2 - 6x - 7}{x^2 - 11x + 28}$$

$$41. \frac{2x^2 - 2x - 12}{6x^2 - 10x - 44}$$

$$42. \frac{x^3 - 6x^2 + 5x}{x^3 + 2x^2 - 35x}$$

$$43. \frac{7x - 2x^2 - 3}{2x^2 + 11x - 6}$$

$$44. \frac{10a^2 - 13ab - 3b^2}{14a^2 - 11ab - 15b^2}$$

$$45. \frac{14a^{2n} - a^nb - 3b^2}{4a^{2n} - 3a^nb + .5b^2}$$

$$46. \frac{6x^2 + 7xy - 20y^2}{15x^2 - 41xy + 28y^2}$$

$$47. \frac{x^2 - 2ax + 3bx}{x^2 - 4a^2 + 12ab - 9b^2}$$

$$48. \frac{10nx + 10ny}{25nx^2 - 25ny^2}$$

$$49. \frac{x^{n+2} - x^n}{x^{n+1} - x^n}$$

$$50. \frac{b^2 + b - 12}{3b^2 + 9b - 54}$$

$$51. \frac{x^2 - 2x^4 + x^6}{x^6 - x^2}$$

$$52. \frac{a^2b(a + 2b)^4}{ab(a^2 - 4b^2)^2}$$

$$53. \frac{x^3 + 5x^2 - 6x}{2x^2 - 2}$$

$$54. \frac{a^3 + 2a^2b + ab^2}{a^5 - 2a^3b^2 + ab^4}$$

$$55. \frac{(a + b)^2 - 1}{a^2c + abc + ac}$$

$$56. \frac{2x^2 - xy - 15y^2}{5x^2 - 16xy + 3y^2}$$

$$57. \frac{8.75x^2 - 3xy - 8y^2}{7.5x^2 + 3.5xy - 2y^2}$$

$$58. \frac{.2x^2 - .3xy - .2y^2}{.2x^2 + .7xy + .3y^2}$$

$$59. \frac{4x^{2n+2} + 4x^{n+2}y^m + x^2y^{2m}}{2x^{2n+1} - x^{n+1}y^m - xy^{2m}}$$

Reduction of a fraction to an equal fraction with a required denominator

In addition and subtraction of fractions, it is more often necessary to change to a higher than to a lower denominator.

EXERCISES

1. Reduce $\frac{a}{a+b}$ to a fraction whose denominator is $a^2 - b^2$.

SOLUTION

$$(a^2 - b^2) \div (a + b) = a - b$$

$$\frac{a}{a+b} = \frac{a(a-b)}{(a+b)(a-b)} = \frac{a^2 - ab}{a^2 - b^2}$$

EXPLANATION. Since the required denominator is $(a - b)$ times the given denominator, in order that the value of the fraction shall not be changed both terms of the fraction must be multiplied by $(a - b)$.

2. Reduce $\frac{5a}{6}$ to a fraction whose denominator is 42.

3. Reduce $\frac{3x}{11b}$ to a fraction whose denominator is 55 b .

4. Reduce $\frac{3a}{14x}$ to a fraction whose denominator is 84 xy .

5. Reduce $\frac{4a^2}{5y}$ to a fraction whose denominator is 20 y^3 .

6. Reduce $\frac{x-3}{x-1}$ to a fraction whose denominator is $(x-1)^2$.

7. Reduce $\frac{2x-5}{2x+5}$ to a fraction whose denominator is $(2x+5)(3x-2)$.

8. Reduce $\frac{-1}{2-x}$ to a fraction whose denominator is $3x^2 - 10x + 8$.

9. Reduce $\frac{ab}{3-b}$ to a fraction whose denominator is $2b^2 - b - 15$.

10. Reduce $\frac{x-5}{1}$ to a fraction whose denominator is $2x+5$.

11. Reduce $3t + 2l$ to a fraction whose denominator is $7l - 5t$.

Reduction of a fraction to an integer or to a mixed expression

This change in form is made in algebra in the same manner as in arithmetic.

EXERCISES

1. Reduce to a mixed number : $\frac{13}{4}$; $\frac{ax+b}{x}$.

SOLUTIONS

$$\frac{13}{4} = 13 \div 4 = 3 + \frac{1}{4} = 3\frac{1}{4}$$

$$\frac{ax+b}{x} = \frac{ax}{x} + \frac{b}{x} = a + \frac{b}{x}$$

EXPLANATION. Since a fraction is an indicated division, by performing the division indicated the fraction is changed to the form of a mixed number.

Reduce to an integral or a mixed expression :

2. $\frac{27a}{9}$

4. $\frac{45x^2y^3z}{15xy}$

6. $\frac{a^2+c^2}{a}$

3. $\frac{26b}{13b}$

5. $\frac{36ac+9c}{9c}$

7. $\frac{12x^2-6x}{6x}$

8. Reduce $\frac{a^3-3a^2-a+1}{a^2}$ to a mixed number.

SOLUTION

$$\frac{a^3-3a^2-a+1}{a^2} = a - 3 + \frac{-a+1}{a^2} = a - 3 - \frac{a-1}{a^2}$$

NOTE. It is not necessary to write the step $(a^3-3a^2-a+1) \div a^2$. The division should be continued until the undivided part of the numerator no longer contains the denominator.

Reduce to an integral or a mixed expression :

9. $\frac{4x^3-8x^2+2x-1}{2x}$

11. $\frac{a^2x^3-ax^2-x-1}{ax}$

10. $\frac{ab-bc-cd+d^2}{b}$

12. $\frac{x^2-x-12}{x-4}$

- | | | |
|---|---|--------------------------------------|
| 13. $\frac{ab + d}{a}$ | 16. $\frac{x^2 + 2x + 1}{x}$ | 19. $\frac{x^2 - 6x + 5}{x - 1}$ |
| 14. $\frac{xy + m}{y}$ | 17. $\frac{b^3 + b^2 + b}{b^2}$ | 20. $\frac{a^2 + 4ab + 4b^2}{a + b}$ |
| 15. $\frac{4x^2 + x}{2x}$ | 18. $\frac{x^2 + x - 12}{x^2}$ | 21. $\frac{x^3 - 2xy - y^3}{x - y}$ |
| 22. $\frac{a^2x^3 - ax^2 - x - 1}{ax}$ | 28. $\frac{x^2 - 6xy + 4y^2}{2xy}$ | |
| 23. $\frac{x^3 - 6x^2 + 14x - 9}{x - 2}$ | 29. $\frac{x^2 - xy - 3y^2 - z}{x + y}$ | |
| 24. $\frac{x^3 + 5x^2 + 3x - 6}{x + 2}$ | 30. $\frac{x^4 - 3x^2 + 5x - 1}{x - 3}$ | |
| 25. $\frac{x^3 - 7x^2 - 4x + 40}{x^2 - 3}$ | 31. $\frac{a^3 + 3a^2b - ab^2 + ab}{a^2 + b}$ | |
| 26. $\frac{a^4 + 3a^2b^2 + b^4}{a^2 + b^2}$ | 32. $\frac{x^4 + 4x^3y + 6x^2y^2 + 4xy^3}{x + y}$ | |
| 27. $\frac{4x^2 + 22x + 21}{2x + 4}$ | 33. $\frac{m^5 - 2m^3n - 3mn^2 - 2mn}{m^3 - mn}$ | |

Reduction to lowest common denominator

In algebra, as in arithmetic, it is frequently desirable to reduce fractions that have different denominators to equal fractions that have a *common* denominator.

In algebra, **lowest common denominator** corresponds to *least common denominator* in arithmetic.

The word "lowest" has reference to the *degree* of the denominator.

It is not always easy to discover by inspection the *lowest common denominator* (L. C. D.), that is, the *lowest common multiple* (L. C. M.) of the given denominators. However, it may be found, as in arithmetic, by factoring the denominators, for it is the product of all their *different* prime factors,

each factor being used the greatest number of times that it occurs in any denominator. Compare the following arithmetic and algebraic methods of finding the L. C. D.

ARITHMETIC

(1) L. C. D. of

$$15, 21, 45 = ?$$

$$15 = 3 \times 5$$

$$21 = 3 \times 7$$

$$45 = 3 \times 3 \times 5$$

$$\text{L. C. D.} = 3 \times 3 \times 5 \times 7$$

ALGEBRA

(2) L. C. D. of

$$ax - bx, a^2 - b^2, a^2 - 2ab + b^2 = ?$$

$$ax - bx = x(a - b)$$

$$a^2 - b^2 = (a + b)(a - b)$$

$$a^2 - 2ab + b^2 = (a - b)(a - b)$$

$$\text{L. C. D.} = x(a - b)^2(a + b)$$

EXERCISES

1. Reduce to equivalent fractions having their lowest common denominator: $\frac{5}{6}$ and $\frac{3}{8}$; $\frac{a}{3bc}$ and $\frac{c}{6ab}$.

SOLUTIONS

$$\text{L. C. D. of } 6 \text{ and } 8 = 24$$

$$\frac{5}{6} = \frac{5 \times 4}{6 \times 4} = \frac{20}{24}$$

$$\frac{3}{8} = \frac{3 \times 3}{8 \times 3} = \frac{9}{24}$$

$$\text{L. C. D. of } 3bc \text{ and } 6ab = 6abc$$

$$\frac{a}{3bc} = \frac{a}{3bc} \times \frac{2a}{2a} = \frac{2a^2}{6abc}$$

$$\frac{c}{6ab} = \frac{c}{6ab} \times \frac{c}{c} = \frac{c^2}{6abc}$$

EXPLANATION. The L. C. M. of the given denominators is found for the L. C. D. Then each fraction is reduced to an equal fraction having this denominator. All fractions should first be reduced to lowest terms.

2. Reduce $2m$ and $\frac{m+n}{m-n}$ to equivalent fractions having their lowest common denominator.

SUGGESTION. First write $2m$ as a fraction with the denominator 1.

Reduce to equivalent fractions having their lowest common denominator:

$$3. \frac{x}{2} \text{ and } \frac{3y}{5}$$

$$4. \frac{2a}{5b} \text{ and } 3x$$

$$5. \frac{a^2b}{c^2d} \text{ and } \frac{ab^2}{cd^2}$$

$$6. \frac{x^2}{2xy} \text{ and } \frac{3}{4ay}$$

$$7. \frac{3xy}{cx} \text{ and } \frac{2ay}{3by}$$

$$8. \frac{3}{x^2y^4}, \frac{-6}{x^3y^3}, \frac{3}{x^4y^2}$$

$$9. \frac{3ab}{8a^2c}, \frac{7a^2}{4b^2c}, \frac{5a^3}{a^3bc^2} \qquad 11. \frac{x+y}{2}, \frac{x-y}{4}, \frac{x^2-y^2}{6}$$

$$10. \frac{m-n}{a}, 2, \frac{a}{m+n} \qquad 12. \frac{x^2}{x^2-1}, \frac{x}{x+1}, \frac{x}{x-1}$$

$$13. \frac{a^3}{a^4-16}, \frac{a}{a^2+4}, \frac{2a}{4-a^2}$$

$$\text{SUGGESTION.} \quad \frac{2a}{4-a^2} = \frac{-2a}{a^2-4}$$

$$14. \frac{4a}{b-a}, \frac{3b}{a+b}, \frac{1}{a^2-b^2}$$

$$15. \frac{a}{1-ax}, \frac{x}{1+ax}, \frac{-ax}{ax-1}$$

$$16. \frac{1}{x^2+7x+10}, \frac{1}{x^2+x-2}, \frac{1}{x^2+4x-5}$$

$$17. \frac{3x}{x^2-3x+2}, \frac{x-1}{x^2-5x+6}, \frac{x+3}{x^2-4x+3}$$

$$18. \frac{a+5}{a^2-4a+3}, \frac{a-2}{a^2-8a+15}, \frac{a+1}{a^2-6a+5}$$

ADDITION AND SUBTRACTION OF FRACTIONS

The method of adding and subtracting fractions is the same in algebra as in arithmetic. In algebra, however, subtraction of fractions practically reduces to addition of fractions, for every fraction to be subtracted is really added with its sign changed. (See page 83.)

$$\text{Just as} \quad \frac{1}{3} + \frac{1}{4} = \frac{4}{12} + \frac{3}{12} = \frac{4+3}{12}$$

$$\text{so} \quad \frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}$$

$$\text{Also as} \quad \frac{1}{3} - \frac{1}{4} = \frac{4}{12} - \frac{3}{12} = \frac{4-3}{12}$$

$$\text{so} \quad \frac{a}{b} - \frac{c}{d} = \frac{ad}{bd} - \frac{bc}{bd} = \frac{ad-bc}{bd}$$

ILLUSTRATIVE EXAMPLES

1. Add $\frac{3x}{4}$, $\frac{7x}{10}$, and $\frac{5z}{12}$.

SOLUTION

Since the fractions have unlike denominators, they must be reduced to fractions having a common denominator. The L. C. D. = 60.

$$\begin{aligned}\frac{3x}{4} + \frac{7x}{10} + \frac{5z}{12} &= \frac{45x}{60} + \frac{42x}{60} + \frac{25z}{60} \\ &= \frac{45x + 42x + 25z}{60} = \frac{87x + 25z}{60}\end{aligned}$$

Check by letting $x = 2$, $z = 3$.

2. Subtract $\frac{x-2}{7}$ from $\frac{5x-1}{8} + \frac{x}{4}$.

SOLUTION

$$\begin{aligned}\frac{5x-1}{8} + \frac{x}{4} - \frac{(x-2)}{7} &= \frac{35x-7}{56} + \frac{14x}{56} - \frac{(8x-16)}{56} \\ &= \frac{35x-7+14x-(8x-16)}{56} \\ &= \frac{35x-7+14x-8x+16}{56} \\ &= \frac{41x+9}{56}\end{aligned}$$

Check as in ex. 1.

NOTE. When a fraction is preceded by the minus sign it is well for the beginner to inclose the numerator, if it is a polynomial, in a parenthesis, as shown above.

RULE. Reduce the fractions to similar fractions having their lowest common denominator for their denominators.

Change the signs of all the terms of the numerators of fractions preceded by the sign $-$, then find the sum of all the numerators, and write it over the common denominator.

Reduce the resulting fraction to its lowest terms, if necessary.

EXERCISES

Add:

1. $\frac{2x}{5}$ and $\frac{3x}{2}$

2. $\frac{4a}{3}$ and $\frac{6b}{5}$

3. $\frac{2a}{3b}$ and $\frac{3a}{2b}$

4. $\frac{-5}{7x}$ and $\frac{-2}{3x}$

Subtract:

5. $\frac{5m}{6}$ from $\frac{4m}{3}$

6. $\frac{4x}{9}$ from $-\frac{x}{2}$

7. $-\frac{y}{3}$ from $\frac{3}{y}$

8. $\frac{a+b}{3}$ from $\frac{a-b}{2}$

Perform the additions and subtractions indicated:

9. $\frac{2x+1}{3} + \frac{x-2}{4} - \frac{x-3}{6} + \frac{5-x}{2}$

10. $\frac{x-2}{6} - \frac{x-4}{9} + \frac{2-3x}{4} - \frac{2x+1}{12}$

11. $\frac{x-1}{3} - \frac{x-2}{18} - \frac{4x-3}{27} + \frac{1-x}{6}$

12. $\frac{2-6x}{5} + \frac{4x-1}{2} - \frac{5x-3}{6} - \frac{1-x}{3}$

13. $\frac{x+3}{4} - \frac{x-2}{5} + \frac{x-4}{10} - \frac{x+3}{6}$

14. $\frac{1-2a}{5} + \frac{2a-1}{4} - \frac{2a-a^2+1}{8}$

15. $\frac{3+x-x^2}{4} - \frac{1-x+x^2}{6} - \frac{1-2x-2x^2}{3}$

16. Reduce $\frac{5a^2 + b^2}{a^2 - b^2} - 2$ to a fraction.

SOLUTION

$$\begin{aligned} \frac{5a^2 + b^2}{a^2 - b^2} - 2 &= \frac{5a^2 + b^2 - 2(a^2 - b^2)}{a^2 - b^2} \\ &= \frac{5a^2 + b^2 - 2a^2 + 2b^2}{a^2 - b^2} \\ &= \frac{3(a^2 + b^2)}{a^2 - b^2} \end{aligned}$$

Reduce the following mixed expressions to fractions :

17. $a + \frac{b}{2}$ 19. $\frac{a^2 - c^2}{c} + 5c$ 21. $a + x - \frac{x^2}{a - x}$
 18. $x - \frac{y}{2}$ 20. $\frac{1 - x}{3} - 4x$ 22. $a - b - \frac{b^2}{a + b}$

Perform the additions and subtractions indicated :

23. $\frac{a - b}{ab} + \frac{b - c}{bc}$ 36. $\frac{ax^2}{a^2 - x^2} - \frac{2ax}{a^2 + ax}$
 24. $\frac{a + b}{a - b} - \frac{a - b}{a + b}$ 37. $\frac{12x}{x^2 - 3x} - \frac{12}{x^2 - 5x + 6}$
 25. $\sqrt{x + y} - \frac{x^2 + y^2}{x - y}$ 38. $\frac{4a + 1}{a^2 - 1} - \frac{2a}{a + a^2}$
 26. $\frac{x}{x - 2} - \frac{x - 2}{x + 2}$ 39. $m - \frac{m^2 + n^2}{m - n} + n$
 27. $x + 1 + \frac{x^2 - 3}{x - 1}$ 40. $\frac{1}{x} + 1 + \frac{2x}{1 + x} - 2$
 28. $\frac{a}{a^2 - ab} - \frac{b - a}{a^2 - 2ab + b^2}$ 41. $3a - 2x - \frac{8a^2 - 4x^2}{3a + 2x}$
 29. $\frac{a + 1}{a^2 + a + 1} + \frac{a - 1}{a^2 - a + 1}$ 42. $\frac{1}{a + b} - \frac{1}{a - b} + \frac{2a}{a^2 - b^2}$
 30. $\frac{a}{\sqrt{a^2 + ab}} + \frac{b}{a^2 - ab}$ 43. $\frac{a + x}{a - x} + \frac{a - x}{a + x} + \frac{4ax}{a^2 - x^2}$
 31. $\frac{1 + a}{1 - a} - \frac{1 - a}{1 + a}$ 44. $\frac{a}{a - 2} - \frac{a - 2}{a + 2} + \frac{3}{4 - a^2}$
 32. $\frac{2 + a}{a - 5} + \frac{a + 3}{a + 2}$ SUGGESTION. $\frac{3}{4 - a^2} = \frac{-3}{a^2 - 4}$
 33. $\frac{3a}{a + 1} + \frac{2a}{2 + a}$ 45. $\frac{a + 1}{a - 1} + \frac{2}{a + 1} + \frac{4a}{1 - a^2}$
 34. $\frac{a - 2}{a + 3} - \frac{2a - 1}{a - 2}$ 46. $\frac{5x + 2}{x^2 - 4} + \frac{2}{x - 2} - \frac{3}{2 - x}$
 35. $\frac{3a - 5}{a + 2} - \frac{a + 3}{4a - 8}$ 47. $\frac{x(a + x)}{a - x} - \frac{3ax - x^2}{x - a}$

$$48. \frac{x^2}{(x+y)^2} + \frac{x}{x+y} + 2 \quad 50. \frac{a-b}{2(a+b)} + \frac{a^2+b^2}{a^2-b^2} + \frac{a}{b-a}$$

$$49. \frac{1}{b} + \frac{b+2}{3b-6} + \frac{b}{4-2b} \quad 51. \frac{a+33}{a^2-9} + \frac{6}{3-a} + \frac{10}{a-3}$$

$$52. \frac{x}{x^2+xy} + \frac{x+y}{xy-y^2} - \frac{y}{x^2-y^2}$$

SUGGESTION. First reduce $\frac{x}{x^2+xy}$ to lowest terms.

MULTIPLICATION OF FRACTIONS

Fractions are multiplied in algebra just as they are in arithmetic.

Thus,
$$\frac{3}{4} \times \frac{5}{2} = \frac{3 \times 5}{4 \times 2} = \frac{15}{8}$$

In general,
$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

PRINCIPLE. *The product of two or more fractions equals the product of their numerators divided by the product of their denominators.*

EXERCISES

1. Multiply $\frac{x-5}{x+5}$ by x^2-25 .

SOLUTION

$$\frac{x-5}{\cancel{x+5}} \cdot \frac{(x-5)\cancel{(x+5)}}{1} = (x-5)^2 = x^2 - 10x + 25$$

Check by substitution of numbers for letters, as in ex. 1, page 164.

2. Multiply $\frac{x+3}{x+2}$ by $1 + \frac{1}{x+1}$.

SOLUTION

$$\begin{aligned} \left(\frac{x+3}{x+2}\right)\left(1 + \frac{1}{x+1}\right) &= \frac{x+3}{x+2} \left(\frac{x+1}{x+1} + \frac{1}{x+1}\right) \\ &= \frac{x+3}{\cancel{x+2}} \cdot \frac{\cancel{x+1}+1}{x+1} \\ &= \frac{x+3}{x+1} \end{aligned}$$

GENERAL SUGGESTIONS. 1. Any integer may be written with the denominator 1.

2. After finding the product of the numerators and the product of the denominators the resulting fraction may be reduced to lowest terms, in many cases, by canceling common factors from numerator and denominator. It is, however, more convenient to remove the common factors before performing the multiplications.

3. Generally, mixed numbers should be reduced to fractions.

Multiply :

$$3. \frac{2}{3} \text{ by } \frac{5}{6}$$

$$8. \frac{4mn}{3xy} \text{ by } -\frac{15bx}{16m^2}$$

$$4. \frac{3}{4} \text{ by } \frac{4}{15}$$

$$9. \frac{2ax}{12by} \text{ by } -\frac{10b^2}{x^2}$$

$$5. \frac{3ab}{4xy} \text{ by } \frac{2y}{3a^2}$$

$$10. \frac{a}{a+b} \text{ by } \frac{b}{a-b}$$

$$6. \frac{5xy}{2ac} \text{ by } \frac{3ax}{10y^2}$$

$$11. \frac{xy^2}{20-8x} \text{ by } \frac{25-10x}{x^2y}$$

$$7. \frac{4ab}{10c^2} \text{ by } \frac{3bc}{a^3}$$

$$12. \frac{1-6x+5x^2}{x^2-3x+2} \text{ by } \frac{2-x}{1-x}$$

Simplify each of the following :

$$13. \frac{a+c}{2} \times \frac{6}{a^2-c^2}$$

$$17. \frac{a^2+2ab+b^2}{5} \times \frac{10}{a+b}$$

$$14. \frac{a^2-b^2}{4} \times \frac{4}{a-b}$$

$$18. \frac{x^2-7x+10}{mn} \times \frac{5mn}{x-5}$$

$$15. \frac{x^2-y^2}{11} \times \frac{22}{x-y}$$

$$19. \frac{(x+y)^2}{x^2+8x+15} \times \frac{x+5}{x+y}$$

$$16. \frac{b+c}{b^2-1} \times \frac{b+1}{b+c}$$

$$20. \frac{a^2-b^2}{a^2+ab} \times \frac{4a}{a-b}$$

$$21. \frac{3a^2}{5a-10} \times \frac{3a-6}{4a^3}$$

$$22. \frac{x^2+2x+1}{y} \times \frac{4y^2}{x^2-1}$$

$$23. \frac{c+d}{5x} \times \frac{10x^2}{c^2-cd-2d^2} \times \frac{ac-2ad}{2x}$$

$$(24.) \frac{a-b}{(a+b)^2} \times \frac{a^2-b^2}{3} \times \frac{4}{a^2-2ab+b^2}$$

$$25. \frac{x^2+x-2}{4x-12} \times \frac{3}{x^2-x-6} \times \frac{x-3}{x-1}$$

$$26. \frac{a-3}{a+7} \times \frac{a^2+4a-21}{a^2-9}$$

$$27. \frac{x}{y} \times \frac{x^2-y^2}{x-y} \times \frac{y^2}{x^2}$$

$$28. \frac{x^2+3x+2}{x^2-3x-10} \times \frac{x^2-6x+5}{x^2+8x+7}$$

$$29. \frac{p+2}{x-3} \times \frac{3x^2-27}{p^2-4} \times \frac{p-2}{3x+9}$$

$$30. \frac{(a-b)^2}{a+b} \times \frac{b}{a^2-ab} \times \frac{(a+b)^2}{a^2-b^2}$$

$$31. \left(\frac{2}{3x} - 1\right) \times \frac{9x^2y}{4-9x^2} \times \frac{2+3x}{4y^3}$$

$$32. \frac{4a-b}{2x+y} \times \frac{2a}{4a^2-ab} \times \frac{4x^2-y^2}{4}$$

$$(33.) \frac{p^4-q^4}{(p-q)^2} \times \frac{p-q}{p^2+pq} \times \frac{p^2}{p^2+q^2}$$

DIVISION OF FRACTIONS

The **reciprocal** of a fraction is the fraction *inverted*, or 1 divided by the fraction.

The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$; of $\frac{x}{z}$ is $\frac{z}{x}$; of m , or $\frac{m}{1}$, is $\frac{1}{m}$.

The **reciprocal** of a number is 1 divided by the number.

EXERCISES

Write the reciprocal of :

1. $\frac{a}{b}$

3. rs

5. $3tu$

7. $\frac{4}{ab}$

2. $\frac{3m}{p}$

4. $\frac{2a^2}{bc}$

6. $\frac{1}{3m}$

8. $\frac{a-x}{b-y}$

To divide one fraction by another

Just as $\frac{3}{4} \div \frac{2}{3} = \frac{3}{4} \times \frac{3}{2} = \frac{3 \times 3}{4 \times 2} = \frac{9}{8}$

so $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$

(1) Factor the numerators and the denominators of the fractions.

(2) Invert the terms of the divisor.

(3) Multiply the dividend by the inverted fraction, canceling all factors which are common.

PRINCIPLE. Dividing by a fraction is equivalent to multiplying by its reciprocal.

EXERCISES

1. Divide $\frac{x^2 - 4}{x^2 - 1}$ by $\frac{x + 2}{x - 1}$.

SOLUTION

$$\frac{x^2 - 4}{x^2 - 1} \div \frac{x + 2}{x - 1} = \frac{(x+2)(x-2)}{(x+1)(x-1)} \times \frac{x-1}{x+2} = \frac{x-2}{x+1}$$

Simplify:

2. $\frac{5mn}{6bx} \div \frac{10m^2n}{3ax^2}$

6. $(4a + 2) \div \frac{2a + 1}{5a}$

3. $\frac{3abm}{7} \div abx$

7. $\frac{a^4 - b^4}{a^2 - 2ab + b^2} \div \frac{a^2 + b^2}{a^2 - ab}$

4. $\frac{my - y^2}{(m + y)^2} \div \frac{y^2}{m^2 - y^2}$

8. $\left(x \div \frac{1}{y}\right) \div \left(y^2 \div \frac{1}{x^2}\right)$

5. $\frac{(a - b)^2}{a + b} \div \frac{a^2 - ab}{b}$

9. $(a + c) \div \left(\frac{a^2 - c^2}{1 + x} \div \frac{a - c}{1 - x^2}\right)$

10. $\frac{a^2 + b^2 - c^2 + 2ab}{a^2 - b^2 - c^2 + 2bc} \div \frac{a + b + c}{a - b + c}$

Complex fractions

A fraction one or both of whose terms contains a fraction is called a **complex fraction**.

EXERCISES

1. Simplify the expression $\frac{\frac{a}{b}}{\frac{x}{y}}$.

FIRST SOLUTION

$$\frac{\frac{a}{b}}{\frac{x}{y}} = \frac{a}{b} \div \frac{x}{y} = \frac{a}{b} \times \frac{y}{x} = \frac{ay}{bx}$$

SECOND SOLUTION

$$\frac{\frac{a}{b} \times by}{\frac{x}{y} \times by} = \frac{ay}{bx}$$

FIRST SOLUTION. Regard the fraction as an expression of division and proceed as in division of fractions.

SECOND SOLUTION. Multiply the numerator and the denominator of the complex fraction by the least common denominator of the denominators, as shown above, and simplify.

Simplify :

2. $\frac{\frac{a}{b}}{\frac{c}{d}}$

5. $\frac{\frac{x^2}{y^2}}{\frac{x}{y}}$

8. $\frac{\frac{2p}{q}}{\frac{p}{t}}$

11. $\frac{p + \frac{a}{b}}{q - \frac{c}{d}}$

3. $\frac{\frac{4}{7}}{\frac{7}{m}}$

6. $\frac{\frac{2a}{3b}}{\frac{4a^2}{9b^2}}$

9. $\frac{\frac{2b}{3c}}{\frac{4b^2}{9c^2}}$

12. $\frac{\frac{x+2y}{w-4z}}{\frac{x-2y}{w+4z}}$

4. $\frac{\frac{3}{p}}{\frac{p}{6}}$

7. $\frac{\frac{y}{2x^2}}{y^2}$

10. $\frac{a + \frac{1}{4}}{b - \frac{1}{2}}$

13. $\frac{\frac{w+y}{x} + \frac{y}{z}}{\frac{a}{b} - \frac{c}{d}}$

EQUATIONS AND PROBLEMS

Clearing equations of fractions

The process of changing an equation containing fractions to an equation without fractions is called **clearing the equation of fractions**.

ILLUSTRATIVE EXAMPLES

1. Solve the equation $\frac{x}{2} = 10 - \frac{x}{3}$.

SOLUTION

$$(1) \quad \frac{x}{2} = 10 - \frac{x}{3}$$

$$(2) \quad 3x = 60 - 2x \quad 6 \times (1)$$

$$(3) \quad 5x = 60 \quad (2) + 2x$$

$$(4) \quad x = 12 \quad (3) \div 5$$

EXPLANATION. Since the first fraction will become an integer if the members of the equation are multiplied by 2 or some number of times 2, and since the second fraction will become an integer if the members are multiplied by 3 or some multiple of 3, the equation may be *cleared of fractions* in a single operation by multiplying both members by some *common multiple* of 2 and 3, as 6 or 12 or 18, etc.

It is best to multiply by the L. C. M. of the denominators, that is, by the L. C. D. of the fractions, which in this case is 6.

CHECK. When 12 is substituted for x , the given equation becomes $6 = 6$; that is, *the equation is satisfied for $x = 12$.*

2. Solve the equation $\frac{x-1}{2} - \frac{x-2}{3} = \frac{2}{3} - \frac{x-3}{4}$.

SOLUTION

$$(1) \quad \frac{x-1}{2} - \frac{x-2}{3} = \frac{2}{3} - \frac{x-3}{4}$$

$$(2) \quad 6(x-1) - 4(x-2) = 8 - 3(x-3) \quad 12 \times (1)$$

$$(3) \quad 6x - 6 - 4x + 8 = 8 - 3x + 9 \quad \text{Parentheses removed from (2)}$$

$$(4) \quad 2x + 2 = 17 - 3x \quad \text{Terms in (3) united}$$

$$(5) \quad 5x = 15 \quad (4) + 3x - 2$$

$$(6) \quad x = 3 \quad (5) \div 5$$

CHECK. Substitute 3 for x in (1):

$$\begin{array}{r|l} \frac{3-1}{2} - \frac{3-2}{3} & \frac{2}{3} - \frac{3-3}{4} \\ \frac{2}{2} - \frac{1}{3} & \frac{2}{3} - \frac{0}{4} \\ 1 - \frac{1}{3} & \frac{2}{3} - 0 \\ \frac{2}{3} & \frac{2}{3} \end{array}$$

RULE. Multiply both members of the equation by the lowest common denominator of the fractions.

CAUTIONS. To insure correct results in solving fractional equations:

(1) Before clearing, reduce all fractions to lowest terms and unite fractions that have like denominators.

(2) If a fraction is negative, the sign of each term of the numerator must be changed when the denominator is removed. If there is more than one term in the numerator of a negative fraction, it is a helpful device to inclose the terms of the numerator in a parenthesis before multiplying. (See example 2, page 164.)

EXERCISES

Solve, and check each result:

1. $2x + \frac{x}{3} = \frac{35}{3}$

3. $\frac{x}{2} + \frac{x}{6} = \frac{10}{3}$

2. $\frac{x}{4} + 10 = 13$

4. $7\frac{1}{2} - \frac{3x}{14} = \frac{x}{7}$

5. $\frac{x}{2} + \frac{x}{3} - \frac{x}{4} + \frac{3x}{10} - \frac{5x}{12} = 7$

6. $\frac{25x}{18} - \frac{5x}{9} + \frac{2x}{3} - \frac{5x}{6} = 2$

Solve, and check each result :

$$(7) \quad \frac{2x}{3} - \frac{7x}{8} + \frac{5x}{18} + \frac{x}{24} = \frac{4}{9}$$

$$(8) \quad \frac{3x}{4} + \frac{7x}{16} - \frac{x}{2} - \frac{9x}{16} = \frac{1}{8}$$

$$9. \quad 1.07x + .52 = .15x + 10.12 + .68x$$

SUGGESTION. Clear of decimal fractions by multiplying by 100.

$$10. \quad \frac{.2x}{7} - \frac{.1x}{4} - \frac{.1x}{2} + \frac{.4x}{7} = \frac{.3}{14}$$

$$11. \quad \frac{3x}{4} - \frac{7x}{12} = \frac{11x}{36} - \frac{8x}{9} + \frac{3}{2}$$

$$12. \quad \frac{n+4}{.3} + \frac{2-2n}{.6} = \frac{n+1}{.2} - \frac{10}{.3}$$

In exercises 13 and 14 find the value of x to the nearest hundredth:

$$13. \quad 2.5x - 3.35 = 2.9 - 3.05x$$

$$14. \quad 3.72x + 1.83 = 1.85x + 4.9$$

$$15. \quad \frac{3x-5}{x+6} = \frac{1}{8}$$

$$(16) \quad \frac{m-2}{m+3} = \frac{3}{4}$$

$$17. \quad \text{Solve} \quad \frac{9x+5}{14} + \frac{8x-7}{6x+2} = \frac{36x+15}{56} + \frac{10\frac{1}{4}}{14}$$

SUGGESTION. First combine the fractions having an arithmetic denominator :

$$\begin{aligned} \frac{36x+15}{56} + \frac{41}{56} - \frac{9x+5}{14} &= \frac{36x+15+41-36x-20}{56} \\ &= \frac{36}{56} = \frac{9}{14} \end{aligned}$$

$$\text{Then} \quad \frac{8x-7}{6x+2} = \frac{9}{14}$$

Solve :

$$18. \quad \frac{3x-2}{2x-5} + \frac{3x-21}{5} = \frac{6x-22}{10}$$

$$19. \quad \frac{4x+3}{9} = \frac{8x+19}{18} - \frac{7x-29}{5x-12}$$

$$20. \frac{6p^2 + p}{15p} - \frac{2p - 4}{7p - 13} = \frac{2p - 1}{5}$$

$$21. \frac{4x}{2x - 3} = \frac{3x}{2x - 3} - \frac{1}{10}$$

$$25. \frac{n}{n + 2} - \frac{n + 20}{n^2 - 4} = \frac{n}{n - 2}$$

$$22. \frac{9n}{n - 2} = \frac{8n}{n - 1} + 1$$

$$26. \frac{2}{n + 1} = \frac{6}{2n + 1} - \frac{1}{n - 3}$$

$$23. \frac{x}{x + 2} = \frac{x + 2}{2x} + \frac{1}{2}$$

$$27. \frac{x - 3}{2x + 6} - \frac{x^2 + 6}{6x^2 - 54} = \frac{x + 3}{3x - 9}$$

$$24. \frac{2n}{n - 4} + \frac{n}{n + 4} = \frac{3n^2 + 16}{n^2 - 16}$$

$$28. \frac{x}{2} - 2\left(\frac{4x}{5} - 3\right) = 4 - \frac{3}{2}\left(\frac{x}{2} + 1\right)$$

$$29. \frac{2 - x}{x - 3} + \frac{7}{5} = \frac{3}{3 - x}$$

SUGGESTION. Change the denominator of the right member to $x - 3$ by multiplying the numerator and the denominator by -1 .

$$30. \frac{2}{x + 3} + \frac{1}{3 - x} = \frac{6x}{x^2 - 9}$$

$$32. \frac{5x^2 + 6}{x^2 - 4} - \frac{5x + 3}{x + 2} = \frac{-3}{2 - x}$$

$$31. \frac{2x + 1}{2x - 1} = \frac{2x - 1}{2x + 1} - \frac{8}{1 - 4x^2}$$

$$33. \frac{x - 1}{x - 2} + \frac{x - 6}{x - 7} = \frac{x - 5}{x - 6} + \frac{x - 2}{x - 3}$$

SUGGESTION. In ex. 33 first subtract the second and the fourth term from both members; then combine the fractions on each side before clearing of fractions.

$$34. \frac{x - 1}{x - 2} + \frac{x - 7}{x - 8} = \frac{x - 5}{x - 6} + \frac{x - 3}{x - 4}$$

ALGEBRAIC REPRESENTATION

1. What part of $m - n$ is p ?
2. Indicate the sum of l and m divided by 2, and that result multiplied by n .
3. Indicate the product of s and $(r - 1)$ divided by the square of the sum of t and v .

4. A boy who had m marbles lost $\frac{1}{a}$ of them. How many marbles had he left?

5. By what number must x be multiplied so that the product shall be z ?

6. Indicate the result when the sum of a , b , and $-c$ is to be divided by the square of the sum of a and b .

7. It is t miles from Albany to Utica. The Empire State Express runs s miles an hour. How long does it take this train to go from Albany to Utica?

8. A cabinetmaker worked x days on two pieces of work. For one he received v dollars, and for the other w dollars. What were his average earnings a day for that time?

9. A train runs x miles an hour and an automobile $(x - y)$ miles an hour. How much longer will it take the automobile to run s miles than the train?

10. Indicate the result when b is added to the numerator and subtracted from the denominator of the fraction $\frac{a}{c}$.

11. A farmer had $\frac{1}{x}$ of his crop in one field, $\frac{1}{y}$ in a second, and $\frac{1}{z}$ in a third. What part of his crop had he in these three fields?

12. A student spends $\frac{1}{m}$ of his income for room rent, $\frac{1}{n}$ for board, $\frac{1}{s}$ for books, and $\frac{1}{r}$ for clothing. If his income is x dollars, how much has he left?

Problems involving fractions

1. The numerator of a certain fraction is 5 smaller than the denominator. If 1 is added to the numerator, the value of the new fraction is $\frac{1}{2}$. Find the original fraction.

2. The denominator of a fraction is 3 larger than the numerator. If 3 is subtracted from the numerator, the value of the new fraction is $\frac{1}{4}$. Find the original fraction.

3. The numerator of a fraction is 1 greater than the denominator. If 1 is subtracted from the numerator and 1 is added to the denominator, the value of the new fraction is $\frac{3}{4}$. Find the original fraction.

4. Two aviators, one in a monoplane and the other in a biplane, flew from Detroit to St. Louis. The biplane arrived 1 hour sooner than the monoplane. How long did it take each, if the time of the biplane was $\frac{5}{8}$ that of the monoplane?

5. The first issue of *The Sun* devoted $\frac{1}{3}$ of its columns to advertisements and $\frac{1}{4}$ to miscellaneous news. The rest of the paper, 5 columns, was devoted to poetry, finance, and shipping news. How many columns did it contain?

6. Of the world's supply of rubber one year, South America produced $\frac{1}{3}$ and Africa $\frac{1}{5}$. How much was produced by each if the rest of the world produced 26,600 tons?

7. A large ear of seed corn exhibited at the Iowa Experiment Station sold for \$150. At the same rate, if it had weighed 8 ounces less, it would have sold for \$90. How much did it weigh?

8. Divide 56 into two parts such that $\frac{3}{8}$ of the greater exceeds $\frac{1}{4}$ of the smaller by 6.

9. The number of vessels entering New York Harbor in one year was 11,399. If $\frac{1}{4}$ the number of steamships was 449 more than $\frac{1}{5}$ the number of sailing vessels, how many of each were there?

10. Find the world's production of nickel in a year when the United States and Canada together produced $\frac{1}{2}$ of it, England $\frac{1}{80}$ of it, and the rest of the world 3800 tons.

11. The sum of two numbers is 72 and their quotient is 8. Find the numbers.

12. The sum of $\frac{1}{5}$ and $\frac{1}{6}$ of a number multiplied by 4 equals 88. Find the number.

13. Separate 54 into two parts such that $\frac{1}{12}$ of the difference between them is $\frac{1}{3}$.

14. Separate m into two parts such that $\frac{1}{n}$ of the difference between them is $\frac{1}{r}$.

15. In a purse containing \$1.45 there are $\frac{1}{2}$ as many quarters as 5-cent pieces and $\frac{2}{3}$ as many dimes as 5-cent pieces. How many coins of each kind are there?

16. The cost per mile of running a train was \$.14 less with electrical equipment than with steam, or $\frac{3}{5}$ as much. What was the cost per mile with electricity?

17. George is $\frac{1}{2}$ as old as his father; a years ago he was $\frac{1}{3}$ as old as his father. What is the present age of each?

18. I have $6\frac{1}{4}$ hours at my disposal. How far can I ride at the rate of 9 miles an hour so that I can return in the given time, walking back at the rate of $3\frac{1}{2}$ miles an hour?

SUGGESTION. Let m = the number of miles I can ride.

Then the equation of the problem is $\frac{m}{9} + \frac{m}{3\frac{1}{2}} = 6\frac{1}{4}$.

19. A motor boat went a certain distance up a river and back in 2 hours and 56 minutes. Its rate per hour was $17\frac{1}{2}$ miles going up and 21 miles returning. How far up the river did it go?

20. A yacht sailed a certain distance up a river and back in r hours. Its rate per hour was s miles going up and t miles returning. How far up the river did it sail?

21. My annual income is m dollars. If $\frac{1}{n}$ of my property is invested at 5% and the remainder at 6%, what is my capital?

REVIEW OF FRACTIONS

1. What three signs are to be considered in connection with a fraction? What is the sign of a fraction?

2. Under what conditions may the sign of the numerator or of the denominator of a fraction be changed?

3. What is the effect of changing the sign of an odd number of factors in either term of a fraction? of an even number of factors?

4. Show that
$$\frac{(x-y)(x-y)}{(z-w)(u-z)} = \frac{-x^2 + 2xy - y^2}{(w-z)(u-z)}.$$

5. When is a fraction in its lowest terms? What principle applies to the reduction of fractions to higher or lower terms?

6. Reduce to lowest terms:

$$\frac{x(4a^2 - 9b^2)}{4a^2x^2 + 12abx^2 + 9b^2x^2}$$

7. In reducing a fraction to an equal fraction having a given denominator, how do you find the number by which both terms are to be multiplied?

8. Reduce $\frac{3b}{a-2b}$ to a fraction whose denominator is $a^2 - 4ab + 4b^2$.

9. (a) What is the reciprocal of a fraction? of any number?

(b) Give the reciprocals of $\frac{x}{y}$, $\frac{a+b}{x-y}$, and x .

10. (a) Define and illustrate complex fraction.

(b) Simplify the fraction you have written.

Reduce the following mixed expressions to fractions:

11. $x - y - \frac{2y}{x+y}$

13. $x - 2 - \frac{6-5x}{x-3}$

12. $\frac{m^2+n^2}{m+n} - n - m$

14. $2x - 3 - \frac{4x^2}{2x+3}$

Simplify :

$$15. \frac{s+t}{2rl} + \frac{s-t}{4rl} - \frac{s^2+t^2}{4rl(s+t)}$$

$$16. \frac{a+x}{x+2} + \frac{a-x}{2-x} - \frac{2(x^2-2a)}{x^2-4}$$

$$17. \frac{c^2ab}{(c-a)(b-c)} - \frac{b^2ca}{(b-a)(b-c)} - \frac{a^2bc}{(a-b)(a-c)}$$

$$18. \frac{b-c}{(b-a)(a-c)} - \frac{c-a}{(b-c)(a-b)} - \frac{a+b}{(a-c)(b-c)}$$

$$19. \frac{a^2b}{x^2y} \times \frac{x^2y^2z}{abc}$$

$$25. \frac{c+d}{5x} \cdot \frac{10x^2}{c^2-cd-2d^2} \cdot \frac{ac-2ad}{2x}$$

$$20. \frac{3a^3c^2}{4xyz} \times \frac{12xz}{27a^4c^3}$$

$$26. \frac{a-b}{(a+b)^2} \cdot \frac{a^2-b^2}{3} \cdot \frac{4}{a^2-2ab+b^2}$$

$$21. \frac{a^2-b^2}{a^2+ab} \times \frac{4a}{a-b}$$

$$27. \frac{x^2+x-2}{4x-12} \cdot \frac{3}{x^2-x-6} \cdot \frac{x-3}{x-1}$$

$$22. \frac{x^2-y^2}{4} \div \frac{x+y}{12a}$$

$$28. \frac{(r-2s)^2}{r-s} \div \frac{r^2-rs-2s^2}{r^2-s^2}$$

$$23. \frac{4-x^2}{2-3x+x^2} \div (2+x)$$

$$29. \frac{x^2-2xy-3y^2}{x^2+2xy+y^2} \div \frac{(x-3y)^2}{x+y}$$

$$24. \frac{xy-x^2y^2}{axy} \div \frac{a-axy}{a^2x+a^2y}$$

$$30. \frac{2a^2+ab-3b^2}{2a^2+5ab+3b^2} \div \frac{a^2-b^2}{a+b}$$

$$31. \left(\frac{a^2+2ab+b^2}{a^2-2ab+b^2} - \frac{a+b}{a-b} \right) \times \left(\frac{3a-3b}{2b} \div \frac{6a+6b}{2a-2b} \right)$$

$$32. \left(4 - \frac{6}{a+1} \right) \div \left(8 - \frac{4a-8}{a^2-1} \right)$$

$$33. \left(\frac{4x}{x^2-4} + \frac{2}{2-x} \right) \div \frac{6x}{x^2-3x-10}$$

$$34. \left(\frac{2x}{x-2} - \frac{x}{x-1} \right) \div \left(\frac{2x}{x-3} - \frac{2x}{x-2} \right)$$

35. What is meant by clearing an equation of fractions?
State the axiom upon which this process is based.

36. What precautions must be taken to secure correct results in solving equations that involve fractions?

Solve, and check each result :

$$37. \quad \frac{5x}{2} - \frac{7x}{4} = 6$$

$$38. \quad \frac{10 - 7x}{6 - 7x} = \frac{5x - 4}{5x}$$

$$39. \quad \frac{4}{x+2} + \frac{7}{x+3} = \frac{37}{x^2 + 5x + 6}$$

$$40. \quad \frac{x+3}{x-3} - \frac{x}{4} = \frac{12}{24} - \frac{x-3}{4}$$

$$41. \quad \frac{x+7}{x+2} + \frac{x+8}{3} = \frac{3x-15}{9} - 3$$

$$42. \quad 2.04x - 3.10 - 2.95x = 8.12 - 5.00x + 1.05$$

$$43. \quad \frac{4}{x-4} = \frac{5x-12}{x^2-16} - \frac{8}{16-x^2}$$

$$44. \quad \frac{3}{x+1} + \frac{24}{1-x^2} = \frac{3}{1-x}$$

$$45. \quad \frac{y-1}{2} + \frac{y-2}{3} + \frac{y-3}{4} = \frac{5y-1}{6}$$

$$46. \quad \frac{v+1}{3} - \frac{v+4}{5} + \frac{v+3}{4} = 16$$

$$47. \quad \frac{7z+2}{6} - \frac{12-z}{4} + \frac{z+2}{2} = 6$$

$$48. \quad \frac{u-3}{7} + \frac{u+5}{3} - \frac{u+2}{6} = 4$$

$$49. \quad \frac{3t-5}{4} - \frac{7t-13}{6} = 3 - \frac{t+3}{2}$$

$$50. \quad \frac{5x+2}{3} - \left(x - \frac{3x-1}{2}\right) = \frac{3x+19}{2} - \left(\frac{x+1}{6} + 5\right)$$

$$51. \quad \frac{3y-2}{y+4} - \frac{y+5}{5-y} = \frac{4y^2+3y-3}{y^2-y-20}$$

CHAPTER IX

RATIO AND PROPORTION

RATIO

The relation of two numbers of the same kind that is expressed by the quotient of the first divided by the second is called the **ratio** of the first to the second.

For example, the ratio of 15 pounds to 3 pounds is $\frac{15}{3}$, or 5, or $\frac{5}{1}$; the ratio of 3 pounds to 15 pounds is $\frac{3}{15}$, or $\frac{1}{5}$.

A ratio may be regarded as a fraction. The first term corresponds to the *numerator* and the second term to the *denominator* of the fraction.

In the ratio $\frac{a}{b}$, a is the first term and b is the second term.

A ratio is usually expressed as shown above, although it may also be indicated by a colon. Thus, $\frac{3}{5} = 3:5$.

EXERCISES

1. What is the ratio of 8 m to 4 m ? of 4 m to 8 m ?
2. Express in its lowest terms the ratio of 6 to 9; of 12 x to 16 y ; of am to bm ; of 20 ab to 10 bc ; of $m + n$ to $m^2 - n^2$.
3. Which is the greater ratio, $\frac{2}{3}$ or $\frac{3}{4}$? $\frac{4}{9}$ or $\frac{2}{5}$? $\frac{4}{9}$ or $\frac{2}{3}$?
4. What is the ratio of $\frac{1}{2}$ to $\frac{1}{4}$? of $\frac{1}{2}$ to $\frac{1}{3}$? of $\frac{2}{3}$ to $\frac{3}{4}$?
5. What is the ratio of 15 days to 30 days? of 21 days to 1 week? of 1 rod to 1 mile?
6. Find the ratio of $\frac{4}{5}x$ to $\frac{2}{5}x^2$; of $\frac{3}{4}ab$ to $\frac{1}{2}ac$; of $2\frac{1}{2}$ to $7\frac{1}{2}$; of .7 m to .8 m ; of $(x^2 - y^2)$ to $(x - y)^2$.

7. Two boys each weigh 120 pounds. What is the ratio of the weight of the first to that of the second?

8. Mary was marked 80% on an algebra examination and John, 90%. What was the ratio of Mary's mark to John's? of John's to Mary's?

9. The ratio of the rate of a local train to that of an express is 1 to 2. If the local train runs 20 miles an hour, find the rate of the express train.

10. The ratio of the consumption of gas in New York one year to that in Chicago was $\frac{7}{4}$. If 12 billion cubic feet were consumed in Chicago, what was the consumption in New York?

PROPORTION

An equality of ratios is called a **proportion**. Thus, $\frac{3}{10} = \frac{6}{20}$ and $\frac{a}{b} = \frac{c}{d}$ are proportions. They are sometimes written $3 : 10 = 6 : 20$ and $a : b = c : d$.

We read the proportion $\frac{3}{10} = \frac{6}{20}$ as follows: "3 is to 10 as 6 is to 20." The proportion $a : b = c : d$ is read " a is to b as c is to d ."

In a proportion, the first and last terms are called the **extremes**; the middle terms are called the **means**.

In the proportion $\frac{3}{4} = \frac{a}{b}$, the terms 3 and b are the extremes and 4 and a are the means.

Since a proportion is really an equation, each member of which is a fraction, it follows that:

GENERAL PRINCIPLE. *The changes that may be made in a proportion without destroying the equality of its ratios correspond to the changes that may be made in the members of an equation without destroying their equality and in the terms of a fraction without altering the value of the fraction.*

From this general principle it follows that we may, by dealing with the fractions in a proportion, discover some general properties of all proportions.

For example, if $\frac{a}{b} = \frac{c}{d}$ and we clear of fractions by multiplying each side of the equation by bd , we have

$$ad = bc \quad \text{Hence}$$

(1) *In any proportion the product of the means is equal to the product of the extremes.*

In the proportion $\frac{a}{m} = \frac{m}{b}$, m is called a **mean proportional** between a and b .

Clearing of fractions in the proportion $\frac{a}{m} = \frac{m}{b}$, we get $m^2 = ab$; $m = \sqrt{ab}$. Hence

(2) *A mean proportional between two numbers is equal to the square root of their product.*

What is the mean proportional in $\frac{3}{8} = \frac{6}{12}$?

Using the axiom relating to division, prove the following principle of proportion:

(3) *If the product of two numbers is equal to the product of two other numbers, one pair of them may be made the extremes and the other pair the means of a proportion.*

For example, since $4 \times 3 = 6 \times 2$, $\frac{4}{6} = \frac{2}{3}$, and $\frac{4}{2} = \frac{6}{3}$. Which are the extremes in these proportions? the means?

If $ax = by$, $\frac{a}{y} = \frac{b}{x}$. (Why?) Also, $\frac{a}{b} = \frac{y}{x}$. (Why?) Which are the means in these proportions? the extremes?

EXERCISES

1. Find the value of x in the proportion $\frac{3}{5} = \frac{x}{55}$.

2. Find the value of a in the proportion $a : 5 = 4\frac{1}{2} : 9$.

Find the value of x in each of the following proportions :

$$3. \frac{2}{3} = \frac{4}{x}$$

$$6. \frac{x+2}{x} = \frac{10}{6}$$

$$4. \frac{5}{x} = \frac{4}{3}$$

$$7. \frac{x}{x-1} = \frac{15}{12}$$

$$5. 1 : \bar{x} = x : 9$$

$$8. x + 2 : x - 2 = 3 : 1$$

9. If $m \times n = p \times q$, form three different proportions with m , n , p , and q . Form three proportions from the equality $8 \times 9 = 24 \times 3$.

10. Find the mean proportional between 16 and 4.

11. Find the mean proportional between 5 and 45.

Test to see whether the following are true proportions :

$$12. \text{ Is } \frac{5\frac{1}{2}}{3} \text{ equal to } \frac{4}{6\frac{1}{2}}?$$

$$14. \text{ Is } \frac{5}{7x} \text{ equal to } \frac{10}{14x}?$$

$$13. \text{ Is } \frac{4}{13} \text{ equal to } \frac{2}{6\frac{1}{2}}?$$

$$15. \text{ Is } \frac{2.4a}{.8a} \text{ equal to } \frac{6a}{2a}?$$

$$16. \text{ When } \frac{a}{b} = \frac{c}{d}, \text{ prove that :}$$

$$(1) \frac{a}{c} = \frac{b}{d}$$

$$(2) \frac{b}{a} = \frac{d}{c}$$

SUGGESTION. Clear of fractions in the original proportion. Use the division axiom.

$$17. \text{ When } \frac{a}{b} = \frac{c}{d}, \text{ prove that :}$$

$$(1) \frac{a+b}{b} = \frac{c+d}{d}$$

$$(2) \frac{a-b}{b} = \frac{c-d}{d}$$

Problems

1. Gun metal consists of 9 parts copper and 1 part zinc. How many pounds of each are there in 1000 pounds of gun metal?

2. In brass there are 2 parts copper and 1 part zinc. How many ounces of each are there in $4\frac{1}{2}$ pounds of the metal?

3. Michigan produces yearly 25% of the iron ore of the United States. The ratio of Michigan's output to Minnesota's is 5 : 8. What per cent of the country's output does Minnesota produce?

4. The United States manufactured 285 million pens one year. The ratio of the steel pens to the whole number of pens was $\frac{18}{9}$. How many steel pens were manufactured that year?

5. A diver descended 210 feet into a lake. The ratio of this distance to the distance that is usually considered the limit for divers is 7 : 5. Find the usual limit for divers.

6. How many pounds of tea are made from 4200 pounds of the green leaf, if the ratio of the weight of the manufactured tea to that of the green leaf is $\frac{5}{21}$?

7. Two machines, one old and one modern, turn out 960 pins a minute. The ratio of the number turned out by the old machine to the number turned out by the modern one is $\frac{1}{15}$. How many are turned out by each machine?

8. Find a number that added to each of the numbers 1, 2, 4, and 7 will give four numbers in proportion.

9. The ratio of the greatest length of Lake Erie to the greatest length of Lake Michigan is 5 : 6. What is the length of each, if Lake Michigan is 50 miles longer than Lake Erie?

10. The ratio of the loss of life in the Lisbon earthquake to that in the Messina earthquake was $\frac{1}{2}$. If 55,000 more lives were lost in the latter than in the former, find the loss of life in each earthquake.

11. The wool sales for one week in New York amounted to 555,000 pounds. The ratio of the domestic sales to the foreign sales was $\frac{1}{2}$. What were the foreign sales?

12. Out of a lot of shell caps, the ratio of 100 times the number rejected by the government inspector for imperfections to the total number was 3 : 11. If 1097 were accepted, how many were rejected?

FORMULAS

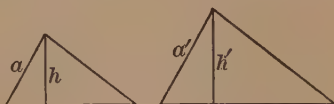
Similar figures have the same shape. In similar figures, the corresponding lines are in proportion and the corresponding angles are equal. For example, in the similar triangles below, a and a' are corresponding sides, and h and h' are corresponding altitudes. Hence $\frac{h}{h'} = \frac{a}{a'}$. The altitudes have the same ratio as the corresponding sides.

Plane figures

The corresponding altitudes of two similar triangles have the same ratio as any two corresponding sides.

FORMULA: $\frac{h}{h'} = \frac{a}{a'}$

- Find a when $a' = 9$ inches, $h = 4\frac{1}{2}$ inches, and $h' = 6\frac{3}{4}$ inches.



The perimeters of two similar polygons have the same ratio as any two corresponding sides.

FORMULA: $\frac{p}{p'} = \frac{a}{a'}$



- Find p' when $\frac{a}{a'} = \frac{2}{3}$ and $p = 210$ rods.

The areas of two rectangles with equal altitudes have the same ratio as their bases.

FORMULA: $\frac{A}{A'} = \frac{b}{b'}$



- Find b when $A = 12$ square inches, $A' = 36$ square inches, and $b' = 7.5$ inches.

The areas of two rectangles with equal bases have the same ratio as their altitudes.

FORMULA: $\frac{A}{A'} = \frac{h}{h'}$

- Find A' when $A = 12$ square feet, $h = 2.3$ feet, and $h' = 6.9$ feet.



The areas of two similar polygons have the same ratio as the squares of their corresponding sides.

FORMULA: $\frac{A}{A'} = \frac{a^2}{a'^2}$



5. Find A when $A' = 112\frac{1}{2}$ square inches, $a = 3$ inches, and $a' = 9$ inches.

6. If a pair of corresponding sides of two similar polygons are $3\frac{1}{3}$ inches and 10 inches, respectively, find the ratio of the areas of the polygons.

The circumferences of two circles have the same ratio as their radii or as their diameters.

FORMULAS:

(a) $\frac{c}{c'} = \frac{r}{r'}$

(b) $\frac{c}{c'} = \frac{d}{d'}$



7. Find c' when $c = 62\frac{2}{7}$ inches, $r = 10$ inches, and $r' = 30$ inches.

8. The diameter of one circle is twice that of a second circle. How do their circumferences compare?

The areas of two circles have the same ratio as the squares of their radii or as the squares of their diameters.

FORMULAS: (a) $\frac{A}{A'} = \frac{r^2}{r'^2}$

(b) $\frac{A}{A'} = \frac{d^2}{d'^2}$

9. Find A when $A' = 616$ square inches, $r = 7$ inches, and $r' = 14$ inches.

10. Find the ratio of the areas of the circles described in example 8.

11. One circle has a diameter of 10 inches and a second circle has a diameter of 8 inches. Find the ratio of the area of the first circle to that of the second.

12. The radius of one circle is r inches and of a second circle, R inches. Find the ratio of the area of the first circle to that of the second.

13. The bases of two similar triangles are 8 inches and 10 inches respectively. The altitude of the first triangle

is $6\frac{1}{2}$ inches. Find the altitude of the second triangle. Find the area of each triangle.

14. Two similar polygons have sides of 3 inches and $4\frac{1}{2}$ inches respectively. What is the ratio of their perimeters? of their areas?

Solids

The areas of the surfaces of two similar solids have the same ratio as the squares of their corresponding lines.

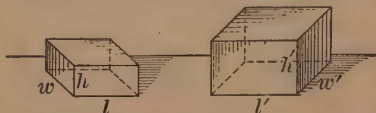
$$\text{FORMULA: } \frac{A}{A'} = \frac{a^2}{a'^2}$$



1. Find A when $A' = 324$ square feet, $a = 21$ feet, and $a' = 42$ feet.

The volumes of any two rectangular solids have the same ratio as the products of their three dimensions.

$$\text{FORMULA: } \frac{V}{V'} = \frac{lwh}{l'w'h'}$$



2. Find the ratio of V to V' if $h = 6$ inches, $w = 12$ inches, $l = 18$ inches, $h' = 12$ inches, $w' = 24$ inches, and $l' = 36$ inches.

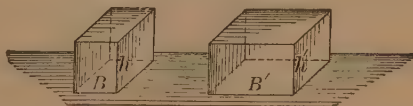
The volumes of two similar solids have the same ratio as the cubes of their corresponding lines.

$$\text{FORMULA: } \frac{V}{V'} = \frac{a^3}{a'^3}$$

3. Find the ratio of the volumes of two cubes, one having an edge of 2 inches and the other an edge of 3 inches.

The volumes of two rectangular solids having equal altitudes have the same ratio as their bases.

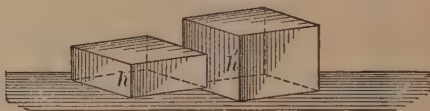
$$\text{FORMULA: } \frac{V}{V'} = \frac{B}{B'}$$



4. Find V' when $V = 640$ cubic inches, $B = 16$ square inches, and $B' = 25$ square inches.

The volumes of two rectangular solids having equal bases have the same ratio as their altitudes.

FORMULA: $\frac{V}{V'} = \frac{h}{h'}$



5. Find V when

$\frac{h}{h'} = \frac{1}{6}$ and $V' = 966$ cubic inches.

The lateral areas of two similar cylinders of revolution have the same ratio as the squares of their radii or as the squares of their diameters or as the squares of their altitudes.

FORMULAS:

(a) $\frac{S}{S'} = \frac{r^2}{r'^2}$

(b) $\frac{S}{S'} = \frac{d^2}{d'^2}$

(c) $\frac{S}{S'} = \frac{h^2}{h'^2}$



6. Find S in terms of π if $S' = 640\pi$ square inches, $r = 8$ inches, and $r' = 16$ inches.

7. Find the ratio of the lateral areas of two similar cylinders if the ratio of the diameters is $\frac{2}{3}$.

8. Find the ratio of the lateral area of a cylinder having an altitude of 18 inches to that of a similar cylinder having an altitude of 36 inches.

9. Find the ratio of the altitudes of two similar cylinders of revolution whose lateral areas are 64π square inches and 256π square inches, respectively.

In comparing the volumes of two similar cylinders of revolution, we may use the following formulas:

(a) $\frac{V}{V'} = \frac{h^3}{h'^3}$

(b) $\frac{V}{V'} = \frac{r^3}{r'^3}$

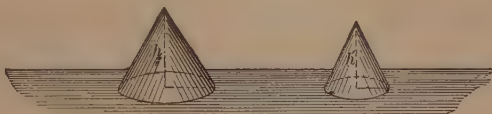
(c) $\frac{V}{V'} = \frac{d^3}{d'^3}$

10. Translate these formulas into verbal statements.

11. The diameter of one cylindrical tank is 15 feet and that of a similar tank is 25 feet. Find the ratio of the volume of the smaller tank to that of the larger.

12. The height of one cylindrical silo is 14 feet and of another similar one, 20 feet. Find the ratio of the number of cubic feet of silage the larger silo will hold to the number the smaller silo will hold.

In comparing the lateral areas and the volumes of two similar cones of revolution, we may use the following formulas:



$$(a) \frac{S}{S'} = \frac{h^2}{h'^2}$$

$$(d) \frac{V}{V'} = \frac{h^3}{h'^3}$$

$$(b) \frac{S}{S'} = \frac{r^2}{r'^2}$$

$$(e) \frac{V}{V'} = \frac{r^3}{r'^3}$$

$$(c) \frac{S}{S'} = \frac{d^2}{d'^2}$$

$$(f) \frac{V}{V'} = \frac{d^3}{d'^3}$$

13. Translate these formulas into verbal statements.



14. What formulas would be used in comparing the surfaces and volumes of spheres?

15. The diameters of two marble spheres are respectively 1 foot and 1 yard in length. Find the ratio of the volumes of these spheres.

16. If a cone of revolution is 8.4 inches high and a similar cone is 5.6 inches high, what is the ratio of their lateral areas? of their volumes?

17. If the volume of a cone, the radius of whose base is 3.5 inches, is 111.65 cubic inches, what is the volume of a similar cone whose radius is 7 inches?

18. If the surface of a sphere is 100 square inches, what is the surface of a sphere whose radius is twice as large?

Problems

1. Divide \$59.50 between two men so that their shares shall be in the ratio of 3 to 4.

2. The distance through which a body falls from rest is proportional to the square of the number of units of time. If a body falls 64.32 feet in the first 2 seconds, how far will it fall during the first 6 seconds? during the 6th second?

3. A line a inches long was divided into two parts in the ratio $m:n$. Find the length of each part.

4. Two partners gained \$6250 in business one year. Find each one's share, their investments being in the ratio 1:4.

5. Two numbers are in the ratio of 3 to 2. If each is increased by 4, the sums will be in the ratio of 4 to 3. What are the numbers?

SUGGESTION. Represent the numbers by $3n$ and $2n$.

6. Divide 25 into two parts such that the greater increased by 1 is to the less decreased by 1 as 4 is to 1.

7. Two trains traveled toward each other from two cities 98 miles apart. If their rates of traveling were as 3 is to 4, how many miles did each travel before they met?

8. A man divided his estate of \$50,000 between two heirs in the ratio of 3 to 5. How much did each heir receive?

9. A dock is divided into two parts so that the length of the longer is to that of the shorter as 11 is to 6. If the total length of the dock is 850 feet, what is the length of each part?

✓
10. The freight earnings of two railroads on a trainload of grain were \$2160. One carried the grain 400 miles; the other, 500 miles. Find the earnings apportioned to each road.

11. Find a number such that if it is subtracted from each of the numbers 7, 9, 10, and 14 it will give four numbers in proportion.

12. What number must be added to each of the numbers 11, 17, 2, and 5 so that the sums shall be in proportion when taken in the order given?

13. If 16 men can do a piece of work in 15 days, how long will it take 20 men to do it? 12 men?

14. The total receipts of a coal mining company one year were \$16,725,000 and the expenses were to the net earnings as 13 is to 2. What were the expenses? the net earnings?

15. Of 24 pounds of salt water, 12% is salt. In order to have a solution that shall contain 4% salt how many pounds of pure water must be added?

SUGGESTION. At present there is 12% of 24 pounds, or 2.88 pounds, of salt. Let w be the number of pounds of pure water to be added so that the ratio of the salt to the mixture shall be 4 : 100. We have, then, the equation $\frac{2.88}{24 + w} = \frac{4}{100}$.

16. How much water must be added to 4 gallons of alcohol which is 95% pure to reduce it to a mixture which is 75% pure?

17. A photographer has two bottles of diluted developer. In one bottle 10% of the contents is developer and the rest is water. In the other the mixture is 50% pure. How much must be drawn from each bottle to make 8 ounces of a mixture 25% pure developer?

18. The radiator of an automobile contains a mixture of alcohol and water which is 20% alcohol. It is desired to raise the percentage of alcohol to 30%. How much pure alcohol must be added to 4 gallons of the 20% mixture to produce the required mixture?

19. If the area of a circle is 5 square inches, what is the area of a circle whose radius is twice the radius of the first circle?



20. The formula $pd = WD$ expresses the physical law that when a lever just balances, the product of the numerical measures of the power (p) and its distance (d) from the fulcrum (F) is equal to the product of the numerical measures of the weight (W) and its distance (D) from the fulcrum. Express this law by means of a proportion.

21. Solve the proportion obtained in exercise 20 for W and find what weight a power of 60 pounds will support by means of a lever if $d = 8$ feet and $D = 3$ feet.

22. A pressure of 50 pounds was exerted upon one end of a 5-foot bar to balance a weight of 200 pounds at the other end of the bar. How far was the weight from the fulcrum?

23. In the figure the altitude h is a mean proportional between the segments a and b of the hypotenuse. Find the length of b if $h = 8$ and $a = 10$.



24. Divide \$180 among four boys, whose ages are 10 years, 10 years, 13 years, and 15 years respectively, in proportion to their ages.

25. The sum of the angles of a quadrilateral is 360° . What is the size of each angle if they are proportional to 5, 20, 1, and 1?

26. If the power required to drive a motor boat is proportional to the square of the speed, how much horse power will it take to drive at 12 miles an hour a boat that a 5-horse-power engine drives at 8 miles an hour?

27. Three men formed a partnership with a capital of \$14,000. Of this sum Mr. Ball contributed \$3200, Mr. Keats \$4800, and Mr. Hunt the rest. Other things being equal, how should they divide among themselves a profit of \$5250?

CHAPTER X

LINEAR, OR SIMPLE, EQUATIONS

ONE UNKNOWN NUMBER

The equation

Linear equations are sometimes called **simple equations**.

From the various equations which you have solved you have learned to appreciate what an important algebraic tool the equation is. You will now learn about some new forms of equations and some new methods of solving them.

Integral and fractional equations

An equation that does not involve an unknown number in any denominator is called an **integral equation**, while an equation that involves an unknown number in any denominator is called a **fractional equation**.

For example, (1) $x + 5 = 8$ and (2) $\frac{2x}{3} + 5 = 8$ are integral equations. Though (2) contains a fraction, the unknown number x does not appear in the denominator.

On the other hand, (1) $x + 5 = \frac{8}{x}$ and (2) $\frac{2}{x-1} = 7$ are fractional equations, since x appears in a denominator.

Review

In Chapter VI, page 98, an order of procedure in the solution of linear equations was suggested, which assists in their solution. This order is reviewed on page 196 and extended to include the steps necessary for the solution of equations containing fractions.

Order of procedure in the solution of linear equations

1. Clear the equation of fractions.
2. Perform any indicated multiplications.
3. Unite like terms in each member of the equation.
4. Apply in one step the addition and subtraction axioms, to get the unknown terms on one side and the known terms on the other side of the equation.

5. Divide each member by the coefficient of the unknown quantity.

6. Check by substituting the result in the first equation.

For example, prove by checking that 1.7 is not the root of $2x + 7 = 10.2$.

Not all equations need every step in the procedure suggested. Omit any step that is not needed and take the next one in order.

EXERCISES (Examples 1 to 7, Oral; 8 to 47, Written)

Solve, and check each result :

- | | |
|-------------------------------------|--|
| 1. $\frac{1}{4}r = 1.5$ | 13. $7x + 6 = 6x + 8$ |
| 2. $\frac{1}{2}x = 2.5$ | 14. $5x - 10 = 2x + 20$ |
| 3. $6r - 7 = 5$ | 15. $4r - 18 = 20 + \frac{1}{5}r$ |
| 4. $\frac{1}{2}b + 3 = 8$ | 16. $5n - (2n + 3) = 12$ |
| 5. $8x - 7 = 3 + 6x$ | 17. $17t + 5(2 - 3t) = 18$ |
| 6. $\frac{1}{x} = 2$ | 18. $5x - [4 - (6x - 3)] = 26$ |
| 7. $\frac{3}{a} = 12$ | 19. $(2w - 1)^2 = 4(w - 3)^2$ |
| 8. $\frac{6}{7x} = 3$ | 20. $21x + (x - 4)^2 = (5 + x)^2$ |
| 9. $\frac{5}{x} + 5 = 20$ | 21. $(12x + 6) \div 3 = 9 - 3x$ |
| 10. $\frac{15}{x + 2} = 3$ | 22. $(x + 1)(x + 2) = 11 + x^2$ |
| 11. $\frac{7}{x} = \frac{3}{x} + 1$ | 23. $\frac{1}{4}x - 4 + \frac{3}{4}x = 16 + \frac{1}{4}x - 10$ |
| 12. $\frac{3}{x} = \frac{8}{x} - 2$ | 24. $.2(.5y - 2) - 2(.6 - .04y) =$
$-.02y$ |
| | 25. $3(2 - x) - 2(x + 3) = 6 - 2x$ |
| | 26. $x - (2 + 4x) = 13 - 5(x + 5)$ |
| | 27. $4x - (2x - 2) = 3x - (3x - 4)$ |

$$28. 3x - \frac{x}{5} = 14$$

$$29. \frac{2x}{3} - \frac{5}{6} = \frac{x}{4}$$

$$30. .25x + 17 = .2x - 15$$

$$31. .3(9 - 2s) - .5(2s - 9) = 0$$

$$32. \frac{2x}{3} - \frac{7x}{8} + \frac{5x}{18} + \frac{x}{24} = \frac{4}{9}$$

$$33. .6x - 1 = .25x + 1.5x - 3.3$$

$$40. 36 + 5x - 22 - (7x - 16) = 5x + 17 - (2x + 22)$$

$$41. 2(r - 5)(r - 4) = (r - 4)(r - 3) + (r - 2)(r - 5)$$

$$42. 5(x + 2)^2 - (x - 5)^2 = (2x - 3)(2x + 3) + 19$$

$$43. \frac{3t - 5}{4} - \frac{7t - 13}{6} = 3 - \frac{t + 3}{2}$$

$$44. \frac{r}{2}(2 - r) - \frac{r}{4}(3 - 2r) = \frac{r + 10}{6}$$

$$45. \frac{6r + 3}{15} - \frac{3r - 1}{5r - 25} = \frac{2r - 9}{5}$$

$$46. \frac{x^3 + 1}{x - 1} - \frac{x^3 - 1}{x + 1} = \frac{8}{x^2 - 1} + 2x$$

$$47. \frac{5x + 2}{3} - \left(x - \frac{3x - 1}{2}\right) = \frac{3x + 19}{2} - \left(\frac{x + 1}{6} + 5\right)$$

$$33. \frac{7x}{3x + 4} = 2$$

$$34. \frac{2}{x - 1} = \frac{7}{3x - 1}$$

$$35. .3x = 8 - .7x$$

$$36. 2.5x - 9 = 1.6x$$

$$37. 7.5x = 23.1 + 4.2$$

$$38. \frac{1.2}{x + 5} = \frac{.3}{x}$$

LITERAL EQUATIONS

Equations in which the unknown quantity has a literal coefficient and in which literal quantities appear as known quantities are solved like those in the preceding exercise.

Equations in which some of or all the known quantities are represented by letters are called **literal equations**.

Usually the letters in the last part of the alphabet represent the unknown quantities, while those in the first part of the alphabet represent the known quantities.

ILLUSTRATIVE EXAMPLES

1. Solve $ax + 2a = 3a$.

SOLUTION

(1) $ax + 2a = 3a$

(2) $ax = a$ (1) $- 2a$

(3) $x = 1$ (2) $\div a$

CHECK, $a + 2a = 3a$
 $3a = 3a$

2. Solve $3bx - 5b^3 = b^3 + bx$.

SOLUTION

(1) $3bx - 5b^3 = b^3 + bx$

(2) $2bx = 6b^3$ (1) $+ 5b^3 - bx$

(3) $x = 3b^2$ (2) $\div 2b$

CHECK, $3b(3b^2) - 5b^3 = b^3 + b(3b^2)$
 $9b^3 - 5b^3 = b^3 + 3b^3$
 $4b^3 = 4b^3$

3. Solve $\frac{x-b}{a} = \frac{x-a}{b}$.

SOLUTION

(1) $\frac{x-b}{a} = \frac{x-a}{b}$

(2) $b(x-b) = a(x-a)$ $ab \times (1)$

(3) $bx - b^2 = ax - a^2$ Parentheses removed from (2)

(4) $bx - ax = b^2 - a^2$ (3) $+ b^2 - ax$

(5) $(b-a)x = (b-a)(b+a)$ (4) factored

(6) $x = b+a$ (5) $\div (b-a)$

CHECK. Substitute $b+a$ for x in (1):

$$\frac{b+a-b}{a} = \frac{b+a-a}{b}$$
$$\frac{a}{a} = \frac{b}{b}$$
$$1 = 1$$

EXERCISES (Examples 1 to 15, Oral; 16 to 37, Written)

Regarding t, w, x, y, z as unknown numbers, solve, and check each result:

- | | | |
|----------------|-----------------|-----------------------|
| 1. $x - c = d$ | 4. $ay = d$ | 7. $2ax + c = b$ |
| 2. $t + a = b$ | 5. $bz = m + n$ | 8. $3bx = b^2 + bx$ |
| 3. $w - 6 = a$ | 6. $by - c = d$ | 9. $6b^2 - 2by = 4by$ |

10. $cx + d = e$ 12. $m = by + n$ 14. $by = b^2$
 11. $r = ax + b$ 13. $2mx = m + n$ 15. $cdx = cd$

Solve for x or y and check each result:

16. $a + 4x = 8 - 3x$ 26. $a^2x - b^4 = b^2x - a^4$
 17. $2(x - b) = 2a$ 27. $\frac{x}{b} - 1 = c$
 18. $(b + c)x = bc$ 28. $\frac{y}{b + c} - c = b$
 19. $3aby = 6ab$ 29. $\frac{bx}{c} = 2 - \frac{cx}{b}$
 20. $2x - c = d$ 30. $\frac{1}{a - y} = \frac{1}{y}$
 21. $x - a = b - x$ 31. $\frac{1}{y + a} = \frac{1}{4a}$
 22. $ax + 6a^2 = 2ax + 4a^2$
 23. $by - 9b^3 = 4by - 15b^3$
 24. $ax - a^2 = cx - c^2$
 25. $rx + s^2 = r^2 - sx$
 32. $\frac{a}{1 - x} = b$ 35. $\frac{a - bx}{ax - b} = \frac{a}{b}$
 33. $\frac{a}{1 + x} = a$ 36. $b(2x - 9c - 14b) = c(c - x)$
 34. $\frac{a + x}{a - x} = \frac{a + b}{a - b}$ 37. $1 - \frac{ab}{x} = \frac{7}{ab} - \frac{49}{abx}$

Formulas are really literal equations. The following formulas are used in later work in mathematics and science.

ILLUSTRATIVE EXAMPLES

1. Solve for h , $A = lh$.

SOLUTION

CHECK

(1) $A = lh$

(2) $\frac{A}{l} = h$

(2) $\div l$

$A \mid l \times \frac{A}{l}$

$A = A$

2. In the formula $c = 2\pi r$, solve for r .

SOLUTION

CHECK

(1) $c = 2\pi r$

(2) $\frac{c}{2\pi} = r$

(1) $\div 2\pi$

$c \mid 2\pi \times \frac{c}{2\pi}$

$c = c$

EXERCISES

1. In the formula $A = \frac{1}{2}bh$, solve for h . Find h to the nearest tenth when $A = 213.8$, $b = 25.3$.

2. $A = \frac{1}{2}h(b_1 + b_2)$. Solve: (1) for h ; (2) for b_1 ; (3) for b_2 . Find b_1 when $h = 7.2$, $b_2 = 6.8$, and $A = 61.2$.

3. $V = \frac{1}{3}bh$. Solve for h . Find h to the nearest tenth when $V = 73.3$ and $b = 27.15$.

4. $a = p(1 + rt)$. Find the value of t in terms of a , p , r . Solve: (1) for p ; (2) for r . Find t when $p = 320$, $r = 4\frac{1}{2}\%$, and $a = 370.40$.

5. $C = \frac{5}{9}(F - 32^\circ)$. Solve for F . Find F to the nearest degree if $C = 5.5^\circ$; if $C = -20^\circ$.

6. $s = \frac{1}{2}gt^2$. Solve for g . Find g to two decimal places when $s = 283.7$ and $t = 4.2$.

7. $C = \frac{E}{R + r}$. Solve for r . Find r when $E = 1.5$, $C = .5$, and $R = .5$.

8. $V_1 P_1 = V_2 P_2$. Solve for P_2 . Find P_2 when $V_1 = 500$, $V_2 = 300$, and $P_1 = 75$.

9. $\frac{1}{f} = \frac{1}{f_1} - \frac{1}{f_2}$. Solve for f_1 . Find f_1 to the nearest tenth when $f = 30.5$ and $f_2 = 15.3$.

10. $\frac{E}{e} = \frac{R + r}{r}$. Solve for e . Find e to the nearest tenth when $E = 1.8$, $R = .7$, and $r = 2.4$.

Application of linear equations in the solution of problems

The general directions for solving problems given in Chapter I, page 22, are reviewed below.

General directions

1. Represent one of the unknown numbers by some letter.

2. From the conditions of the problem find an expression for each of the other unknown numbers.

3. From the conditions of the problem find two expressions that are equal and write the equation.

4. Solve the equation.

5. Check the result by reading the problem again and substituting the results in the conditions of the problem to see if these conditions are met. Remember that it is not sufficient to check the equation, since you may have written a wrong equation.

Problems

1. John and his father went fishing and together caught 12 fish. Three times the number of fish that John caught exceeds 12 by as much as 5 times the number that his father caught exceeds 8. How many fish did each catch?

2. In a baseball game $\frac{1}{2}$ the number of runs scored by the winning team exceeds 5 by as much as $\frac{1}{3}$ the losing team's runs exceeds 1. If the total number of runs scored is 18, what is the final score?

3. The powder and the shell used in a twelve-inch gun weigh 1265 pounds. The powder weighs 15 pounds more than $\frac{1}{4}$ as much as the shell. Find the weight of each.

4. One ton of coal will make 8.7 tons of steam. If an ocean liner requires 1200 tons of coal a day for steam, how many tons of steam are required an hour?

5. A grocer paid \$14 for a molasses pump and 5 feet of tubing. He paid 15 times as much for the pump as he paid for each foot of tubing. How much did the pump cost? the tubing?

6. In lighting a hall a certain number of 20-candle-power electric lamps and twice as many 25-candle-power lamps were used. The total illumination amounted to 280 candle power. Find the number of lamps of each kind used.

7. At the waterworks 2 large pumps and 4 small ones delivered 4800 gallons of water a minute. Each of the large pumps delivered 4 times as much water as each small pump. How many gallons per minute did each pump deliver?

8. The crew of a United States battleship in target practice made 11 hits in less than a minute. If $\frac{3}{4}$ of the number of shots fired was 9 times the number of misses, how many shots were fired?

9. A playground is 101 feet longer than it is wide. If its width were decreased 25 feet, its length would be twice its width. Find the dimensions of the playground.

10. A resolution was passed by a plurality of 20 votes. If $\frac{1}{4}$ of those voting for it had voted against it, it would have been lost by 12 votes. How many votes were there?

11. A merchant bought 62 barrels of flour, part at \$10.50 a barrel, the rest at \$8.50 a barrel. If he paid \$541 for the flour, how many barrels of each grade did he buy?

12. A dealer paid \$185 for 25 boxes of candles. If he paid \$9 a box for part of them and \$6.50 a box for the rest, how many did he buy at each price?

13. A sporting goods merchant purchased an assortment of tennis rackets for \$480. By selling $\frac{1}{4}$ of them at \$6 each, $\frac{1}{8}$ of them at \$7 each, $\frac{1}{3}$ of them at \$5 each, and the rest, or $\frac{1}{4}$ of them, at \$8 each, he gained \$128. How many did he sell at each price?

14. In a recent balloon race, the sum of the distances covered by the *Lotus II* and the *United States* was 1025 miles. The distance covered by the former was 50 miles more than $\frac{1}{2}$ of that covered by the latter. How far did each travel?

15. A newspaper reporter saved $\frac{1}{5}$ of his weekly salary. This was \$2 more than was saved by an artist on the same paper, whose weekly salary was \$5 greater but who saved only $\frac{1}{7}$ of it. How much did the reporter earn per week? the artist?

16. During a year of 365 days one locality had 6 days less of clear weather than of cloudy weather and 4 days more of clear weather than of partly cloudy weather. Find the number of days of each kind of weather during the year.

17. Find a fraction whose value is $\frac{4}{7}$ and whose denominator is 15 greater than its numerator.

18. Find a fraction whose value is $\frac{2}{3}$ and whose numerator is 3 greater than half its denominator.

19. The numerator of a certain fraction is 8 less than the denominator. If each term of the fraction is decreased by 5, the resulting fraction equals $\frac{1}{3}$. What is the fraction?

20. An acre of wheat yielded 2000 pounds more of straw than of grain. The weight of the grain was .3 of the total weight of grain and straw. How many 60-pound bushels of wheat were produced?

21. A grocer bought a box of soap containing 72 cakes for \$4.50. Some of the soap he sold at 3 cakes for \$.25, and the rest at \$.10 a cake. This gave a profit of \$1.90. How many cakes did he sell at each price?

22. It cost 3.6¢ less to travel 100 miles in one automobile than it did to travel 80 miles in another. The average cost per mile of the former was 1.9¢ less than the average cost per mile of the latter. Find the cost per mile of each.

23. John can do a piece of work in 8 days. If Paul can do it in 10 days, in how many days can both together do it?

SOLUTION

Let t = the required number of days (time)

Then $\frac{1}{t}$ = the part of the work both can do in 1 day

$\frac{1}{8}$ = the part of the work John can do in 1 day

$\frac{1}{10}$ = the part of the work Paul can do in 1 day

$$\therefore \frac{1}{t} = \frac{1}{8} + \frac{1}{10}$$

$t = 4\frac{4}{7}$, the required number of days

24. Samuel can do a piece of work in 10 days, Henry in 12 days, and George in 8 days. In how many days can all together do it?

25. It takes a man 6 days to make a Panama hat, and a boy 7 days. How long would it take them if they could work together?

26. The average amount of coal blasted out by a keg of powder can be mined by one man in 2 days and by another in 3 days. How long would it take them to mine the coal if the two men worked together?

✓ 27. Arthur and Louis can dig a ditch in 10 days, Louis and John in 6 days, and Arthur and John in $7\frac{1}{2}$ days. In what time can all do the work together? each alone?

SUGGESTION. Since Arthur and Louis can dig $\frac{1}{10}$ of the ditch in 1 day, Louis and John $\frac{1}{6}$ of it in 1 day, and Arthur and John $\frac{2}{15}$ of it in 1 day, $\frac{1}{10} + \frac{1}{6} + \frac{2}{15}$ is twice the part all can dig in 1 day.

28. Albert and Benjamin can load a car in $1\frac{3}{4}$ hours, Benjamin and Charles in $2\frac{1}{2}$ hours, and Albert and Charles in $2\frac{1}{3}$ hours. How long will it take each alone to load it?

29. Of the population of Mexico at one time the per cent of whites was $\frac{1}{2}$ that of Indians. Mixed races formed 5% more than the per cent of Indians. Find the per cent of each.

30. The units' digit of a two-digit number exceeds the tens' digit by 5. If the number increased by 63 is divided by the sum of its digits, the quotient is 10. Find the number.

SOLUTION

Let t = the digit in tens' place

Then $t + 5$ = the digit in units' place

and $10t + (t + 5)$ = the number

$$\therefore \frac{10t + (t + 5) + 63}{2t + 5} = 10$$

$$t = 2$$

$$t + 5 = 7$$

Therefore the number is 27.

31. The tens' digit of a two-digit number is 3 times the units' digit. If the number diminished by 33 is divided by the difference of the digits, the quotient is 10. Find the number.

32. The tens' digit of a two-digit number is $\frac{1}{2}$ the units' digit. If the number increased by 27 is divided by the sum of its digits, the quotient is $6\frac{1}{4}$. Find the number.

33. Mr. Wright invested \$800, part at 6%, the rest at 5%. The total annual interest was \$45. Find how much money he invested at each rate.

SUGGESTION. Let p = the number of dollars invested at 6%

Then $800 - p$ = the number of dollars invested at 5%

$$\therefore .06p + .05(800 - p) = 45$$

34. Mr. Worth's annual income from his investments is \$8920; 25% of his property is invested at 5%, 40% at 4%, $16\frac{2}{3}\%$ at 3%, and the rest at 2%. How much has he invested?

35. Mr. Bailey lent some money at 4% interest, but received \$48 less interest on it annually than was received by Mr. Day, who had lent $\frac{4}{5}$ as much at 6%. How much did each man lend?

36. A New York Central train went 905.4 miles in a certain length of time. Another train with a speed 3 miles greater per hour covered 54 miles more in the same length of time. What was the speed of each train?

37. An express train on the Union Pacific Railroad whose rate is 40 miles an hour starts 1 hour 4 minutes after a freight train and overtakes it in 1 hour 36 minutes. How many miles does the freight train run per hour?

38. A western train traveled a certain distance in 5 minutes. If its rate per minute had been $\frac{1}{5}$ of a mile more, it would have covered the distance in $4\frac{1}{2}$ minutes. Find its rate per minute.

39. A well was deepened to 4 feet more than twice its original depth. If it was finally 22 feet deeper than it was at first, what was its depth after it was deepened?

40. A loin of beef was sold for 11¢ more than twice its cost. If the gain was $101\frac{3}{8}\%$, what was the cost?

41. A house cost a dollars and rented for n dollars a month. What per cent per annum was the income from the investment?

42. Mr. Evans has $\frac{2}{3}$ of his property invested at 4% , $\frac{1}{3}$ at 5% , and the remainder at $4\frac{1}{2}\%$. For how much is his property valued, if his annual income is \$562.80?

43. Two men starting from Albany travel east and west, respectively, one traveling 3 times as fast as the other. In 6 hours they are 360 miles apart. Find the rate of each.

44. A man invests $\frac{4}{5}$ of his savings at $4\frac{1}{2}\%$ and the rest at 4% . His annual income from these investments is \$176. Find his savings.

45. A cubic inch of aluminum weighs .092 pound and a cubic inch of copper weighs .31 pound. Find the percentage of composition by volume of a mixture of the two metals weighing .276 pound to the cubic inch.

46. When a certain number of radio cabinets are sold at \$12 each, the same amount is realized as when all but 9 of the cabinets are sold at \$15 each. How many cabinets are there?

47. An alloy of tin and lead, weighing 40 pounds, loses 4 pounds in weight when immersed in water. Find the amount of tin and of lead in the alloy if 10 pounds of tin lose 1.375 pounds and 5 pounds of lead lose .375 pounds when immersed in water.

48. A cistern can be filled in 4 hours by two pipes running together and in $6\frac{1}{4}$ hours by one of the pipes alone. In how many hours can the other pipe fill this cistern?

CHAPTER XI.

SIMULTANEOUS LINEAR EQUATIONS

TWO UNKNOWN NUMBERS — GRAPHIC SOLUTION

In Chapter III we saw how a graph may be used to represent statistical data and to compare related quantities. In Chapter IV we made graphs of formulas. In addition to making graphs of equations, as represented by these formulas, we shall now see that we can represent other equations graphically and even solve them graphically.

Drawing graphs of equations

Let x and y be two algebraic quantities so related that $y = 2x - 3$. It is evident that we may give x a series of values, and obtain a corresponding series of values of y ; and that the number of such pairs is unlimited.

For example, in the equation $y = 2x - 3$, if we let $x = 5$, the equation becomes $y = 2(5) - 3$, or $y = 10 - 3$, or $y = 7$.

The table at the right gives a number of values of y for different values of x in this equation.

In the figure on page 208, XX' is called the x -axis, and YY' the y -axis. The point of intersection of these axes, O , is called the **origin**. The values of x are represented by lines laid off on the x -axis or parallel to it, and the values of y by lines

$$y = 2x - 3$$

x	y
5	7
4	5
3	3
2	1
1	-1
0	-3
-1	-5
-2	-7

laid off on the y -axis or parallel to it. In these graphs the x -axis and the y -axis are perpendicular to each other.

The position of P indicates the point at which $x = 3$ and $y = 3$; the position of Q shows the point at which $x = 4$ and $y = 5$; the position of R shows the point at which $x = 5$, $y = 7$; etc.

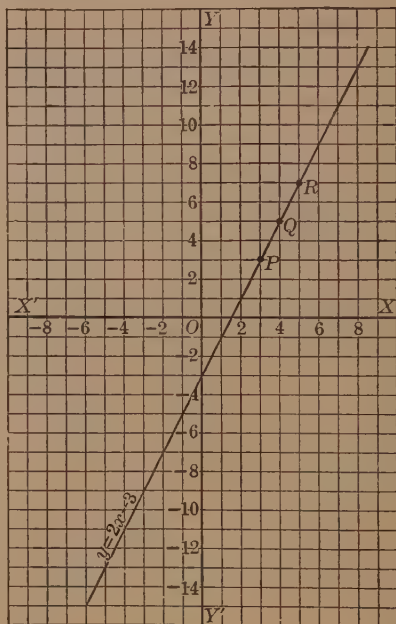
Plotting points

NOTE. The use of paper ruled in small squares, called *squared paper*, is convenient in plotting graphs.

Every point on the graph of the equation gives a pair of corresponding values of x and y .

Conversely, to locate any point with reference to two axes for the purpose of representing a pair of corresponding values of x and y , the value of x may be laid off on the x -axis as an x -distance, or *abscissa*, and that of y on the y -axis as a y -distance, or *ordinate*. If from each of the points on the axes obtained by these measurements, a line parallel to the other axis is drawn, the intersection of these two lines locates the point.

Thus, in the graph above, to represent the corresponding values $x = 3$, $y = 3$, a point P is located by measuring 3 units from O to the right on the x -axis and 3 units above O on the y -axis, and from these points drawing lines parallel to OY and to OX and extending these lines until they intersect.



The x -distance and the y -distance of a point referred to two perpendicular axes are called the **rectangular coördinates**, or simply the **coördinates**, of the point.

The x -distance is always written first.

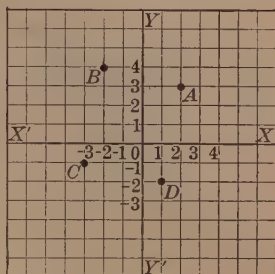
Thus, in the graph on page 208, the coördinates of P are $(3, 3)$ and of Q are $(4, 5)$.

By universal custom *positive* values of x are laid off from O as a zero-point, or **origin**, toward the *right*, and *negative* values toward the *left*.

Also *positive* values of y are laid off *upward* and *negative* values *downward*.

The point A in the figure above may be designated as "the point $(2, 3)$."

Similarly, B is the point $(-2, 4)$, C is the point $(-3, -1)$, and D is the point $(1, -2)$.



EXERCISES

Draw two axes at right angles to each other and locate the following points:

- | | |
|------------------|-------------------|
| 1. $A = (3, 2)$ | 7. $G = (-2, 5)$ |
| 2. $B = (3, -2)$ | 8. $H = (-3, -4)$ |
| 3. $C = (4, 3)$ | 9. $L = (0, 4)$ |
| 4. $D = (4, -3)$ | 10. $M = (0, -5)$ |
| 5. $E = (5, 5)$ | 11. $N = (3, 0)$ |
| 6. $F = (-5, 5)$ | 12. $P = (-6, 0)$ |

13. Where do all points having the x -distance 0 lie? the y -distance 0?

14. What are the coördinates of the origin?

15. Construct the graph of the equation $2y - x = 2$.

SOLUTION

$$(1) 2y - x = 2$$

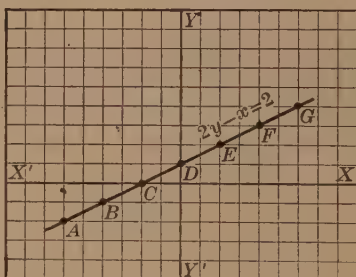
$$(2) 2y - 2 = x$$

Values are now given to y and corresponding values are computed for x by means of this equation. The numbers substituted for y need not be large. Convenient numbers to be substituted for y in this instance are the even integers from -2 to $+4$.

A record of the work should be kept as follows:

$$x = 2y - 2$$

y	x	POINT
-2	-6	A
-1	-4	B
0	-2	C
1	0	D
2	2	E
3	4	F
4	6	G



A line drawn through A, B, C, D , etc. is the graph of $x = 2y - 2$.

Construct the graph of each of the following:

16. $y = 3x - 7$

21. $x - 2y = 2$

17. $y = 2x + 1$

22. $3x = 2y$

18. $y = 2x - 1$

23. $2x + y = 1$

19. $3x - y = 4$

24. $2x + 3y = 6$

20. $4x - y = 10$

25. $3x + 2y = 12$

PRINCIPLE. The graph of an equation involving two unknown numbers of the first degree is a straight line.

For this reason, as has been stated on page 63, such equations are called **linear equations**.

Since a straight line is determined by two points, to plot the graph of a linear equation, plot two points and draw a

straight line through them. The accuracy of the work may be checked by plotting a third point. This should be on the same straight line as the first two points plotted.

It is often convenient to plot the points where the graph intersects the axes. To find where it intersects the x -axis, let $y = 0$; to find where it intersects the y -axis, let $x = 0$.

Thus, in the equation on page 210, $x = 2y - 2$, when $y = 0$, $x = -2$, locating C ; when $x = 0$, $y = 1$, locating D .

Draw a straight line through C and D .

If the equation has no *constant* term (a term like -2 above, which does not involve an unknown number), $x = 0$ when $y = 0$, and this method gives only one point. In any case it is desirable, *for the sake of accuracy*, to plot points some distance apart, as A and G .

EXERCISES

Construct the graph of each of the following:

- | | |
|-------------------|---------------------------------------|
| 1. $2x - 3y = 6$ | 6. $-2x + y = -3$ |
| 2. $2x + 3y = 0$ | 7. $-3x + 4y = 8$ |
| 3. $3x + 4y = 12$ | 8. $5x + 3y = 7\frac{1}{2}$ |
| 4. $5x - 2y = 10$ | 9. $x - \frac{1}{2}y = 3$ |
| 5. $8x - 3y = -6$ | 10. $\frac{1}{2}x + \frac{1}{3}y = 2$ |

Various types of systems of linear equations

In the equation $x + y = 12$,
 x and y may have an unlimited number of pairs of values,
 as $x = 1$ and $y = 11$;
 or $x = 2$ and $y = 10$; etc.

For since $y = 12 - x$, if any value is assigned to x , a corresponding value of y may be obtained.

An equation that is satisfied not by just one set of values but by an unlimited number of sets of values of its unknown numbers is called an **indeterminate equation** because it does not determine any one set of values.

PRINCIPLE. Any single equation involving two or more unknown numbers is indeterminate.

$$\begin{array}{l} \text{The equations} \quad 2x + 2y = 10 \\ \quad \quad \quad 3x + 3y = 15 \end{array} \}$$

express but one relation between x and y ; namely, that their sum is 5. In fact, the equations are *equivalent* to

$$x + y = 5$$

and to each other. Such equations are often called **dependent equations**, for either may be *derived* from the other.

$$\begin{array}{l} \text{The equations} \quad x + y = 5 \\ \quad \quad \quad x - y = 1 \end{array} \}$$

express two distinct relations between x and y ; namely, that their sum is 5 and their difference is 1. The equations cannot be reduced to the same equation; that is, they are not *equivalent*.

Equations that express different relations between the unknown numbers involved and therefore cannot be reduced to the same equation are called **independent equations**.

Each of the equations

$$\begin{array}{l} x + y = 5 \\ x - y = 1 \end{array} \}$$

is satisfied separately by an unlimited number of sets of values of x and y ; but these letters have only one set of values in both equations; namely,

$$x = 3 \text{ and } y = 2.$$

Two or more equations are consistent if they are satisfied by the same set or sets of values of the unknown numbers. Such equations are called **simultaneous equations**, or **systems of equations**.

$$\begin{array}{l} \text{The equations} \quad x + y = 5 \\ \quad \quad \quad x + y = 7 \end{array} \}$$

have no set of values of x and y in common.

Such equations are called **inconsistent equations**.

Graphic solution of simultaneous linear equations

1. Let it be required to solve graphically the equations

$$\begin{cases} y = 2 + x & (1) \\ y = 6 - x & (2) \end{cases}$$

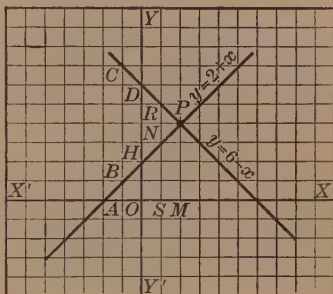
Construct the graph of each equation.

When $x = -1$, the value of y in (1) is represented by AB , and in (2) by AC .

Therefore when $x = -1$, the equations are not satisfied by the same values of y .

Compare the values of y when $x = 0$; when $x = 1$; 2.

For what value of x are the values of y in the two equations equal, or coincident?



What values of x and y will satisfy *both* equations? The required values of x and y , then, are represented graphically by the coördinates of P , the *intersection of the graphs*.

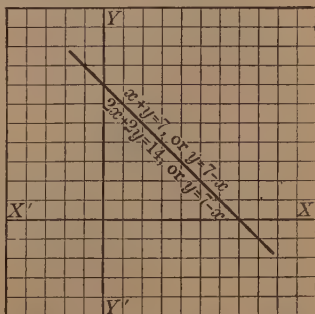
2. Let the given equations be
$$\begin{cases} x + y = 7 \\ 2x + 2y = 14 \end{cases}$$

What happens if we try to eliminate either x or y ?

Since $y = 7 - x$ in both equations, what will be the relative positions of any two points plotted for the same value of x ? the relative positions of the two graphs?

The algebraic analysis shows that the equations are **indeterminate**.

The graphic analysis shows also that the equations are *indeterminate*, for *their graphs coincide*.



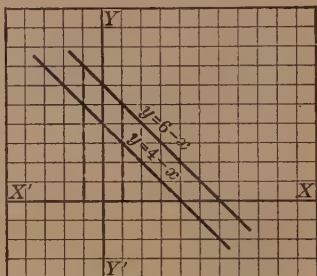
3. Let the given equations be $\begin{cases} y = 6 - x & (1) \\ y = 4 - x & (2) \end{cases}$

When $x = -1$, how much greater is the value of y in (1) than in (2), as shown both by the equations and by their graphs?

Compare the y 's for other values of x .

For every value of x the values of y in the two equations differ by 2, and the graphs are 2 units apart, vertically.

In algebraic language, the equations do not form a system. These are called **inconsistent equations**.



In graphic language, their graphs *cannot intersect*, as they are parallel straight lines.

PRINCIPLES. 1. A single linear equation involving two unknown numbers is *indeterminate*.

2. Two linear equations involving two unknown numbers are *determinate*, provided the equations are independent and form a system. They are satisfied by one, and only one, pair of common values.

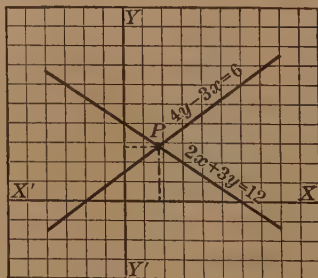
3. The pair of common values is represented graphically by the *coordinates of the intersection of their graphs*.

4. Solve graphically the equations $\begin{cases} 4y - 3x = 6 \\ 2x + 3y = 12 \end{cases}$

SOLUTION

When the graphs of both equations are plotted, it is found that they intersect at a point P , whose coordinates are 1.8 and 2.8, approximately.

Hence, to the nearest tenth, $x = 1.8$ and $y = 2.8$.



NOTE. In solving systems of equations by the graphic method the same axes must be used for the graphs of both equations. Why?

EXERCISES

Construct the graphs of each of the following systems of equations. Solve, if possible. If there is no solution, tell why. (Preserve the graphs of examples 1 to 10 for comparison with the corresponding examples on page 222.)

$$1. \begin{cases} x - y = 1 \\ x + y = 9 \end{cases}$$

$$2. \begin{cases} 2x - y = 5 \\ 4x + y = 16 \end{cases}$$

$$3. \begin{cases} 3x = y + 9 \\ 3y = 6x - 15 \end{cases}$$

$$4. \begin{cases} y = 4x \\ x - y = 3 \end{cases}$$

$$5. \begin{cases} x + y = 4 \\ y = 2 - x \end{cases}$$

$$6. \begin{cases} 2x - 5y = 5 \\ 10y = 2x + 1 \end{cases}$$

$$7. \begin{cases} 2x + 3y = 8 \\ 3x + 2y = 8 \end{cases}$$

$$8. \begin{cases} 4y + 3x = 5 \\ 4x - 3y = 3 \end{cases}$$

$$9. \begin{cases} x + 3y = -6 \\ 2x - 4y = -12 \end{cases}$$

$$10. \begin{cases} 3x + 4y = 10 \\ 6x + 8y = 20 \end{cases}$$

$$11. \begin{cases} x + y = 3 \\ x + 2y = 4 \end{cases}$$

$$12. \begin{cases} x = 4 + y \\ y = 3 + x \end{cases}$$

$$13. \begin{cases} x = \frac{1}{2}(y + 4) \\ y = 2(x - 2) \end{cases}$$

$$14. \begin{cases} x + y = -3 \\ x - 2y = -12 \end{cases}$$

$$15. \begin{cases} x = 2(y + 1) \\ 21 = 2(2x + y) \end{cases}$$

$$16. \begin{cases} x + y = 8 \\ 2x - 6y = -9 \end{cases}$$

$$17. \begin{cases} 3y = 2x - 7 \\ 2x = 6 + 3y \end{cases}$$

$$18. \begin{cases} 3(x - 4) = 2y \\ 6(y + 6) = 9x \end{cases}$$

$$19. \begin{cases} 10x + y = 14 \\ 8x - 5y = -2 \end{cases}$$

$$20. \begin{cases} 4x - 10y = 0 \\ 2x + y = 12 \end{cases}$$

$$21. \begin{cases} x - 2y = 2 \\ 2y - 6x = 3 \end{cases}$$

$$22. \begin{cases} \frac{3}{2}x + \frac{5}{2}y = 3\frac{1}{2} \\ 10x - 2y = 14 \end{cases}$$

$$23. \begin{cases} 8x - 3y = -.5 \\ 2x + 3y = 3.5 \end{cases}$$

$$24. \begin{cases} .2x + .5y = 1.1 \\ .3x - .2y = -1.2 \end{cases}$$

TWO UNKNOWN NUMBERS — ALGEBRAIC SOLUTION

You are familiar with the methods of solving linear equations of one unknown number. The general method of solving a system of two independent equations in two unknown numbers, as

$$\begin{cases} x + y = 5 \\ x - y = 3 \end{cases}$$

is to combine the equations, using the axioms already learned, in such a way as to obtain one equation involving x alone and another involving y alone. Each of these new equations may be solved separately by previous methods.

The process of deriving from a system of equations another system with fewer unknown numbers is called **elimination**.

Elimination by addition and subtraction

EXERCISES

1. Solve the equations :

$$\begin{aligned} (1) \quad & \begin{cases} 2x + 3y = 7 \\ 3x + 4y = 10 \end{cases} \\ (2) \quad & \end{aligned}$$

SOLUTION

$$\begin{aligned} (3) \quad & 8x + 12y = 28 && 4 \times (1) \\ (4) \quad & 9x + 12y = 30 && 3 \times (2) \\ (5) \quad & x = 2 && (4) - (3) \\ (6) \quad & 4 + 3y = 7 && (5) \text{ substituted in } (1) \\ (7) \quad & 3y = 3 && (6) - 4 \\ (8) \quad & y = 1 && (7) \div 3 \end{aligned}$$

CHECK. Substitute 2 for x and 1 for y in each of the given equations. It is not sufficient to substitute in one equation alone.

$$\begin{array}{r|l} 4 + 3 & 7 \\ 6 + 4 & 10 \end{array} \qquad \begin{array}{l} 7 = 7 \\ 10 = 10 \end{array}$$

RULE. (1) *If necessary, multiply or divide the equations by such numbers as will make the coefficients of the quantity to be eliminated numerically equal.*

(2) *Eliminate by addition if the resulting coefficients have unlike signs or by subtraction if they have like signs.*

Solve by addition or subtraction, and check results:

$$2. \begin{cases} 7x - 5y = 52 \\ 2x + 5y = 47 \end{cases}$$

$$3. \begin{cases} 3x + 2y = 23 \\ x + y = 8 \end{cases}$$

$$4. \begin{cases} 3x - 4y = 7 \\ x + 10y = 25 \end{cases}$$

$$5. \begin{cases} 2x - 10y = 15 \\ 2x - 4y = 18 \end{cases}$$

$$6. \begin{cases} 2a + 3b = 17 \\ 3a + 2b = 18 \end{cases}$$

$$7. \begin{cases} 3d + 4y = 25 \\ 4d + 3y = 31 \end{cases}$$

$$8. \begin{cases} 5p + 6q = 32 \\ 7p - 3q = 22 \end{cases}$$

$$9. \begin{cases} 3a + 6z = 39 \\ 9a - 4z = 51 \end{cases}$$

$$10. \begin{cases} 6x - 5y = 33 \\ 4x + 4y = 44 \end{cases}$$

$$11. \begin{cases} 3m + 11n = 67 \\ 5m - 3n = 5 \end{cases}$$

Elimination by substitution

EXERCISES

1. Solve by substitution the equations:

$$(1) \begin{cases} 3x + 2y = 27 \\ x - y = 4 \end{cases}$$

$$(2) \begin{cases} x - y = 4 \end{cases}$$

SOLUTION

$$(3) \quad x = 4 + y$$

$$(4) \quad 3(4 + y) + 2y = 27$$

$$(5) \quad 12 + 5y = 27$$

$$(6) \quad 5y = 15$$

$$(7) \quad y = 3$$

$$(8) \quad x = 7$$

$$(2) \div y$$

$$(3) \text{ substituted in (1)}$$

$$\text{Terms in (4) united}$$

$$(5) - 12$$

$$(6) \div 5$$

$$(7) \text{ substituted in (3)}$$

$$\text{CHECK.} \quad \begin{array}{r|l} 21 + 6 & 27 \\ 7 - 3 & 4 \end{array}$$

$$27 = 27$$

$$4 = 4$$

RULE. Solve one of the equations for one of the unknown quantities in terms of the other unknown quantity. Substitute this value for the unknown quantity in the other equation. Solve the resulting equation.

Solve by substitution, and check results:

$$2. \begin{cases} x - y = 4 \\ 4y - x = 14 \end{cases}$$

$$3. \begin{cases} x + y = 10 \\ 6x - 7y = 34 \end{cases}$$

$$4. \begin{cases} 6y - 10x = 14 \\ y - x = 3 \end{cases}$$

$$5. \begin{cases} y + 1 = 3x \\ 5x + 9 = 3y \end{cases}$$

Solve by substitution, and check results :

$$6. \begin{cases} 3x - 4y = 26 \\ x - 8y = 22 \end{cases}$$

$$7. \begin{cases} 4y = 10 - x \\ y - x = 5 \end{cases}$$

$$8. \begin{cases} 7z - 3x = 18 \\ 2z - 5x = 1 \end{cases}$$

$$9. \begin{cases} 17 = 3x + z \\ 7 = 3z - 2x \end{cases}$$

$$10. \begin{cases} 3 - 15y = -x \\ 3 + 15y = 4x \end{cases}$$

$$11. \begin{cases} 1 - x = 3y \\ 3(1 - x) = 40 - y \end{cases}$$

Choice of methods

Two methods of elimination have been given. Select the method better adapted to the example you are solving.

EXERCISES

Solve by either method, checking all results :

$$1. \begin{cases} x + z = 1.3 \\ x - z = .5 \end{cases}$$

$$2. \begin{cases} 3x + y = 10 \\ x + 3y = 6 \end{cases}$$

$$3. \begin{cases} 4x + 5y = -2 \\ 5x + 4y = 2 \end{cases}$$

$$4. \begin{cases} .5x - y = 2.8 \\ .3x + 5y = 2.8 \end{cases}$$

$$5. \begin{cases} x + 3 = y - 3 \\ 2(x + 3) = 6 - y \end{cases}$$

$$6. \begin{cases} 5x - y = 12 \\ x + 3y = 12 \end{cases}$$

$$7. \begin{cases} 4(2 - x) = 3y \\ 2(2 - x) = 2(y - 2) \end{cases}$$

$$8. \begin{cases} (x + 1) + (y - 2) = 7 \\ (x + 1) - (y - 2) = 5 \end{cases}$$

Eliminate before or after clearing of fractions, as may be more advantageous :

$$9. \begin{cases} x + \frac{y}{3} = 11 \\ \frac{x}{3} + 3y = 21 \end{cases}$$

$$11. \begin{cases} \frac{x}{2} - \frac{2y}{3} = -2 \\ \frac{5x}{2} + \frac{y}{3} = 12 \end{cases}$$

$$10. \begin{cases} \frac{3x}{4} + \frac{4y}{5} = 21 \\ \frac{2x}{3} + \frac{3y}{5} = 17 \end{cases}$$

$$12. \begin{cases} \frac{x+y}{2} - \frac{x-y}{3} = 8 \\ \frac{x+y}{3} + \frac{x-y}{4} = 11 \end{cases}$$

$$13. \text{ Solve: } \begin{cases} x + \frac{1}{2}(3x - y - 1) = \frac{1}{4} + \frac{3}{4}(y - 1) \\ \frac{1}{5}(4x + 3y) = \frac{1}{10}(7y + 24) \end{cases}$$

$$14. \begin{cases} \frac{x-y}{2} - \frac{x-3y}{5} = y-3 \\ \frac{3}{4}(x-y) + \frac{5}{6}(x+y) = 18 \end{cases}$$

$$15. \begin{cases} \frac{x}{2} - 12 = \frac{y+32}{4} \\ \frac{y}{8} + \frac{3x-2y}{5} = 25 \end{cases}$$

$$16. \begin{cases} \frac{x+2}{3} - \frac{y-2}{2} = \frac{3(y-4x)}{4} \\ \frac{x-3}{2} - 5 = \frac{y+5}{3} - 3(y-x) \end{cases}$$

$$17. \begin{cases} \frac{x}{8} + 4y = 15 \\ \frac{x}{6} + \frac{2y}{3} = 6 \end{cases}$$

$$18. \begin{cases} \frac{x}{3} = \frac{y}{2} \\ \frac{x}{3} - \frac{y}{3} = 1 \end{cases}$$

$$19. \begin{cases} \frac{x-1}{4} + y = 3 \\ \frac{x-1}{4} + 4y = 9 \end{cases}$$

Equations of the form $\frac{a}{x} + \frac{b}{y} = c$, though not linear equations, may be solved as linear equations by first regarding $\frac{1}{x}$ and $\frac{1}{y}$ as the unknown numbers.

20. Solve the equations:

$$(1) \begin{cases} \frac{4}{x} - \frac{3}{y} = \frac{14}{5} \end{cases}$$

$$(2) \begin{cases} \frac{4}{x} + \frac{10}{y} = \frac{50}{3} \end{cases}$$

SOLUTION

$$(3) \quad \frac{13}{y} = \frac{208}{15}$$

$$(4) \quad \frac{1}{y} = \frac{16}{15}$$

$$(5) \quad \frac{4}{x} - \frac{48}{15} = \frac{14}{5}$$

$$(6) \quad \frac{4}{x} = \frac{30}{5} = 6$$

$$(7) \quad 4 = 6x$$

$$(8) \quad \frac{2}{3} = x$$

$$(9) \quad 15 = 16y$$

$$(10) \quad \frac{15}{16} = y$$

$$(2) - (1)$$

$$(3) \div 13$$

$$(4) \text{ substituted in (1) }$$

$$(5) + \frac{48}{15}$$

$$x \times (6)$$

$$(7) \div 6$$

$$15y \times (4)$$

$$(9) \div 16$$

CHECK

(1)

$$\begin{array}{r|l} \frac{4}{x} - \frac{3}{y} & \frac{14}{5} \\ \frac{4}{x} - \frac{15}{15} & \frac{5}{5} \\ 6 - \frac{16}{5} & \frac{14}{5} \\ \frac{14}{5} & = \frac{14}{5} \end{array}$$

(2)

$$\begin{array}{r|l} \frac{4}{x} + \frac{10}{y} & \frac{50}{3} \\ \frac{4}{x} + \frac{15}{15} & \frac{5}{3} \\ 6 + \frac{32}{3} & \frac{50}{3} \\ \frac{50}{3} & = \frac{50}{3} \end{array}$$

Solve, and check results :

$$21. \begin{cases} \frac{5}{x} - \frac{3}{y} = -2 \\ \frac{25}{x} + \frac{1}{y} = 6 \end{cases}$$

$$28. \begin{cases} \frac{2}{x} - \frac{3}{y} = 5 \\ \frac{5}{x} - \frac{2}{y} = 7 \end{cases}$$

$$22. \begin{cases} \frac{7}{x} - \frac{8}{y} = -1 \\ \frac{1}{x} + \frac{3}{y} = \frac{25}{28} \end{cases}$$

$$29. \begin{cases} \frac{4}{x} + \frac{3}{y} = \frac{9}{8} \\ \frac{3}{x} + \frac{4}{y} = \frac{11}{12} \end{cases}$$

$$23. \begin{cases} \frac{3}{2x} - \frac{1}{y} = -3 \\ \frac{5}{2x} + \frac{3}{y} = 23 \end{cases}$$

$$30. \begin{cases} \frac{5}{3x} + \frac{4}{y} = 3 \\ \frac{7}{y} - \frac{1}{6x} = 2\frac{1}{6} \end{cases}$$

$$24. \begin{cases} \frac{6}{x} + \frac{4}{y} = 4 \\ \frac{45}{x} - \frac{10}{y} = 15 \end{cases}$$

$$31. \begin{cases} x - 2y = -4xy \\ \frac{12}{y} - \frac{9}{x} = 27 \end{cases}$$

$$25. \begin{cases} \frac{2}{3x} - \frac{1}{2y} = \frac{1}{6} \\ \frac{6}{x} - \frac{3}{y} = 2 \end{cases}$$

$$32. \begin{cases} \frac{2}{x} + \frac{9}{y} = \frac{16}{15} \\ \frac{1}{x} - \frac{1}{3y} = \frac{1}{20} \end{cases}$$

$$26. \begin{cases} \frac{x+y}{7} + y = \frac{3x}{2} + 3 \\ \frac{3x+y}{2} + \frac{17}{6} = \frac{5y}{3} \end{cases}$$

$$33. \begin{cases} \frac{2}{x} - \frac{1}{y} = \frac{2}{3} \\ 3 + \frac{3}{y} = \frac{5}{2} \end{cases}$$

$$27. \begin{cases} \frac{4x-7y}{3y} + \frac{1}{3} = 0 \\ 5x - 17 = \frac{4x}{3} - 3y \end{cases}$$

$$34. \begin{cases} \frac{x+y}{x-y} = 7 \\ \frac{x+y}{x-y+5} = 2 \end{cases}$$

Simultaneous literal equations

EXERCISES

1. Solve the equations:

(1) $\begin{cases} ax + by = m \\ cx + dy = n \end{cases}$

(2) $\begin{cases} ax + by = m \\ cx + dy = n \end{cases}$

SOLUTION

(3) $adx + bdy = dm$

$d \times (1)$

(4) $bcx + bdy = bn$

$b \times (2)$

(5) $(ad - bc)x = dm - bn$

(3) - (4)

(6) $x = \frac{dm - bn}{ad - bc}$

(5) $\div (ad - bc)$

(7) $acx + bcy = cm$

$c \times (1)$

(8) $acx + ady = an$

$a \times (2)$

(9) $(ad - bc)y = an - cm$

(8) - (7)

(10) $y = \frac{an - cm}{ad - bc}$

(9) $\div (ad - bc)$

CHECK. These equations may be checked in the usual way by substituting the values of x and y in the given equations. Sometimes a less involved check is to assign suitable values for the letters other than those representing unknown quantities. For example, let $a = 1$, $b = 2$, $c = 3$, $d = 4$, $m = 5$, and $n = 6$.

$$x = \frac{20 - 12}{4 - 6}, \text{ or } -4$$

$$y = \frac{6 - 15}{4 - 6}, \text{ or } \frac{9}{2}$$

Substitute these numerical values for x and y in (1) and (2):

(1) $-4 + 9 \mid 5 \quad 5 = 5$

(2) $-12 + 18 \mid 6 \quad 6 = 6$

Solve for x and y , and check results:

2. $\begin{cases} ax + by = m \\ bx - ay = c \end{cases}$

6. $\begin{cases} a(x - y) = 5 \\ bx - cy = n \end{cases}$

3. $\begin{cases} ax - by = m \\ cx - dy = r \end{cases}$

7. $\begin{cases} a(a - x) = b(y - b) \\ ax = by \end{cases}$

4. $\begin{cases} ax = by \\ x + y = ab \end{cases}$

8. $\begin{cases} x + y = b - a \\ bx - ay + 2ab = 0 \end{cases}$

5. $\begin{cases} m(x + y) = a \\ n(x - y) = 2a \end{cases}$

9. $\begin{cases} x - y = a - b \\ ax + by = a^2 - b^2 \end{cases}$

Solve for x and y :

$$10. \begin{cases} \frac{x}{a} + \frac{y}{b} - 2 = 0 \\ bx - ay = 0 \end{cases}$$

$$14. \begin{cases} \frac{x+1}{y+1} = \frac{a+b+1}{a-b+1} \\ x - y = 2b \end{cases}$$

$$11. \begin{cases} \frac{1}{x} + \frac{1}{y} = \frac{1}{a} \\ \frac{1}{x} - \frac{1}{y} = \frac{1}{b} \end{cases}$$

$$15. \begin{cases} \frac{1}{x-a} = \frac{1}{a-y} \\ \frac{x+y}{x-y} = a \end{cases}$$

$$12. \begin{cases} \frac{a}{x} - \frac{b}{y} = -1 \\ \frac{b}{x} - \frac{a}{y} = -1 \end{cases}$$

$$16. \begin{cases} \frac{x}{a} + \frac{y}{b} = c \\ \frac{x}{b} + \frac{y}{c} = d \end{cases}$$

$$13. \begin{cases} bx + ay = 2ab \\ ax + by = a^2 + b^2 \end{cases}$$

$$17. \begin{cases} mx - ny = m^2 + n^2 \\ x - y = 2n \end{cases}$$

$$18. \begin{cases} (a-b)x + (a+b)y = 2a^2 - 2b^2 \\ (a+b)x + (a-b)y = 2(a^2 + b^2) \end{cases}$$

To check algebraic solutions by the use of graphs

EXERCISES

Solve the following simultaneous equations algebraically, if possible. Check by comparing with the graphs constructed for the corresponding equations in the exercise on page 215.

$$1. \begin{cases} x - y = 1 \\ x + y = 9 \end{cases}$$

$$6. \begin{cases} 2x - 5y = 5 \\ 10y = 2x + 1 \end{cases}$$

$$2. \begin{cases} 2x - y = 5 \\ 4x + y = 16 \end{cases}$$

$$7. \begin{cases} 2x + 3y = 8 \\ 3x + 2y = 8 \end{cases}$$

$$3. \begin{cases} 3x = y + 9 \\ 3y = 6x - 15 \end{cases}$$

$$8. \begin{cases} 4y + 3x = 5 \\ 4x - 3y = 3 \end{cases}$$

$$4. \begin{cases} y = 4x \\ x - y = 3 \end{cases}$$

$$9. \begin{cases} x + 3y = -6 \\ 2x - 4y = -12 \end{cases}$$

$$5. \begin{cases} x + y = 4 \\ y = 2 - x \end{cases}$$

$$10. \begin{cases} 3x + 4y = 10 \\ 6x + 8y = 20 \end{cases}$$

Problems

1. The length of a lot is 20 yards more than its width, and its perimeter is 360 yards. Find its dimensions.

SUGGESTION.

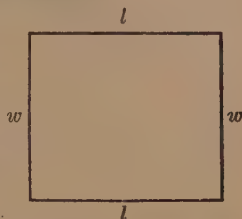
Let l = the number of yards in length of lot

w = the number of yards in width of lot

Then (1) $l - w = 20$

(2) $2l + 2w = 360$

Solve the equations and check the results by finding whether they satisfy the conditions given in the problem.



2. Some Panama hats bought in Colombia cost \$5 each, and others cost \$.50 each. If 18 hats of these two grades cost \$45, how many hats of each kind were bought?

3. A grocer sold 2 boxes of raspberries and 3 boxes of cherries to one customer for \$1.08, and 3 boxes of raspberries and 2 boxes of cherries to another customer for \$1.12. Find the price of each per box.

4. On the Fourth of July, 850 glasses of soda water were sold at a soda fountain, some at \$.10 each, the others at \$.15 each. The receipts were \$110. How many were sold at each price?

5. A fruit dealer bought 36 pineapples. He sold some at \$.15 each and the rest at \$.20 each, thereby receiving \$6.20. How many did he sell at each price?

6. The receipts from 300 tickets for a high-school entertainment were \$125. Adults were charged \$.50 each and children, \$.25 each. How many tickets of each kind were sold?

7. A dealer packed 1800 Christmas wreaths in 6 barrels and 6 cases. Later he packed 2250 similar wreaths in 9 barrels and 7 cases. Find the capacity of a barrel; of a case.

8. A natural bridge in Utah has a span of 60 feet more than its height. If its height were 200 feet less, it would be $\frac{1}{3}$ of its span. Find the height and the span of the bridge.

9. A chimney in Detroit is 50 feet lower than one in Chicago, and $\frac{1}{3}$ of the height of the latter is 25 feet more than $\frac{1}{4}$ of the height of the former. Find the height of each.

10. An errand boy went to the bank to deposit some bills for his employer. Some of them were 1-dollar bills and the rest were 2-dollar bills. The number of bills was 38 and their value was \$50. Find the number of each.

11. A grocer bought 1290 oranges of two sizes. Of one kind it took 120 oranges to fill a box and of the other kind, 150 oranges. If there were 10 boxes in all, find the number of boxes of each kind.

12. The cost of firing 20 shots from a battleship was \$4040. The shots from one gun cost \$400 each and every shot from a smaller gun cost \$70. How many shots of each kind were fired?

13. The champion National League baseball team one year won 62 games more than it lost. The team that came second played 154 games, winning 16 games less and losing 18 games more than the first team. How many games did each team win and how many did each lose?

14. A steam pipe was inclosed in a case. The diameter of the pipe was $\frac{2}{3}$ of the diameter of the case. The radius of the case was 2 inches less than the diameter of the pipe. What was the diameter of the pipe? of the case?

15. In a department store where 1000 boys and girls were employed, the average daily wage was \$2.50 for a boy and \$1.50 for a girl. If the amount paid to boys and girls together was \$2340 a day, how many boys were employed? how many girls?

16. It required 60 inches of tape to bind the four edges of a card on which a photograph was mounted. The length of the card was 6 inches greater than the width. How many inches long was the card? how many inches wide?

17. In one hour 1375 vehicles passed a merchant's door in New York. The motor trucks would have equaled the pleasure cars in number if there had been 50 more of the former and 25 less of the latter. How many of each passed the merchant's door?

18. The American and British tourists to Japan during a recent year numbered 2794. If there had been twice as many Americans and 3 times as many British, the number of both would have been 6682. How many tourists were there from each country?

19. The number of students attending an eastern university at one time was 3000 more than the number attending a western university, and $\frac{1}{2}$ the number of students in the eastern university plus $\frac{1}{3}$ the number in the western university equaled 6500. How many students were attending each university?

20. Under the present contract, it costs \$24.15 less a year per lamp to maintain electric lights in a certain city than it did under the previous contract, and the expense of 6 lamps under the previous contract was \$46.35 more than that of 7 lamps now. Find the present yearly expense per lamp.

21. Mr. Card invested \$4000, part at 5% and the rest at 4%. If the annual income from both investments was \$175, what was the amount of each investment?

22. On the same day two seats in the New York Stock Exchange were sold for \$192,500. If one of them had been sold for \$1500 less and the other for \$1000 more, the prices of the two would have been equal. Find the price of each.

23. A man had 10 fox skins, some of which were silver fox, worth \$600 a pelt, and some black fox, worth \$1500 a pelt. If he had owned 3 less of the former and 3 more of the latter, the total value would have been \$12,300. How many silver fox skins did he have? how many black fox skins?

24. The quantity of peanuts raised in the United States in a year is 135 million pounds more than the quantity of all nuts imported, and $\frac{1}{5}$ of the former equals $\frac{1}{2}$ of the latter. Find the number of pounds of nuts imported and of peanuts raised in the United States.

25. The receipts from a football game were \$700. Admission tickets to the grounds were sold for \$.50, and to the grand stand for \$.25 extra. If twice as many persons had purchased tickets for the grand stand, the receipts would have been \$800. How many tickets of each kind were sold?

26. A train of 25 cars loaded with iron ore was run out on a dock and the ore was emptied into pockets beneath the tracks. The ore filled 7 pockets and $\frac{1}{2}$ of another. To fill this last pocket required 16 tons less than 2 extra carloads. What was the capacity of a car? of a pocket?

27. If 100 pounds of soft coal in burning can evaporate 50 pounds more water than 6 gallons of oil, and if 60 pounds of coal can evaporate 10 pounds less water than 4 gallons of oil, how many pounds of water can 1 pound of coal evaporate? 1 gallon of oil?

28. If 1 is added to the numerator of a certain fraction, the value of the fraction becomes $\frac{3}{4}$; if 2 is added to the denominator, the value becomes $\frac{1}{2}$. What is the fraction?

SUGGESTION. Let n = the numerator and d = the denominator.

29. If the numerator of a certain fraction is decreased by 4, the value of the fraction becomes $\frac{1}{2}$; but if the numerator is increased by 2, the value of the fraction becomes $\frac{3}{4}$. What is the fraction?

30. A certain number expressed by two digits is equal to 7 times the sum of its digits. If 27 is subtracted from the number, the difference will be expressed by reversing the order of the digits. What is the number?

SUGGESTION. The sum of t tens and u units is $(10t + u)$ units; of u tens and t units, $(10u + t)$ units.

✓
31. A bridge contains 5600 tons of steel and iron together. If 17 times the weight of the steel equals 11 times the weight of the iron, what is the number of tons of each metal in the bridge?

32. Mr. Perkins deposited in the bank 40 bills valued at \$108. Some were 1-dollar bills and the rest were 5-dollar bills. How many bills of each kind did he deposit?

33. Two women shelled 165 pounds of hazelnuts. If one had shelled 10 pounds more and the other 5 pounds less, they would have shelled equal amounts. How many pounds did each shell?

34. Two apples together weighed 64 ounces. If the weight of the smaller apple had been 13 ounces less, it would have been $\frac{1}{2}$ of the weight of the larger. Find the weight of each.

35. Alvah earned \$6 more a week than Malcolm. If the former earned in 5 weeks as much as the latter did in 8 weeks, how much was the weekly wage of each?

36. The sum of 2 times one number and 5 times another is 20, and the sum of 3 times the former and 4 times the latter is 23. Find the numbers.

37. On its first trip, a steamer made 762 knots in 2 days. If the second day's run was 152 knots more than the first day's run, how many knots did the steamer make each day?

38. A large canal lock is $7\frac{1}{2}$ times as long as it is wide. If the width had been 5 feet less, it would have been $\frac{1}{8}$ as much as the length. Find the length and the width of the lock.

39. A man can row 12 miles downstream in 2 hours and 12 miles upstream in 6 hours. Find his rate of rowing in still water; the rate of the current.

SUGGESTION. If r = the rate in still water and c = the rate of the current, $2(r + c) = 12$ and $6(r - c) = 12$. Why?

✓
40. A bird flying with the wind goes 55 miles an hour, and flying against a wind twice as strong it goes 30 miles an hour. What is the rate of the wind in each case?

SIMULTANEOUS LINEAR EQUATIONS

41. A man can row m miles downstream in c hours, and m miles upstream in d hours. Find his rate of rowing in still water; the rate of the current.

42. The sum of the reciprocals of two numbers is $4\frac{1}{8}$, and one number is 32 times the other. Find the numbers.

43. The greater of two numbers divided by the less gives a quotient of 4 and a remainder of 2. If the greater exceeds the less by 50, what are the numbers?

44. The greater of two numbers divided by the less gives a quotient of 7 and a remainder of 4; the less divided by the greater gives $\frac{2}{15}$. Find the numbers.

45. If 1 is subtracted from the numerator of a certain fraction, its value becomes $\frac{5}{6}$; if 10 is added to its denominator, its value becomes $\frac{1}{2}$. What is the fraction?

46. A quantity of wheat could be thrashed by two machines in 6 days, but the larger machine worked alone for 8 days and was then replaced by the smaller, which finished the work in 3 days. How long would it have taken the larger machine to thrash all the wheat? the smaller machine?

47. If the bleachers at the ball grounds seated 2000 less, they would provide $\frac{1}{5}$ of the total seating capacity. If the total seating capacity were 1000 more, it would be 4 times that of the bleachers. How many do the bleachers seat?

48. A man observes the velocity of the report of a cannon with the wind to be 1175 feet a second. Another man observes the velocity of the same report against the wind to be 1085 feet a second. Find the rate of the wind.

49. The perimeter of a rectangular city lot was 420 feet. A builder divided the lot into two equal lots having the same width and half the length of the original lot. Find the dimensions of the smaller lots if their perimeters were 260 feet each.

THREE UNKNOWN NUMBERS

(Pages 229 to 231 may be postponed until after Chapter XV without interrupting the sequence.)

You have solved systems of *two* independent simultaneous linear equations involving *two* unknown numbers. In general:

PRINCIPLE. *Every system of independent linear equations involving the same number of unknown numbers as there are equations can be solved, and is satisfied by one and only one set of values of its unknown numbers.*

ILLUSTRATIVE EXAMPLE

Solve the equations:

$$\begin{aligned} (1) \quad & \begin{cases} x + 2y + 3z = 14 \\ 2x + y + 2z = 10 \\ 3x + 4y - 3z = 2 \end{cases} \\ (2) \quad & \\ (3) \quad & \end{aligned}$$

SOLUTION

(4)	$4x + 6y = 16$	(1) + (3)
(5)	$6x + 3y + 6z = 30$	$3 \times (2)$
(6)	$6x + 8y - 6z = 4$	$2 \times (3)$
(7)	$12x + 11y = 34$	(5) + (6)
(8)	$12x + 18y = 48$	$3 \times (4)$
(9)	$7y = 14$	(8) - (7)
(10)	$y = 2$	From (9)
(11)	$4x + 12 = 16$	(10) substituted in (4)
(12)	$x = 1$	From (11)
(13)	$1 + 4 + 3z = 14$	(10) and (12) substituted in (1)
(14)	$z = 3$	From (13)

CHECK. Substitute $x = 1$, $y = 2$, and $z = 3$ in the given equations.

(1) becomes	$1 + 4 + 9$	14	$14 = 14$
(2) becomes	$2 + 2 + 6$	10	$10 = 10$
(3) becomes	$3 + 8 - 9$	2	$2 = 2$

That is, the given equations are satisfied for $x = 1$, $y = 2$, and $z = 3$.

EXERCISES

Solve, and check all results:

$1. \quad \begin{cases} x + 3y - z = 10 \\ 2x + 5y + 4z = 57 \\ 3x - y + 2z = 15 \end{cases}$	$2. \quad \begin{cases} x + y + z = 53 \\ x + 2y + 3z = 105 \\ x + 3y + 4z = 134 \end{cases}$
---	---

$$3. \begin{cases} x - y + z = 30 \\ 3y - x - z = 12 \\ 7z - y + 2x = 141 \end{cases}$$

$$4. \begin{cases} 8x - 5y + 2z = 53 \\ x + y - z = 9 \\ 13x - 9y + 3z = 71 \end{cases}$$

$$(5) \begin{cases} x + 3y + 4z = 83 \\ x + y + z = 29 \\ 6x + 8y + 3z = 156 \end{cases}$$

$$6. \begin{cases} 2x + 3y + 4z = 29 \\ 3x + 2y + 5z = 32 \\ 4x + 3y + 2z = 25 \end{cases}$$

$$7. \begin{cases} \frac{1}{8}x - \frac{1}{5}y + \frac{1}{4}z = 3 \\ \frac{1}{5}x - \frac{1}{4}y + \frac{1}{5}z = 1 \\ \frac{1}{4}x - \frac{1}{3}y + \frac{1}{2}z = 5 \end{cases}$$

$$8. \begin{cases} x + y = 9 \\ y + z = 7 \\ z + x = 5 \end{cases}$$

$$9. \begin{cases} 2x - 3y + z = -2 \\ 4x - 4y - 3z = 2 \\ 6x + y - 4z = 6 \end{cases}$$

$$10. \begin{cases} x + y + z = a + b + c \\ x + 2y + 3z = b + 2c \\ x + 3y + 4z = b + 3c \end{cases}$$

$$11. \begin{cases} \frac{xy}{x+y} = \frac{1}{5} \\ \frac{yz}{y+z} = \frac{1}{6} \\ \frac{zx}{z+x} = \frac{1}{7} \end{cases}$$

$$12. \begin{cases} \frac{1}{x} + \frac{1}{y} = 6 \\ \frac{1}{y} + \frac{1}{z} = 10 \\ \frac{1}{z} + \frac{1}{x} = 8 \end{cases}$$

$$13. \begin{cases} \frac{x+y}{3} + 3z = 29 \\ \frac{2x-y}{2} + 2z = 22 \\ 3x - y = 3(z-1) \end{cases}$$

$$14. \begin{cases} \frac{1}{x} + \frac{1}{y} - 1 = 0 \\ \frac{1}{y} + \frac{1}{z} + 3 = 0 \\ \frac{1}{z} + \frac{1}{x} - 2 = 0 \end{cases}$$

$$15. \begin{cases} x + y - z = 0 \\ x - y = 2b \\ x + z = 3a + b \end{cases}$$

$$16. \begin{cases} ax + by + cz = 3 \\ x + y = \frac{a+b}{ab} \\ y + z = \frac{b+c}{bc} \end{cases}$$

$$17. \begin{cases} \frac{xy}{x-y} = \frac{1}{8} \\ \frac{yz}{y-z} = \frac{1}{4} \\ \frac{zx}{z+x} = \frac{1}{16} \end{cases}$$

SUGGESTION FOR EX. 11. If $\frac{xy}{x+y} = \frac{1}{5}$, $\frac{x+y}{xy} = 5$; whence $\frac{1}{y} + \frac{1}{x} = 5$.

Problems

1. The sum of three numbers is 38. The sum of $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third is 12; of $\frac{1}{3}$ of the first, $\frac{1}{4}$ of the second, and $\frac{1}{5}$ of the third is 9. Find the numbers.

2. Separate 800 into three parts, such that the sum of the first, $\frac{1}{2}$ of the second, and $\frac{2}{5}$ of the third is 400; and the sum of the second, $\frac{3}{4}$ of the first, and $\frac{1}{4}$ of the third is 400.

3. John and Paul can do a piece of work in 10 days; John and Guy can do it in 8 days; Paul and Guy can do it in 12 days. How long will it take each to do it alone?

4. Fred and Hugh can do a piece of work in r days; Fred and David can do it in s days; Hugh and David can do it in t days. How long will it take each to do it alone?

5. A certain number is expressed by three digits whose sum is 14. If 693 is added to the number, the digits will appear in reverse order. If the units' digit is equal to the tens' digit increased by 6, what is the number?

6. Ann, Eva, and Dora have certain sums of money. If Ann gives Eva \$100, they will have the same amount; if Ann gives Dora \$100, Dora will have twice as much as Ann; if Eva gives Dora \$100, Dora will have 4 times as much as Eva. What sum has each?

7. A quantity of water sufficient to fill three jars of different sizes will fill the smallest jar 4 times, the largest jar twice with 4 gallons to spare, or the second jar 3 times with 2 gallons to spare. What is the capacity of each jar?

8. A contractor used three scows, a , b , and c , to convey sand from his dredge to the dumping ground. He was credited as follows. Find the capacity of each scow.

April 20, scows a , b , c , a , b , c , a , and b conveyed 3230 cu. yd.

April 21, scows c , a , b , c , a , b , and c conveyed 2820 cu. yd.

April 22, scows a , b , c , a , b , c , and a conveyed 2870 cu. yd.

REVIEW

1. Distinguish between integral and fractional equations; between dependent and independent equations.
2. What is meant by the root of an equation? by solving an equation? by checking the roots?
3. Define and illustrate equivalent equations.
4. What is a formula? Give five simple formulas and tell what each represents.
5. Find three pairs of values for x and y in $x + y = 15$. What kind of equation is this, and why?
6. Define simultaneous equations; elimination. State the axiom upon which elimination by addition is based; elimination by substitution. What other expression is used for simultaneous equations?
7. Outline the method of elimination by addition or subtraction; by substitution.
8. State what is meant by a graph. Of what practical use are graphs?
9. Define coördinates; origin; abscissa; ordinate.
10. In making graphs, where are positive values of x and y laid off? negative values? Interpret the statement $A = (-4, 3)$.
11. What is the x -distance of any point of the y -axis? the y -distance of any point of the x -axis? What are the coördinates of the origin?
12. Why are linear equations so called? What other term is applied to them?
13. Construct the graph of $2y = 3x - 4$.
14. How many points is it necessary to plot in drawing the graph of a linear equation? Why?
15. Tell how to determine where a graph crosses the x -axis; the y -axis.

16. In drawing the graph of the equation $3y = 2x$, what is the result if the only points plotted are those where the graph intersects the axes? What must be done in a case like this?

17. Of what does the graphic solution of a system of two linear equations consist?

18. Solve graphically and algebraically:

$$\begin{cases} 2x - 3y = 10 \\ 5x + 2y = 6 \end{cases}$$

Compare the results obtained.

19. If two linear simultaneous equations are indeterminate, how will this fact be shown by the graphs of the equations referred to the same axes?

20. Draw the graphs of the two equations:

$$\begin{cases} x + y = 6 \\ x = 13 - y \end{cases}$$

and tell the algebraic meaning of the graphs.

21. From the following select the integral equations; the fractional equations; the numerical equations; the literal equations; the indeterminate equations:

(a) $3x + 5y = 19$

(d) $\frac{2x}{3} + \frac{25x}{9} = 31$

(b) $\frac{7}{2x} + \frac{15}{3x} = 51$

(e) $5x + 2 = 3x - 10$

(c) $ax + bx = c$

(f) $\frac{a}{bx} + \frac{c}{dy} = a^2b^2$

22. Classify the following sets of equations as (1) dependent equations or independent equations; (2) simultaneous equations or inconsistent equations:

(a) $\begin{cases} 3x = 18 \\ x + 5 = 11 \end{cases}$

(c) $\begin{cases} x + y = 6 \\ 3x - y = 2 \end{cases}$

(b) $\begin{cases} x + y = 7 \\ x + y = 11 \end{cases}$

(d) $\begin{cases} x - y = 2 \\ 3x - 3y = 6 \end{cases}$

23. Solve: $\begin{cases} x:y = 2:3 \\ x+5:x-y = 3:1 \end{cases}$

24. The sum of two acute angles of a right triangle is 90° . If they are in the ratio 5:3, what is the value of each?

25. John is 13 years 6 months old and his brother Paul is 15 years old. Their father divided \$1.71 between them in proportion to their ages. How much did each receive?

26. The sum of the angles of a triangle is 180° . The largest angle is 4 times as large as the smallest one and equal to the sum of the two smaller angles. Find the size of each angle.

27. A steamer made a trip up a river and back in 24 hours, allowing 6 hours for loading and unloading the freight. A passenger observed that the boat averaged $10\frac{1}{2}$ miles an hour on the up trip and $17\frac{1}{2}$ miles an hour on the return trip. Find the distance the boat went up the river and the number of hours the trip up the river took.

28. A banker has two piles of paper money, each containing 5-dollar and 10-dollar bills. The value of the first pile is \$2500 and of the second, \$2000. If each pile contains the same number of bills and if the number of 5-dollar bills in the first pile is the same as the number of 10-dollar bills in the second pile, find the number of each kind in each pile.

29. The perimeter of a certain triangle is 20 inches. Find the length of each side if the sum of the two shorter sides is 50% more than the longest side and their difference is 25% of the longest side.

30. The bisector of an angle of a triangle divides the opposite side into two parts which are proportional to the adjacent sides of the triangle. If the sides are a inches, b inches, and c inches respectively, find the lengths of the two parts of the side c inches long made by the bisector of the opposite angle. Solve when $a = 9$, $b = 5$, $c = 7$.

CHAPTER XII

POWERS AND ROOTS

Squares and square roots

1. What is the product of 2×2 ? of $(-2) \times (-2)$?
2. What is the product of $a \times a$? of $(-a) \times (-a)$?
3. What number multiplied by itself will give 4? Is there more than one such number?

If 36 is the square of 6 and of -6 , then 6 and -6 are square roots of 36.

The symbol $\sqrt{\quad}$ indicates the **square root** of a number. The sign $\sqrt{\quad}$ is called a **radical sign**, while the number beneath the sign is called the **radicand**.

4. Read $\sqrt{36}$; $\sqrt{49}$. What name do we give to 36?

Every positive number has a positive and a negative square root. The positive root is called the **principal square root**. To avoid confusion, we use the symbol $\sqrt{36}$ to mean the principal square root of 36, namely, $+6$; and the symbol $-\sqrt{36}$ for the negative root, namely, -6 . If both roots are required, we use the double sign $\pm$ (read "plus or minus").

5. Complete the following table:

SQUARE ROOT	SQUARE	SQUARE ROOT	SQUARE	SQUARE ROOT	SQUARE
± 1	1	± 9	81	± 18	324
± 2	4	± 10	?	± 19	?
± 3	?	± 11	?	± 20	?
± 4	?	± 12	?	± 21	?
± 5	?	± 13	?	± 22	?
± 6	?	± 14	?	± 23	?
± 7	?	± 15	?	± 24	?
± 8	?	± 16	?	± 25	?
		± 17	?		

6. Since $4 \times 4 = 16$, therefore $2^2 \times 2^2 = 2^4$. That is, we use 2 as a factor 4 times to obtain 16. What is the result of multiplying $2^3 \times 2^3$? What do we do to the exponents?

7. We see from the above that $(2^2)^2 = 2^4$ and $(2^3)^2 = 2^6$. What is $(3^2)^2$? $(4^3)^2$? $(5^2)^2$? $(a^2)^2$? $(a^3)^2$? $(x^4)^2$? $(a^4b^4)^2$? $(x^5y^5)^2$?

8. When we square a quantity with an exponent, what do we do to the exponent?

Powers of powers

$$\begin{array}{lll} 2^2 = 4 & (2^2)^3 = 2^6 = 64 & (2^2)^5 = 2^{10} = 1024 \\ (2^2)^2 = 2^4 = 16 & (2^2)^4 = 2^8 = 256 & (2^2)^6 = 2^{12} = 4096 \end{array}$$

To raise a power of a number to any power

RULE. *Multiply the exponent of the power of the number by the exponent of the required power.*

Thus, $(a^m)^n = a^{mn}$

Powers of positive and negative numbers

POWERS OF A
POSITIVE NUMBER

$$2^1 = 2$$

$$2^2 = 4$$

$$2^3 = 8$$

$$2^4 = 16$$

$$2^5 = 32$$

$$2^6 = 64$$

$$2^7 = 128$$

$$2^8 = 256$$

POWERS OF A
NEGATIVE NUMBER

$$(-2)^1 = -2$$

$$(-2)^2 = 4$$

$$(-2)^3 = -8$$

$$(-2)^4 = 16$$

$$(-2)^5 = -32$$

$$(-2)^6 = 64$$

$$(-2)^7 = -128$$

$$(-2)^8 = 256$$

Law of signs for powers

From these examples, we note the following law of signs:

(1) *All powers of a positive number are positive.*

(2) *Even powers of a negative number are positive and odd powers are negative.*

Powers of products

$$(2 \cdot 3)^2 = (2 \cdot 3)(2 \cdot 3) = 2 \cdot 2 \cdot 3 \cdot 3 = 2^2 \cdot 3^2$$

Powers of quotients

$$\left(\frac{2}{3}\right)^2 = \frac{2}{3} \cdot \frac{2}{3} = \frac{2^2}{3^2}$$

We see also from the last two examples an illustration of the following principles :

PRINCIPLES. (1) *Any power of a product is equal to the product of its factors each raised to that power.*

(2) *Any power of a quotient of two numbers is equal to the quotient of the numbers each raised to that power.*

Powers of monomials

ILLUSTRATIVE EXAMPLES

1. What is the cube of $4 a^3 b$?

SOLUTION

$$(4 a^3 b)^3 = (4 a^3 b)(4 a^3 b)(4 a^3 b) = 64 a^9 b^3$$

2. What is the fifth power of $-2 ab^2$?

SOLUTION

$$(-2 ab^2)^5 = (-2 ab^2)(-2 ab^2)(-2 ab^2)(-2 ab^2)(-2 ab^2) = -32 a^5 b^{10}$$

To raise an integral term to any power

RULE. *Raise the numerical coefficient to the required power and annex to it each letter with an exponent equal to the product of its exponent by the exponent of the required power.*

Make the power positive or negative according to the law of signs.

EXERCISES

Raise to the power indicated :

- | | | |
|------------------|-------------------|-------------------|
| 1. $(ab^2c^3)^2$ | 5. $(-4c^2y^5)^3$ | 9. $(-1)^9$ |
| 2. $(a^3b^2c)^4$ | 6. $(-2a^3n^5)^4$ | 10. $(-1)^{10}$ |
| 3. $(2a^2c)^3$ | 7. $(abcx)^m$ | 11. $(3bc)^n$ |
| 4. $(7a^2m^5)^2$ | 8. $(2e^2x^c)^6$ | 12. $(2a^2x^3)^n$ |

To raise a fraction to any power

ILLUSTRATIVE EXAMPLE

What is the square of $-\frac{5a^3x^2}{7b^2c}$?

SOLUTION

$$\left(-\frac{5a^3x^2}{7b^2c}\right)^2 = -\frac{5a^3x^2}{7b^2c} \times -\frac{5a^3x^2}{7b^2c} = \frac{25a^6x^4}{49b^4c^2}$$

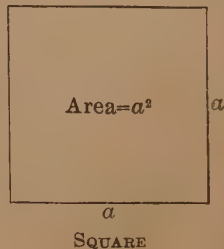
RULE. Raise both numerator and denominator to the required power and prefix the proper sign to the result.

EXERCISES

Raise to the power indicated :

- | | | |
|--|------------------------------------|--|
| 1. $\left(\frac{1}{4b}\right)^2$ | 4. $\left(-\frac{5}{ab}\right)^2$ | 7. $\left(-\frac{2a}{b}\right)^6$ |
| 2. $\left(\frac{2x}{7y}\right)^2$ | 5. $\left(-\frac{2}{3x}\right)^4$ | 8. $\left(-\frac{b^3c^2}{a^2x}\right)^2$ |
| 3. $\left(\frac{3x^2}{10y^3}\right)^2$ | 6. $\left(-\frac{3x}{2y}\right)^3$ | 9. $\left(\frac{a^2b^3}{xy^4}\right)^n$ |

10. The area of the square at the right is a^2 , obtained by multiplying together the equal sides, the length of each being a . Find the area of a square if the length of each of the sides is as follows :



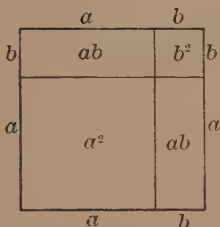
- (1) $2.3ab$; (2) $3.8x^2$; (3) $.7a$.

Powers of polynomials

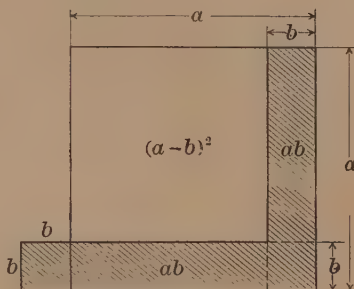
We have already learned that:

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$



REPRESENTATION OF $(a + b)^2$



REPRESENTATION OF $(a - b)^2$

In the right-hand figure, the area of the largest square is a^2 . The area of $(a - b)^2 = a^2 - ab - ab + b^2 = a^2 - 2ab + b^2$

By multiplication we may show that:

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$$

In like manner we may show that:

$$(a + b - c)^2 = a^2 + b^2 + c^2 + 2ab - 2ac - 2bc$$

$$(a - b - c)^2 = a^2 + b^2 + c^2 - 2ab - 2ac + 2bc$$

$$(a - b + c)^2 = a^2 + b^2 + c^2 - 2ab + 2ac - 2bc$$

PRINCIPLE. *The square of any polynomial is equal to the sum of the squares of each of the terms added to twice the product of each term by each term that follows.*

EXERCISES

Square the following polynomials:

- | | | |
|------------------|---------------------|----------------------|
| 1. $3x + 4y^3$ | 4. $4r^2 + s^2 + t$ | 7. $5a - 2c - 3e$ |
| 2. $5m^a - 4n^b$ | 5. $2a + b + c$ | 8. $3r^2 + 2s + t^4$ |
| 3. $5m^3 - 6a^x$ | 6. $3a + 2b - 4c$ | 9. $2x^2 - 3y + z^3$ |

By multiplication, raise the following quantities to the required power:

$$10. (x + y)^3 \quad 12. (a + b)^4 \quad 14. (a + b)^5$$

$$11. (x - y)^3 \quad 13. (a - b)^4 \quad 15. (a - b)^5$$

16. The formula $V = e^3$, which expresses the volume of a cube in terms of its edge, was given on page 72. Express this formula in words.

17. (a) Find the volume of a cube the length of whose edge is $(a + b)$.

(b) Find the volume of a cube the length of whose edge is $(a - b)$.

(c) State a rule for finding the cube of a binomial such as $(a + b)$.

(d) State a rule for finding the cube of a binomial such as $(a - b)$.

18. What is the volume of a cube whose edge is $(a + 2b)$?

19. What is the volume of a cube whose edge is $(2a - b)$?

20. Learn the cubes of all numbers from 1 through 12. (See Table, page 251.)

Expand:

$$21. (3a + 5b)^3 \quad 25. (7x - 2y + 3z)^2 \quad 29. (4x^2 - 3x + 2)^2$$

$$22. (2x - 7y)^3 \quad 26. (5a + 3b - c)^2 \quad 30. (3 - 2a - a^2)^2$$

$$23. (\frac{1}{2}x + \frac{1}{3}y)^3 \quad 27. (3a - 2b - 7c)^2 \quad 31. (.2x + .3y - z)^2$$

$$24. (4a + 3b)^3 \quad 28. (5a - 4b + 3c)^2 \quad 32. \left(\frac{a}{2} + \frac{b}{3} - \frac{c}{4}\right)^2$$

Process of obtaining roots of numbers

One of two equal factors of a number is called its **square root**; one of the three equal factors, its **third** or **cube root**; one of the four equal factors, its **fourth root**; etc.

Find by inspection the square root of 49; 81; 64; the cube root of 27; 64; 125; the fourth root of 16; 81.

The square root of a is written $\sqrt{a}$; the cube root of a is written $\sqrt[3]{a}$; the fourth root of a is written $\sqrt[4]{a}$; etc.

The small number in the opening of the radical sign, as 3 in $\sqrt[3]{a}$, is called the **index** of the root. When no index is written, the second root, or square root, is meant. What does the index show?

The process of finding a root of a number is the inverse of the process of finding a power of a number. For example $\sqrt[3]{8} = \sqrt[3]{2 \cdot 2 \cdot 2} = 2$.

Roots of numbers

Since $2^3 = 8$, we know that $\sqrt[3]{8} = 2$.

Since $(-2)^3 = -8$, we know that $\sqrt[3]{-8} = -2$.

Since $2^4 = 16$ and $(-2)^4 = 16$, we know that $\sqrt[4]{16} = \pm 2$.*

Since $2^5 = 32$, we know that $\sqrt[5]{32} = 2$.

Since $(-2)^5 = -32$, we know that $\sqrt[5]{-32} = -2$.

A root is called *odd* or *even* according as its index is odd or even.

The real root of a number, if it has the same sign as the number, is called the **principal root** of the number.

The principal square root of 25 is 5, not -5 . The principal cube root of 8 is 2, and of -8 is -2 .

Since $(2^2)^3 = 2^{2 \times 3} = 2^6$, the principal cube root of 2^6 is 2^2 .

$$\sqrt[3]{2^6} = 2^{\frac{6}{3}} = 2^2$$

Hence the following law:

Law of exponents

The exponent of any root of a number is equal to the exponent of the number divided by the index of the root.

From the preceding illustrations, we observe the following law of signs:

* As in the case of square roots, the principal root, $+2$, is meant when we write $\sqrt[4]{16}$.

Law of signs

An odd root of a number has the same sign as the number. An even root may have either sign.

EXERCISES

What are the principal indicated roots of the following numbers?

1. $\sqrt{625}$

5. $\sqrt{484}$

9. $\sqrt{289}$

2. $\sqrt[3]{729}$

6. $\sqrt[3]{512}$

10. $\sqrt[3]{1000}$

3. $\sqrt[3]{343}$

7. $\sqrt{576}$

11. $\sqrt[4]{81}$

4. $\sqrt{441}$

8. $\sqrt[3]{216}$

12. $\sqrt[5]{1}$

Since $(5a)^2 = 5^2 a^2 = 25 a^2$, the principal square root of $25 a^2$ is $5 a$.

$$\sqrt{25 a^2} = \sqrt{25} \sqrt{a^2} = 5 a$$

Any root of a product may be obtained by taking that root of each of the factors and finding the product of the results.

Since $\left(\frac{2}{3}\right)^4 = \frac{2^4}{3^4} = \frac{16}{81}$, the principal fourth root of $\frac{16}{81}$ is $\frac{2}{3}$.

$$\sqrt[4]{\frac{16}{81}} = \frac{\sqrt[4]{16}}{\sqrt[4]{81}} = \frac{\sqrt[4]{2^4}}{\sqrt[4]{3^4}} = \frac{2}{3}$$

Any root of the quotient of two numbers is equal to that root of the dividend divided by that root of the divisor.

Roots of monomials

EXERCISES

1. Find the square root of $36 a^6 b^2$.

SOLUTION

Since in squaring a monomial the coefficient is squared and the exponents of the letters are multiplied by 2, therefore to find the square root, the *square root* of the coefficient must be found, and to it must be annexed the letters, each with its exponent *divided* by 2.

The square root of 36 is 6, and the square root of the literal factors is $a^3 b$. Therefore the *principal* square root of $36 a^6 b^2$ is $6 a^3 b$.

The square root may also be $-6 a^3 b$, since $(-6 a^3 b)(-6 a^3 b) = 36 a^6 b^2$.

$$\therefore \sqrt{36 a^6 b^2} = \pm 6 a^3 b$$

2. Find the cube root of $-125 x^6 y^{21}$.

SOLUTION

$$\sqrt[3]{-125 x^6 y^{21}} = -5 x^2 y^7$$

To find the root of an integral term

RULE. Find the required root of the numerical coefficient, annex to it the letters, each with its exponent divided by the index of the root sought, and prefix the proper sign to the result.

Find the indicated roots:

- | | | |
|----------------------------------|----------------------------------|------------------------------------|
| 3. $\sqrt[3]{a^3 b^6 c^9}$ | 8. $\sqrt[3]{-8 a^6 b^{15}}$ | 13. $\sqrt{(-mb^3)^2}$ |
| 4. $\sqrt{a^6 b^4 c^8}$ | 9. $\sqrt[5]{-32 x^{10} y^{10}}$ | 14. $\sqrt[3]{(-a^2 b)^9}$ |
| 5. $\sqrt[5]{a^{10} x^5 y^{15}}$ | 10. $\sqrt{16 x^4 y^2}$ | 15. $-\sqrt[9]{a^{18} b^9 c^{27}}$ |
| 6. $\sqrt[4]{a^{4n} b^8 c^{12}}$ | 11. $\sqrt[7]{-a^7 b^{14} x^7}$ | 16. $-\sqrt[3]{-27 p^9 r^3}$ |
| 7. $\sqrt{x^{4n} y^8 z^{2m}}$ | 12. $\sqrt[5]{-243 y^{5n}}$ | 17. $\sqrt[3]{8 a^6 c^9 d^9}$ |

To find the root of a fractional term

RULE. Find the required root of both numerator and denominator and prefix the proper sign to the resulting fraction.

18. Find the cube root of $\frac{-8 x^9 y^6}{27 m^3 n^{12}}$.

SOLUTION

$$\sqrt[3]{\frac{-8 x^9 y^6}{27 m^3 n^{12}}} = \frac{\sqrt[3]{-8 x^9 y^6}}{\sqrt[3]{27 m^3 n^{12}}} = \frac{-2 x^3 y^2}{3 mn^4}$$

Find the indicated roots:

- | | | |
|---|--|---|
| 19. $\sqrt{\frac{64 a^4 b^6}{81 m^2 n^8}}$ | 24. $\sqrt[4]{\frac{16 a^8}{81 b^4}}$ | 29. $\sqrt{\frac{289(2a-b)^4}{169 a^{6n}}}$ |
| 20. $\sqrt[7]{\frac{(x-y)^{14}}{128 x^{14}}}$ | 25. $\sqrt[3]{\frac{216 a^{3m+3}}{c^{6n}}}$ | 30. $\sqrt[4]{\frac{441 a^4}{4 a^{4n+2}}}$ |
| 21. $\sqrt[3]{-125 \frac{x^3}{y^9}}$ | 26. $\sqrt{\frac{16(a-b)^2}{9(a+3b)^2}}$ | 31. $\sqrt[3]{\frac{-8a^3(3a^2-b)^3}{27 a^6}}$ |
| 22. $\sqrt[5]{\frac{-32 a^5 x^{10}}{243 y^{15}}}$ | 27. $\sqrt[3]{\frac{8(2a-3b)^3}{216}}$ | 32. $\sqrt[5]{\frac{-243 a^{10} b^{15}}{32 x^5}}$ |
| 23. $\sqrt[3]{\left(\frac{-27 a^3}{64 b^6}\right)^2}$ | 28. $\sqrt{\frac{625 a^{2n+2}}{121 b^{4n-2}}}$ | 33. $\sqrt[4]{\frac{16(a^2-2ab+b^2)^2}{(9 x^2 y^4)^2}}$ |

To find the square root of a polynomial

EXERCISES

1. Find the square root of
- $a^2 + 2ab + b^2$
- .

SOLUTION

$$\begin{array}{r}
 a^2 + 2ab + b^2 \overline{) a + b} \\
 a^2 \pm (a + b), \text{ Ans.} \\
 \hline
 \text{Trial divisor, } 2a \overline{) 2ab + b^2} \\
 \text{Complete divisor, } 2a + b \overline{) 2ab + b^2}
 \end{array}$$

EXPLANATION. Since $a^2 + 2ab + b^2$ is the square of $(a + b)$, we know that the square root of $a^2 + 2ab + b^2$ is $a + b$.

Since the first term of the root is a , it may be found by taking the square root of a^2 , the first term of the power. On subtracting a^2 , there is a remainder of $2ab + b^2$.

The second term of the root is known to be b , and this may be found by dividing the first term of the remainder by twice the part of the root already found. This divisor is called a *trial* divisor.

Since $2ab + b^2$ is equal to $b(2a + b)$, the complete divisor which multiplied by b produces the remainder $2ab + b^2$ is $2a + b$; that is, the complete divisor is found by adding the second term of the root to twice the root already found.

On multiplying the complete divisor by the second term of the root and subtracting, there is no remainder. Hence $a + b$ is the required root.

NOTE. The sign $\pm$ should be prefixed to the result, since $-(a + b)$ as well as $+(a + b)$ is the square root of $a^2 + 2ab + b^2$.

2. Find the square root of
- $4x^4 + 12x^3 - 3x^2 - 18x + 9$
- .

SOLUTION

$$\begin{array}{r}
 4x^4 + 12x^3 - 3x^2 - 18x + 9 \overline{) 2x^2 + 3x - 3} \\
 4x^4 \pm (2x^2 + 3x - 3) \\
 \hline
 \text{Trial divisor, } 4x^2 \overline{) 12x^3 - 3x^2} \\
 \text{Complete divisor, } 4x^2 + 3x \overline{) 12x^3 + 9x^2} \\
 \hline
 \text{Trial divisor, } 4x^2 + 6x \overline{) -12x^2 - 18x + 9} \\
 \text{Complete divisor, } 4x^2 + 6x - 3 \overline{) -12x^2 - 18x + 9}
 \end{array}$$

NOTE. Arrange the terms according to the ascending or descending powers of some letter.

Check by substituting 2 for x or by squaring the result.

Find both square roots of each of the following and check the first 8 examples by numerical substitution :

3. $4x^2 + 12xy + 9y^4$
4. $16x^2 - 56xy + 49y^2$
5. $\frac{4a^2}{b^2} + 8 + \frac{4b^2}{a^2}$
6. $\frac{a^2}{9} + \frac{10a}{3} + 25$
7. $\frac{25}{4n^2} + 15 + 9n^2$
8. $\frac{25d^2}{16r^2} - \frac{5d}{r} + 4$
9. $1 + 6x + 5x^2 - 12x^3 + 4x^4$
10. $4a^4 + 5a^2 + 1 - 4a^3 - 2a$
11. $\frac{a^4}{9} + \frac{4a^3}{3} + 2a^2 - 12a + 9$
12. $\frac{x^4}{64} + \frac{x^3}{8} - x + 1$
13. $x^2 + 2x - 1 - \frac{2}{x} + \frac{1}{x^2}$
14. $x^4 + x^3 + \frac{13x^2}{20} + \frac{x}{5} + \frac{1}{25}$
15. $a^4 - 2a^2b + 2a^2c^2 - 2bc^2 + b^2 + c^4$
16. $4a^2 - 12ab + 16ac + 9b^2 + 16c^2 - 24bc$
17. $9x^2 + 25y^2 + 9z^2 - 30xy + 18xz - 30yz$
18. $a^4 + 53a^2 + 14a^3 + 28a + 4$
19. $25x^4 - 30ax^3 + 49a^2x^2 - 24a^3x + 16a^4$
20. $4x^6 - 20x^4 - 4x^3 + 25x^2 + 10x + 1$
21. $1 + 10x + 25x^2 - 4x^3 - 20x^4 + 4x^6$
22. $4c^4 - 4c^3 + 5c^2 - 2c + 1$
23. $.04x^4 + .08x^3 - .16x^2 - .2x + .25$
24. $\frac{a^4}{9} - \frac{4a^3}{3} + \frac{a^2b}{3} + 4a^2 - 2ab + \frac{b^2}{4}$
25. $\frac{4m^6}{9} - \frac{4m^5}{3} + \frac{19m^4}{15} + \frac{3m^3}{5} - \frac{73m^2}{50} + \frac{3m}{10} + \frac{9}{16}$

26. Compare the square roots of examples 20 and 21 and explain why the signs appear as they do.

Compare the number of digits:

NUMBER	Sq. ROOT	NUMBER	Sq. ROOT	NUMBER	Sq. ROOT
1	1	1'00	10	1'00'00	100
25	5	10'24	32	56'25'00	750
81	9	98'01	99	99'80'01	999

PRINCIPLE. *If a number is separated into periods of two digits each, beginning at the decimal point, its square root will have as many digits as the number has periods.*

The left-hand period may be incomplete, consisting of only one digit.

First method — Using the formula

If the number of units expressed by the tens' digit is represented by t and the number of units expressed by the units' digit by u , any number consisting of tens and units will be represented by $t + u$, and its square by

$$(t + u)^2, \text{ or } t^2 + 2tu + u^2$$

$$\text{Since } 25 = 20 + 5, 25^2 = (20 + 5)^2 = 20^2 + 2(20 \times 5) + 5^2 = 625$$

ILLUSTRATIVE EXAMPLES

1. Find the square root of 3844.

A. FIRST METHOD

$$\begin{array}{r|l} 38'44 & \underline{60 + 2} \\ t^2 = & 36\ 00 \\ \hline 2\ t = 120 & 2\ 44 \\ u = & 2 \\ \hline 2\ t + u = 122 & 2\ 44 \end{array}$$

EXPLANATION. Separating the number into periods of two digits each, we find that the root is composed of two digits, tens and units. Since the largest square in 38 is 6, the tens of the root cannot

be greater than 6 tens, or 60. Writing 6 tens in the root, squaring, and subtracting from 3844, we have a remainder of 244.

Since the square of a number composed of tens and units is equal to (the square of the tens) + (twice the product of the tens and the units) + (the square of the units), when the square of the tens has been subtracted the remainder, 244, is twice the product of the tens and the units, plus the square of the units, or only a little more than twice the product of the tens and the units.

Therefore 244 divided by twice the tens is approximately equal to the units. 2×6 tens, or 120, then, is a *trial*, or *partial*, *divisor*. On dividing 244 by the trial divisor, the units' figure is found to be 2.

Since twice the tens are to be multiplied by the units, and the units also are to be multiplied by the units to obtain the square of the units, in order to abridge the process the tens and units are first added, forming the *complete divisor* 122, which is then multiplied by the units. Thus, $(120 + 2)$ multiplied by 2 = 244.

Therefore the square root of 3844 is 62.

Check by squaring 62. The result will be 3844.

B. ABBREVIATION OF FIRST METHOD

$$38'44 \overline{)62}$$

$$t^2 = 36$$

$$2t = 120 \quad 244$$

$$u = 2$$

$$2t + u = 122 \quad 244$$

EXPLANATION. In practice it is usual to place the figures of the same order in the same column and to disregard the zeros on the right of the products.

Since any number may be regarded as composed of tens and units, the foregoing processes have a general application.

Thus, $346 = 34 \text{ tens} + 6 \text{ units}$; $2377 = 237 \text{ tens} + 7 \text{ units}$.

2. Find the square root of 859.765 to the nearest tenth.

SOLUTION

$$8'59.76'50 \overline{)29.32}$$

4 Ans. 29.3

$$\text{Trial divisor} = 2 \times 20 = 40$$

$$\text{Complete divisor} = 40 + 9 = 49$$

$$\text{Trial divisor} = 2 \times 290 = 580$$

$$\text{Complete divisor} = 580 + 3 = 583$$

$$\text{Trial divisor} = 2 \times 2930 = 5860$$

$$\text{Complete divisor} = 5860 + 2 = 5862$$

$$4 \ 59$$

$$4 \ 41$$

$$18 \ 76$$

$$17 \ 49$$

$$1 \ 27 \ 50$$

$$1 \ 17 \ 24$$

+

RULE. Separate the number into periods of two figures each, left and right from the decimal point.

Find the greatest square in the left-hand period and write its square root for the first figure of the required root.

Square this root, subtract the result from the left-hand period, and annex to the remainder the next period for a new dividend.

Double the root already found, with a zero annexed, for a trial divisor, and by it divide the dividend. The quotient, or the quotient diminished, will be the second figure of the root. Add to the trial divisor the figure last found, multiply this complete divisor by the figure of the root last found, subtract the product from the dividend, and to the remainder annex the next period for the next dividend.

Proceed in this manner until all the periods have been used. The result will be the square root sought.

Second method — Using factoring

This method consists in factoring, grouping the factors, and taking the root of each group.

$$\text{Thus, } \sqrt{256} = \sqrt{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}$$

$$= \sqrt{2^8} = 2^4 = 16$$

$$\sqrt{576} = \sqrt{3 \cdot 3 \cdot 8 \cdot 8} = \sqrt{3^2 \cdot 8^2} = 3 \cdot 8 = 24$$

NOTE. The square root of a common fraction may be obtained by finding the square root of both numerator and denominator separately or by reducing the fraction to a decimal and then finding the root.

EXERCISES

Find the square roots of the following, and check :

1. 529

9. 6889

17. $\frac{625}{729}$

2. 2209

10. 5041

18. $\frac{576}{841}$

3. 4761

11. 7569

19. $\frac{169}{225}$

4. 9025

12. 57121

20. $\frac{196}{1156}$

5. 9409

13. 42025

21. $\frac{289}{324}$

6. 9801

14. 2480.04

22. $\frac{361}{400}$

7. 7921

15. 10.9561

23. $\frac{576}{784}$

8. 6561

16. .001225

24. $\frac{289}{961}$

$$25. \frac{15a \times 24 \times 10a^3}{88b \times 98 \times 11b}$$

SOLUTION

$$\begin{aligned} \sqrt{\frac{15a \times 24 \times 10a^3}{88b \times 98 \times 11b}} &= \sqrt{\frac{5a \times 3 \times 3 \times 8 \times 2 \times 5a^3}{11b \times 8 \times 2 \times 49 \times 11b}} \\ &= \sqrt{\frac{5^2 a^4 \times 3^2 \times 4^2}{11^2 \times 4^2 \times 7^2 b^2}} = \frac{5 \times 3 a^2}{11 \times 7 b} = \frac{15 a^2}{77 b} \end{aligned}$$

$$26. \frac{30 \times 4 \times 6a \times 20a^3}{6 \times 18b^2 \times 12b^{2n}}$$

$$29. \frac{98 \times 75 \times 8v^3 \times 3v}{16a^4}$$

$$27. \frac{45 \times 6 \times 9 \times 30x^{4n-8}}{(9a^3)^2}$$

$$30. \frac{2304a^{4+2n}}{576a^{4n-2}}$$

$$28. \frac{98 \times 24 \times 12 \times 9y^8}{18 \times 32y \times 9y^{2n+1}}$$

$$31. \frac{.4 \times .24 \times 3.75z^6}{12.25 \times 30.25z^2}$$

32. 1332.25

33. 1376.41

34. 9.8596

Find the square roots to three decimal places :

35. .002209

38. $\frac{3}{4}$

41. $\frac{2}{9}$

36. .091601

39. $\frac{4}{5}$

42. $\frac{5}{16}$

37. .954814

40. $\frac{5}{8}$

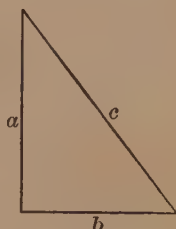
43. $\frac{5}{12}$

In the right triangle at the right we know (see page 71) that $c^2 = a^2 + b^2$.

44. If $a = 44$ and $b = 33$, $c = ?$

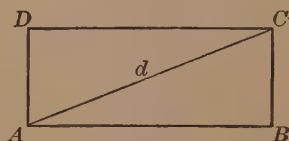
45. If $c = 30$ and $b = 18$, find a .

46. In the rectangle below, ABC is what kind of triangle? If $CB = a$ and $AB = b$, express d in terms of a and b .



RIGHT TRIANGLE

47. In this figure let $CB = a$ and $AB = b$. If $a = 1\frac{1}{2}$ inches and $b = 2$ inches, find the length of d .



RECTANGLE

48. The area of a square lot is equal to 110.25 square rods. What is the length of each side of the lot?

To find cube roots by factoring

Cube roots, like square roots, may sometimes be found by factoring.

$$\begin{aligned}\text{Thus, } \sqrt[3]{42875} &= \sqrt[3]{5 \cdot 5 \cdot 5 \times 7 \cdot 7 \cdot 7} = \sqrt[3]{5^3 \times 7^3} \\ &= 5 \times 7 = 35\end{aligned}$$

EXERCISES

Find, by the factoring method, the following cube roots :

- | | | | |
|---------------------|---------------------|-----------------------|------------------------------|
| 1. $\sqrt[3]{512}$ | 4. $\sqrt[3]{2744}$ | 7. $\sqrt[3]{4096}$ | 10. $\sqrt[3]{4096 a^6 b^9}$ |
| 2. $\sqrt[3]{1000}$ | 5. $\sqrt[3]{8000}$ | 8. $\sqrt[3]{125000}$ | 11. $\sqrt[3]{21.952}$ |
| 3. $\sqrt[3]{1728}$ | 6. $\sqrt[3]{3375}$ | 9. $\sqrt[3]{13824}$ | 12. $\sqrt[3]{15.625}$ |

Practice in the use of the table of powers and roots

ILLUSTRATIVE EXAMPLES

1. $27^2 = 729$

4. $27^3 = 19,683$

2. $2.7^2 = 7.29$

5. $2.7^3 = 19.683$

3. $.27^2 = .0729$

6. $\sqrt{27.04} = 5.2$

EXERCISES

Using the table on page 251, write the results of the first 16 exercises:

1. 7.2^2

9. $\sqrt[3]{456.533}$

2. 840^2

10. $\sqrt[3]{474.552}$

3. 9.8^3

11. $\sqrt[3]{551.368}$

4. $\sqrt{26.01}$

12. $.79^2$

5. $\sqrt{.9409}$

13. $\sqrt{.9216}$

6. 9.7^2

14. $.94^3$

7. $\sqrt{98.01}$

15. $\sqrt[3]{132.651}$

8. $\sqrt{.6889}$

16. $\sqrt[3]{753.571}$

17. If the legs of a right triangle are 3.5 inches and 4.2 inches long, find the length of the hypotenuse.

18. The perimeter of a rectangle is 28 inches. Find the diagonal if the sides are in the ratio $\frac{2}{1}$.

19. Find the value of m in the formula

$$m = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2} \text{ when } a = 9, b = 5, c = 7.$$

20. If $t = \frac{2}{a+b}\sqrt{abs(s-c)}$, find the value of t when

$$a = 10, b = 12, c = 16, s = \frac{a+b+c}{2}.$$

21. (a) Find the value of m in example 19 when $a = 25$, $b = 20$, $c = 15$.

(b) Find the value of t in example 20 when $a = 10$, $b = 24$, $c = 26$.

TABLE OF POWERS AND ROOTS

In practical work in powers and roots, tables like the following are used:

No.	SQUARES	CUBES	SQUARE ROOTS	CUBE ROOTS	No.	SQUARES	CUBES	SQUARE ROOTS	CUBE ROOTS
1	1	1	1.000	1.000	51	2,601	132,651	7.141	3.708
2	4	8	1.414	1.259	52	2,704	140,608	7.211	3.732
3	9	27	1.732	1.442	53	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	54	2,916	157,464	7.348	3.779
5	25	125	2.236	1.709	55	3,025	166,375	7.416	3.802
6	36	216	2.449	1.817	56	3,136	175,616	7.483	3.825
7	49	343	2.645	1.912	57	3,249	185,193	7.549	3.848
8	64	512	2.828	2.000	58	3,364	195,112	7.615	3.870
9	81	729	3.000	2.080	59	3,481	205,379	7.681	3.892
10	100	1,000	3.162	2.154	60	3,600	216,000	7.745	3.914
11	121	1,331	3.316	2.223	61	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	62	3,844	238,328	7.874	3.957
13	169	2,197	3.605	2.351	63	3,969	250,047	7.937	3.979
14	196	2,744	3.741	2.410	64	4,096	262,144	8.000	4.000
15	225	3,375	3.872	2.466	65	4,225	274,625	8.062	4.020
16	256	4,096	4.000	2.519	66	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	67	4,489	300,763	8.185	4.061
18	324	5,832	4.242	2.620	68	4,624	314,432	8.246	4.081
19	361	6,859	4.358	2.668	69	4,761	328,509	8.306	4.101
20	400	8,000	4.472	2.714	70	4,900	343,000	8.366	4.121
21	441	9,261	4.582	2.758	71	5,041	357,911	8.426	4.140
22	484	10,648	4.690	2.802	72	5,184	373,248	8.485	4.160
23	529	12,167	4.795	2.843	73	5,329	389,017	8.544	4.179
24	576	13,824	4.898	2.884	74	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	75	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	76	5,776	438,976	8.717	4.235
27	729	19,683	5.196	3.000	77	5,929	456,533	8.774	4.254
28	784	21,952	5.291	3.036	78	6,084	474,552	8.831	4.272
29	841	24,389	5.385	3.072	79	6,241	493,039	8.888	4.290
30	900	27,000	5.477	3.107	80	6,400	512,000	8.944	4.308
31	961	29,791	5.567	3.141	81	6,561	531,441	9.000	4.326
32	1,024	32,768	5.656	3.174	82	6,724	551,368	9.055	4.344
33	1,089	35,937	5.744	3.207	83	6,889	571,787	9.110	4.362
34	1,156	39,304	5.830	3.239	84	7,056	592,704	9.165	4.379
35	1,225	42,875	5.916	3.271	85	7,225	614,125	9.219	4.396
36	1,296	46,656	6.000	3.301	86	7,396	636,056	9.273	4.414
37	1,369	50,653	6.082	3.332	87	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.361	88	7,744	681,472	9.380	4.447
39	1,521	59,319	6.244	3.391	89	7,921	704,969	9.433	4.464
40	1,600	64,000	6.324	3.419	90	8,100	729,000	9.486	4.481
41	1,681	68,921	6.403	3.448	91	8,281	753,571	9.539	4.497
42	1,764	74,088	6.480	3.476	92	8,464	778,688	9.591	4.514
43	1,849	79,507	6.557	3.503	93	8,649	804,357	9.643	4.530
44	1,936	85,184	6.633	3.530	94	8,836	830,584	9.695	4.546
45	2,025	91,125	6.708	3.556	95	9,025	857,375	9.746	4.562
46	2,116	97,336	6.782	3.583	96	9,216	884,736	9.797	4.578
47	2,209	103,823	6.855	3.608	97	9,409	912,673	9.848	4.594
48	2,304	110,592	6.928	3.634	98	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	99	9,801	970,299	9.949	4.626
50	2,500	125,000	7.071	3.684	100	10,000	1,000,000	10.000	4.641

CHAPTER XIII

EXPONENTS AND RADICALS

THEORY OF EXPONENTS

The laws of exponents

- I. $a^m \times a^n = a^{m+n}$ when m and n are positive integers.
- II. $a^m \div a^n = a^{m-n}$ when m and n are positive integers and m is greater than n .
- III. $(a^m)^n = a^{mn}$ when m and n are positive integers.
- IV. $\sqrt[n]{a^m} = a^{m \div n}$ when m and n are positive integers and m is a multiple of n .
- V. $(ab)^n = a^n b^n$ when n is a positive integer.

These laws may be *proved* as follows :

- I. Let m and n be any positive integers, and let a be any number.
 By notation, $a^m = a \cdot a \cdot a \cdots$ to m factors
 and $a^n = a \cdot a \cdot a \cdots$ to n factors

$$\begin{aligned} \therefore a^m \cdot a^n &= (a \cdot a \cdot a \cdots \text{to } m \text{ factors})(a \cdot a \cdot a \cdots \text{to } n \\ &\quad \text{factors}) \\ &= a \cdot a \cdot a \cdots \text{to } (m + n) \text{ factors} \\ &= a^{m+n} \end{aligned}$$

By notation,

- II. Let m and n be positive integers, m being greater than n , and let a be any number.

By notation, $a^m = a \cdot a \cdot a \cdots$ to m factors
 and $a^n = a \cdot a \cdot a \cdots$ to n factors

$$\therefore \frac{a^m}{a^n} = \frac{a \cdot a \cdot a \cdots \text{to } m \text{ factors}}{a \cdot a \cdot a \cdots \text{to } n \text{ factors}}$$

Remove equal factors from dividend and divisor.

Then $a^m \div a^n = a \cdot a \cdot a \cdots$ to $(m - n)$ factors

By notation, $\quad = a^{m-n}$

III. Let m and n be positive integers, and let a be any number.

By notation, $(a^m)^n = a^m \cdot a^m \cdot a^m \cdots$ to n factors

By law I, $= a^{m+m+m+\cdots}$ to n terms

By notation, $= a^{mn}$

IV. Let r and s be positive integers, and let a be any number.

By law III, $(a^r)^s = a^{rs}$ (1)

Indicating the s th root of each member, we have

$$\sqrt[s]{(a^r)^s} = \sqrt[s]{a^{rs}} \quad (2)$$

But $\sqrt[s]{(a^r)^s} = a^r$; $\therefore$ from (2), Equality Axiom, $\sqrt[s]{a^{rs}} = a^r$

That is, $\sqrt[s]{a^{rs}} = a^{rs \div s} = a^r$

In general, when m and n are positive integers, m being a multiple of n , and a any number,

$$\sqrt[n]{a^m} = a^{m \div n}$$

V. Let n be a positive integer, and let a and b be any numbers.

By notation, $(ab)^n = ab \cdot ab \cdot ab \cdots$ to n factors

$= (a \cdot a \cdot a \cdots \text{to } n \text{ factors})(b \cdot b \cdot b \cdots \text{to } n \text{ factors})$

By notation, $= a^n b^n$

Assuming that the law of exponents for multiplication

$$a^m \times a^n = a^{m+n}$$

is true for *all* values of m and n , the meanings of *zero*, *negative*, and *fractional* exponents may be seen readily by substituting these different kinds of exponents for m and n or both, and observing to what conclusions we are led.

A zero exponent

We have assumed that any new kind of exponent shall have its meaning determined in harmony with the law of exponents for multiplication, expressed by the formula,

$$a^m \times a^n = a^{m+n}$$

If $n = 0$, $a^m \times a^0 = a^{m+0}$, or a^m

Dividing by a^m , we have $a^0 = \frac{a^m}{a^m} = 1$

Any number (not zero) with a zero exponent is equal to 1.

A negative exponent

Since $a^m \times a^n = a^{m+n}$ is to hold true for all values of m and n if $m = -n$,

$$a^{-n} \times a^n = a^{-n+n} = a^0$$

But

$$a^0 = 1$$

Hence

$$a^{-n} \times a^n = 1$$

Dividing by a^n , we have

$$a^{-n} = \frac{1}{a^n}$$

Any number with a negative exponent is equal to the reciprocal of the same number with a numerically equal positive exponent.

By the definition of negative exponent just given,

$$a^{-m} = \frac{1}{a^m} \text{ and } b^{-n} = \frac{1}{b^n}$$

Therefore
$$\frac{a^{-m}}{b^{-n}} = \frac{\frac{1}{a^m}}{\frac{1}{b^n}} = \frac{1}{a^m} \times \frac{b^n}{1} = \frac{b^n}{a^m}$$

Any factor may be transferred from one term of a fraction to the other without changing the value of the fraction, provided the sign of the exponent is changed.

A fractional exponent

Since the first law of exponents is to hold true for all exponents,

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$$

That is, $a^{\frac{1}{2}}$ is one of the two equal factors of a , or is a square root of a .

Again,
$$a^{\frac{3}{2}} \times a^{\frac{3}{2}} = a^{\frac{3}{2}+\frac{3}{2}} = a^3$$

That is, $a^{\frac{3}{2}}$ is a square root of the cube of a .

Or, similarly,
$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}} = a^{\frac{3}{2}}$$

That is, $a^{\frac{3}{2}}$ is the cube of the square root of a .

In general, confining the discussion to principal roots, let p and q be any two positive integers. By the first law of exponents, $a^{\frac{p}{q}} \cdot a^{\frac{p}{q}} \cdots$ to q factors $= a^{\frac{p}{q}+\frac{p}{q}+\cdots \text{to } q \text{ terms}} = a^p$.

Therefore $a^{\frac{p}{q}}$, one of the q equal factors of a^p , is a q th root of the p th power of a .

Similarly, $a^{\frac{1}{q}}$ is a q th root of a .

Also, since $a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \cdots$ to p factors $= a^{\frac{1}{q} + \frac{1}{q} + \cdots \text{to } p \text{ terms}} = a^{\frac{p}{q}}$, $a^{\frac{p}{q}}$ is the p th power of a q th root of a .

The numerator of a fractional exponent with positive integral terms indicates a power and the denominator indicates a root.

The fraction as a whole indicates a root of a power or a power of a root.

The meaning of the fractional exponent just given is derived from law IV (page 253), when the restriction that m is a multiple of n is removed. Thus,

$$\sqrt[n]{a^m} = a^{m \div n} = a^{\frac{m}{n}}$$

Any fractional exponent that does not itself involve a root sign may be reduced to one of the forms $\frac{p}{q}$ or $-\frac{p}{q}$.

Thus, $8^{-\frac{3}{2}} = 8^{-\frac{3}{2}}$

But $a^{-\frac{p}{q}} = \frac{1}{a^{\frac{p}{q}}} = \frac{1^{\frac{p}{q}}}{a^{\frac{p}{q}}} = \left(\frac{1}{a}\right)^{\frac{p}{q}}$

EXERCISES

Find a simple value for:

1. 3^{-2} 3. $(-4)^{-3}$ 5. $\frac{3}{5^{-2}}$ 7. $(-\frac{1}{2})^{-3}$

2. 4^0 4. $(-8)^0$ 6. $\frac{3^{-2}}{4}$ 8. $(x^3y^{2n}z^m)^0$

9. Find the value of:

$2^3 - 3 \cdot 2^2 + 5 \cdot 2^1 - 7 \cdot 2^0 + 4 \cdot 2^{-1} - 2^{-2}$

10. Find the value of $x^2 - 3x^1 + 4x^0 + x^{-1} - 5x^{-2} + x^{-3}$ when $x = \frac{1}{2}$; when $x = -\frac{1}{2}$; when $x = 1$.

11. Which is the greater, $(-\frac{1}{3})^{-3}$ or $(\frac{1}{3})^3$? $(-\frac{1}{3})^{-4}$ or $(\frac{1}{3})^4$?

Write with negative exponents:

- | | | |
|------------------|---------------------|------------------------|
| 12. $1 \div 5$ | 15. $a \div x^3$ | 18. $c^2d \div a^2b^3$ |
| 13. $1 \div a^2$ | 16. $m \div bn^2$ | 19. $am^3 \div bx^n$ |
| 14. $1 \div 2^n$ | 17. $c \div a^2x^3$ | 20. $a^mc^x \div b^n$ |

Write with positive exponents:

- | | | |
|---|-----------------------|---------------------|
| 21. $3y^{-2}$ | 23. $xy^{-1}z^{-4}$ | 25. a^nb^{-3n} |
| 22. $2ax^{-1}$ | 24. $a^{-1}b^2c^{-3}$ | 26. $5a^2b^{-4}c^5$ |
| 27. $4x^3 - 2x^2 + 5x^1 - 6x^0 + 3x^{-1} - 5x^{-2}$ | | |
| 28. $a^3b^{-3} - a^2b^{-2} + ab^{-1} - 1 + a^{-1}b - a^{-2}b^2 + a^{-3}b^3$ | | |

Write without a denominator:

- | | | | |
|-----------------------------|------------------------------|------------------------|------------------------------------|
| 29. $\frac{c^2d^2}{a^2b^2}$ | 31. $\frac{1}{m^{-3}n}$ | 33. $\frac{4}{x^{-4}}$ | 35. $\left(\frac{a}{b^2}\right)^3$ |
| 30. $\frac{a^2m^3}{b^3n^2}$ | 32. $\frac{3a^2c^{-2}}{x^2}$ | 34. $\frac{x^{-1}}{y}$ | 36. $\frac{a^{-3}}{b^{-3}}$ |
37. Find the value of $27^{\frac{2}{3}}$.

SOLUTION

$$27^{\frac{2}{3}} = (\sqrt[3]{27})^2 = 3^2 = 9$$

Simplify, taking only principal roots:

- | | | | |
|-----------------------|------------------------|--------------------------------------|----------------------------|
| 38. $4^{\frac{1}{2}}$ | 40. $16^{\frac{1}{4}}$ | 42. $(-8)^{-\frac{1}{3}}$ | 44. $81^{.75}$ |
| 39. $4^{\frac{3}{2}}$ | 41. $16^{\frac{3}{4}}$ | 43. $(15\frac{5}{8})^{-\frac{2}{3}}$ | 45. $(-64)^{-\frac{2}{3}}$ |
46. Which is the greater, $64^{-\frac{2}{3}}$ or $(\frac{1}{64})^{\frac{2}{3}}$? $64^{\frac{2}{3}}$ or $(\frac{1}{64})^{-\frac{2}{3}}$?
47. Find the value of $x^{-\frac{4}{3}} - 4x^{-\frac{2}{3}} + 4$ when $x = -\frac{1}{125}$.
48. Express $\sqrt[3]{x^{-1}y^{-4}z^2}$ with positive fractional exponents.

SOLUTION

$$\sqrt[3]{x^{-1}y^{-4}z^2} = x^{-\frac{1}{3}}y^{-\frac{4}{3}}z^{\frac{2}{3}} = \frac{z^{\frac{2}{3}}}{x^{\frac{1}{3}}y^{\frac{4}{3}}}$$

Express with positive fractional exponents :

49. $\sqrt{cd}$ 51. $(\sqrt{xy})^3$ 53. $\sqrt{x^3y^{-1}z^{-3}}$
 50. $\sqrt[3]{ab}$ 52. $(\sqrt[3]{mn})^4$ 54. $3\sqrt[3]{a^{-2}b^xc^{-1}}$

Express roots with radical signs and powers with positive exponents :

55. $x^{\frac{1}{2}}$ 57. $a^{\frac{1}{2}}b^{\frac{3}{2}}$ 59. $m^{\frac{3}{5}}n^{\frac{4}{5}}$ 61. $a^{\frac{2}{3}} \div b^{\frac{1}{3}}$
 56. $a^{\frac{3}{2}}$ 58. $x^{\frac{2}{3}}y^{\frac{1}{3}}$ 60. $a^{\frac{2}{3}}b^{-\frac{4}{3}}$ 62. $c^{\frac{1}{4}} \div d^{\frac{5}{4}}$

63. Simplify $\sqrt[3]{x^2} + x^{\frac{1}{3}} + 8^{\frac{2}{3}} + 3x^{\frac{2}{3}} - 5\sqrt[3]{x} - \sqrt[3]{27^2}$.

64. Simplify $4\sqrt[5]{x} + 5x^0 - 3x^{-\frac{1}{5}} + 2\sqrt[5]{x^{-1}} - 8^{-\frac{2}{3}} - 2x^{\frac{1}{5}}$.

Operations involving positive, negative, zero, and fractional exponents

Since zero, negative, and fractional exponents have been defined in conformity with the law of exponents for multiplication, this law holds true for all exponents so far encountered.

EXERCISES

Multiply :

1. a by a^{-4} 3. $a^{\frac{1}{2}}b^{\frac{1}{3}}$ by $a^{\frac{1}{2}}b^{\frac{2}{3}}$ 5. n^{-2} by $an^{\frac{5}{2}}$
 2. a^2 by a^0 4. $m^{\frac{2}{5}}n$ by $m^{\frac{3}{5}}n^{-1}$ 6. a^{m-n} by a^{n-p}
 7. $x^{\frac{4}{5}}y^{-\frac{1}{5}} + x^{\frac{3}{5}} + x^{\frac{2}{5}}y^{\frac{1}{5}} + x^{\frac{1}{5}}y^{\frac{2}{5}} + y^{\frac{3}{5}}$ by $x^{\frac{1}{5}}y^{\frac{1}{5}}$
 8. $y^n + x^{-1}y^{n+1} + x^{-2}y^{n+2} + x^{-3}y^{n+3}$ by x^ny^{-n}
 9. Expand $(a^{\frac{1}{5}}b^{-\frac{1}{2}} + 1 + a^{-\frac{1}{5}}b^{\frac{1}{2}})(a^{\frac{1}{5}}b^{-\frac{1}{2}} - 1 + a^{-\frac{1}{5}}b^{\frac{1}{2}})$.

SOLUTION

$$a^{\frac{1}{5}}b^{-\frac{1}{2}} + 1 + a^{-\frac{1}{5}}b^{\frac{1}{2}}$$

$$a^{\frac{1}{5}}b^{-\frac{1}{2}} - 1 + a^{-\frac{1}{5}}b^{\frac{1}{2}}$$

$$a^{\frac{2}{5}}b^{-1} + a^{\frac{1}{5}}b^{-\frac{1}{2}} + a^0b^0$$

$$- a^{\frac{1}{5}}b^{-\frac{1}{2}} - 1 - a^{-\frac{1}{5}}b^{\frac{1}{2}}$$

$$+ a^0b^0 + a^{-\frac{1}{5}}b^{\frac{1}{2}} + a^{-\frac{2}{5}}b$$

$$a^{\frac{2}{5}}b^{-1} + 1 + a^{-\frac{2}{5}}b$$

Expand :

10. $(a^{\frac{1}{2}} + b^{\frac{1}{2}})(a^{\frac{1}{2}} - b^{\frac{1}{2}})$

11. $(x^{\frac{2}{3}} + y^{\frac{2}{3}})(x^{\frac{2}{3}} - y^{\frac{2}{3}})$

12. $(x^{\frac{2}{5}} - 4)(x^{\frac{2}{5}} + 5)$

13. $(x^{\frac{2}{5}} - x^{\frac{1}{5}}y^{-\frac{1}{5}} + y^{-\frac{2}{5}})(x^{\frac{1}{5}} + y^{-\frac{1}{5}})$

14. $(1 - x + x^2)(x^{-3} + x^{-2} + x^{-1})$

15. $(a^{-1} + b^{-\frac{1}{2}} + c^{\frac{1}{2}})(a^{-1} - b^{-\frac{1}{2}} + 2c^{\frac{1}{2}})$

Divide :

16. n^3 by n^4

18. n^4 by n^{-4}

20. s^{-2} by $s^{-\frac{7}{2}}$

17. n^2 by n^0

19. $r^{\frac{3}{2}}$ by $r^{-\frac{5}{2}}$

21. x^{1-n} by x^{n-1}

22. $x^4 + x^2y^2 + y^4$ by x^2y^2

23. $a^{-4} - a^{-2}b + b^2$ by $a^{-2}b$

24. $x^4 - 2ax^3 + a^3x - a^4$ by a^4x^4

25. Divide $b^{-1} + 3a^{-\frac{1}{2}} - 10a^{-1}b$ by $a^{\frac{1}{2}}b^{-1} - 2$.

SOLUTION

$$\begin{array}{r}
 b^{-1} + 3a^{-\frac{1}{2}} - 10a^{-1}b \quad | \quad a^{\frac{1}{2}}b^{-1} - 2 \\
 \hline
 b^{-1} - 2a^{-\frac{1}{2}} \quad | \quad a^{-\frac{1}{2}} + 5a^{-1}b \\
 \hline
 5a^{-\frac{1}{2}} - 10a^{-1}b \\
 \hline
 5a^{-\frac{1}{2}} - 10a^{-1}b
 \end{array}$$

Divide :

26. $a - b$ by $a^{\frac{1}{3}} - b^{\frac{1}{3}}$

28. $x - 1$ by $x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1$

27. $a + b$ by $a^{\frac{1}{5}} + b^{\frac{1}{5}}$

29. $x^{\frac{4}{7}} - 2 + x^{-\frac{4}{7}}$ by $x^{\frac{2}{7}} - x^{-\frac{2}{7}}$

Simplify the following :

30. $(a^{\frac{1}{2}})^2$

34. $(-a^2)^4$

38. $\sqrt[3]{a^{-\frac{1}{2}}b^{-3}}$

31. $(a^{-\frac{1}{3}})^6$

35. $(-a^{\frac{4}{5}})^{-1}$

39. $(-\frac{1}{3^2}a^{10})^{-\frac{3}{5}}$

32. $(a^{-4})^2$

36. $\sqrt[4]{a^{-\frac{8}{3}}}$

40. $(\frac{1}{3^6}a^{-2}b^{\frac{1}{3}})^{-\frac{3}{2}}$

33. $(-a^{\frac{1}{2}})^3$

37. $\sqrt{x^{\frac{4}{3}}y^{-3}}$

41. $(\frac{1}{9}m^{-1}n^{-\frac{1}{2}})^{\frac{1}{2}}$

42. Factor $4x^{-2} - 9y^{-2}$, and express the result with positive exponents.

SOLUTION

$$4x^{-2} - 9y^{-2} = (2x^{-1} + 3y^{-1})(2x^{-1} - 3y^{-1}) = \left(\frac{2}{x} + \frac{3}{y}\right)\left(\frac{2}{x} - \frac{3}{y}\right)$$

Factor, expressing results with positive exponents:

43. $a^{-2} - b^{-2}$

46. $8x^2 - 18x^{-4}$

49. $a^2 + 2 + a^{-2}$

44. $9 - x^{-2}$

47. $27 - 48x^{-6}$

50. $b^4 - 8 + 16b^{-4}$

45. $16 - a^{-4}$

48. $25b^3 - by^{-2}$

51. $12 - x^{-1} - x^{-2}$

Simplify, expressing results with positive exponents:

52. $\left(\frac{2^{-2}}{2^{-3}}\right)^2$

57. $\left(\frac{a^4b^{-3}}{a^3b^{-2}}\right)^2$

62. $\frac{3a^{\frac{1}{4}} \times 4a^{-1}}{6\sqrt[4]{a^5}}$

53. $\left(\frac{2^{\frac{3}{4}}}{4^{\frac{1}{4}}}\right)^{-12}$

58. $\left(\frac{16m^{-\frac{2}{3}}}{r^{-4}}\right)^{-\frac{3}{4}}$

63. $\frac{6x^{-2} \div 3x^{-\frac{1}{2}}}{2x^{-3} \times \sqrt{x}}$

54. $\left(\frac{\sqrt{3^{\frac{1}{3}}}}{\sqrt[3]{2^{-2}}}\right)^{-6}$

59. $\left(\frac{m^{-2}n^{-\frac{2}{3}}}{x^{\frac{1}{6}}}\right)^{-3}$

64. $\frac{2^{-1} \times 2^{-2}}{4^{-2} \times 4^{-3}}$

55. $\left(\frac{9^{-3}}{x^{-4}y^{-2}}\right)^{-\frac{1}{2}}$

60. $\left(\frac{x^{-3} \div x^{-5}}{4x^{-4}}\right)^{-\frac{1}{2}}$

65. $\frac{32^{\frac{2}{3}} + 125^{\frac{2}{3}}}{81^{\frac{3}{4}} + 216^{\frac{1}{3}}}$

56. $\left(\frac{27^{-\frac{1}{3}}}{a}\right)^{-1}$

61. $\sqrt[3]{\frac{9^r \times 3^{2+r}}{27^r}}$

66. $\frac{\sqrt[3]{a^2} \times \sqrt{b^3}}{\sqrt[4]{b^6} \times \sqrt[6]{a^{-2}}}$

67. $\frac{x^2 - y^2}{(x^{\frac{1}{2}} - y^{\frac{1}{2}})(x^{\frac{1}{2}} + y^{\frac{1}{2}})}$

68. $\frac{-8^{\frac{2}{3}} + (-27)^{-\frac{2}{3}}}{(-8)^{-\frac{2}{3}} - 27^{\frac{2}{3}}}$

RADICALS

An indicated root of a number is called a **radical**; the number whose root is required is called the **radicand**.

$\sqrt{5a}$, $(x^6)^{\frac{1}{3}}$, $\sqrt[3]{a^2 + 2}$, and $(x + y)^{\frac{1}{4}}$ are radicals whose radicands are, respectively, $5a$, x^6 , $a^2 + 2$, and $x + y$. (See page 235.)

The **order** of a radical is shown by the index of the root or by the denominator of the fractional exponent.

$\sqrt{a + z}$ and $(b - x)^{\frac{1}{2}}$ are radicals of the second order.

In the discussion and treatment of radicals only *principal roots* will be considered.

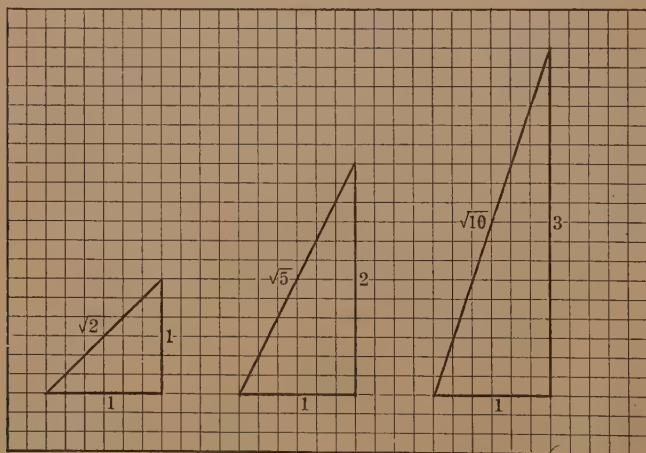
Thus, $\sqrt{16}$ will be taken to represent only the principal square root of 16, or 4. The other square root will be denoted by $-\sqrt{16}$.

Graphic representation of radicals of the second order

As you have learned, in a right triangle the square of the length of the longest side (the hypotenuse) is equal to the sum of the squares of the lengths of the other two sides.

It follows, then, that *the length of the hypotenuse of a right triangle is equal to the square root of the sum of the squares of the lengths of the two legs*.

Hence we may graphically represent a radical of the second order, as shown below.



EXERCISES

Represent graphically :

1. $\sqrt{26}$

3. $\sqrt{17}$

5. $\sqrt{\frac{1}{2}}$

2. $\sqrt{20}$

4. $\sqrt{32}$

6. $\sqrt{34}$

Rational and irrational numbers. Surds

A number that is, or can be, expressed as an integer or as a fraction with integral terms is called a **rational number**.

3 , $\frac{1}{2}$, $\sqrt{25}$, and $.33$ are rational numbers.

A number that cannot be expressed as an integer or as a fraction with integral terms is called an **irrational number**.

$\sqrt{2}$, $4\frac{1}{2}$, $1 + \sqrt{3}$, and $\sqrt{1 + \sqrt{3}}$ are irrational numbers.

It will be observed that the irrational number $\sqrt{5}$ can be represented graphically by a line of *exact* length, though it cannot be represented exactly by decimal figures, for $\sqrt{5} = 2.236 \dots$, which is an endless decimal.

When the indicated root of a *rational* number cannot be obtained exactly, the expression is called a **surd**.

$\sqrt{2}$ is a surd, since 2 is rational but has no rational square root.

$\sqrt{1 + \sqrt{3}}$ is irrational but is not a surd, because $1 + \sqrt{3}$ is not rational. Note that all surds are irrational; but some irrational numbers are not surds.

Radicals may be either rational or irrational. Both $\sqrt{4}$ and $\sqrt{3}$ are radicals; but $\sqrt{4}$ is rational while $\sqrt{3}$ is irrational.

A surd may contain a *rational factor*, that is, a factor whose radicand is a perfect power of a degree corresponding to the order of the surd.

The rational factor may be removed and written as the coefficient of the irrational factor.

In $\sqrt{8} = \sqrt{4 \times 2}$, the rational factor is $\sqrt{4}$; that is, $\sqrt{8} = 2\sqrt{2}$.
In $\sqrt[3]{54} = \sqrt[3]{27 \times 2}$, the rational factor is $\sqrt[3]{27}$; that is, $\sqrt[3]{54} = 3\sqrt[3]{2}$.

In the following pages it will be assumed that irrational numbers obey the same laws as rational numbers.

A surd that has a rational coefficient is called a **mixed surd**.

$2\sqrt{2}$, $a\sqrt[3]{x^2}$, and $(a + b)\sqrt{a + b}$ are mixed surds.

A surd that has no rational coefficient except unity is called an **entire surd**.

$\sqrt{5}$, $\sqrt[3]{11}$, and $\sqrt{a^2 + x^2}$ are entire surds.

REDUCTION OF RADICALS

A radical is *reduced* or *simplified* when the radicand contains no rational factor, when the radicand is integral, and when the index of the root is as small as possible.

To simplify a radical by writing the rational factor as the coefficient of the irrational factor

EXERCISES

1. Simplify $\sqrt{20 a^6}$.

SOLUTION

$$\sqrt{20 a^6} = \sqrt{4 a^6 \times 5} = \sqrt{4 a^6} \times \sqrt{5} = 2 a^3 \sqrt{5}$$

EXPLANATION. Since the highest factor of $20 a^6$ that is a square is $4 a^6$, $\sqrt{20 a^6}$ is separated into two factors, a rational factor, $\sqrt{4 a^6}$, and an irrational factor, $\sqrt{5}$; that is, $\sqrt{20 a^6} = \sqrt{4 a^6} \times \sqrt{5}$.

Finding the square root of $4 a^6$ and prefixing the root to the irrational factor as a coefficient gives the result $2 a^3 \sqrt{5}$.

2. Write the rational factor of $\sqrt[3]{-864}$ as the coefficient of the irrational factor.

SOLUTION

$$\sqrt[3]{-864} = \sqrt[3]{-216 \times 4} = \sqrt[3]{-216} \times \sqrt[3]{4} = -6 \sqrt[3]{4}$$

RULE. Separate the radical into two factors, one of which is its highest rational factor.

Find the required root of the rational factor, multiply the result by the coefficient, if any, of the given radical, and place the product as the coefficient of the irrational factor.

Simplify:

3. $\sqrt{12}$

5. $\sqrt[3]{16}$

7. $\sqrt[3]{250}$

4. $\sqrt{75}$

6. $\sqrt{128}$

8. $\sqrt{54}$

9. Simplify $2\sqrt{99}$.

SOLUTION

$$2\sqrt{9 \times 11} = 2\sqrt{9} \times \sqrt{11} = 2 \cdot 3\sqrt{11} = 6\sqrt{11}$$

Simplify:

10. $\sqrt{80}$

11. $\sqrt{125}$

12. $\sqrt{45}$

13. $3\sqrt{108}$

14. $4\sqrt{175}$

15. $2\sqrt{160}$

16. $3\sqrt{162}$

17. $5\sqrt{72}$

18. $2(98)^{\frac{1}{2}}$

19. $4(128)^{\frac{1}{2}}$

20. $3(180)^{\frac{1}{2}}$

21. $\sqrt[4]{32}$

22. $\sqrt{48 a^3 b}$

23. $(25 b)^{\frac{1}{2}}$

24. $\sqrt{98 c^3}$

25. $(50 a)^{\frac{1}{2}}$

26. $\sqrt{75 a^4 x^{10}}$

27. $(72 a^6 b^3)^{\frac{1}{3}}$

28. $\sqrt{20 a^4 b}$

29. $\sqrt{12(x-4)^2}$

30. $\sqrt[3]{x^6 - x^3}$

31. $\sqrt{16 a^2 - 48 a^2 b}$

32. $\sqrt{9 a b^2 - 27 b^3}$

33. $\sqrt{27 x^3 y^2 - 36 x^2 y^3}$

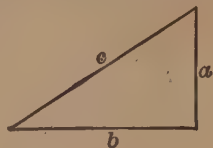
34. $\sqrt{12 a^2 b^2 - 28 a^2 b}$

35. $\sqrt{16 x^2 - 16}$

36. $\sqrt{36 a^2 + 72 a^3}$

37. In a right triangle (see p. 71), find the lengths of sides not given:

	a	b	c
(1)	10	4	?
(2)	3	6	?
(3)	?	7	25
(4)	5	?	18

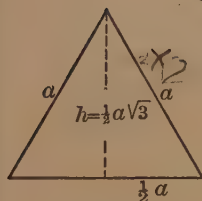


Simplify:

38. $\sqrt{by^2 + 4 by + 4 b}$

39. $\sqrt{5 x^2 - 10 xy + 5 y^2}$

40. $(27 c^6 - 27 c^3)^{\frac{1}{3}}$

EQUILATERAL
TRIANGLE

The figure at the left is an equilateral triangle; that is, all the sides are equal.

If a represents one side, $\frac{1}{2} a$ or $\frac{a}{2}$ represents

$\frac{1}{2}$ the side. The altitude $h = \sqrt{a^2 - \left(\frac{a}{2}\right)^2}$

$$= \sqrt{a^2 - \frac{a^2}{4}} = \sqrt{\frac{3 a^2}{4}} = \frac{a}{2} \sqrt{3}. \text{ In an equi-}$$

lateral triangle, the length of the altitude is always one half the length of a side multiplied by $\sqrt{3}$. Or, by the formula: $h = \frac{a}{2} \sqrt{3}$.

41. Find the length of the altitude of the following equilateral triangles, the length of a side of each being as follows :

$$12 \quad 24 \quad 15 \quad \frac{2}{3} \quad m \quad 4m \quad a+b$$

42. In the formula $s = \frac{1}{2}at^2$, if $s = 12$ $a = 3$, find the value of t .

43. In $a = \frac{y^2}{r}$, if $a = 6$, $r = 4$, find the value of y .

44. In the formula $F = \frac{mv^2}{r}$, if $F = 40$, $m = 12$, $r = 6$, find the value of v .

45. In the formula $A = \pi r^2$, if $A = 250\pi$, find r .

46. In $V = \frac{1}{3}\pi r^2 h$, if $V = 120\pi$ and $h = 6$, find r .

To simplify a radical having a fractional radicand by reducing it to a radical having an integral radicand

ILLUSTRATIVE EXAMPLE

Simplify $\sqrt{\frac{a^2}{2y^3}}$

SOLUTION

$$\sqrt{\frac{a^2}{2y^3}} = \sqrt{\frac{a^2 \times 2y}{2y^3 \times 2y}} = \sqrt{\frac{a^2}{4y^4}} \times \sqrt{2y} = \frac{a}{2y^2} \sqrt{2y}$$

RULE. Multiply the numerator and the denominator of the radical by the smallest factor which will make the denominator a power of a degree corresponding to the order of the surd.

EXERCISES

Simplify by reducing to radicals with integral radicands :

- | | | |
|-----------------------------|--------------------------------|--|
| 1. $\sqrt{\frac{1}{2}}$ | 8. $\sqrt{\frac{8}{7}}$ | 13. $(a+b)\sqrt{\frac{a+b}{a-b}}$ |
| 2. $\sqrt{\frac{2}{3}}$ | 9. $4\sqrt{\frac{x^2}{2}}$ | 14. $\frac{2y}{x-2y}\sqrt{\frac{x-2y}{2y}}$ |
| 3. $\sqrt{\frac{3}{8}}$ | 10. $6a\sqrt{\frac{2}{3}}$ | 15. $\frac{(a+b)^2}{a-b}\sqrt[3]{\frac{a+b}{(a-b)^2}}$ |
| 4. $\sqrt[3]{\frac{4}{5}}$ | 11. $12\sqrt{\frac{4x^2}{3}}$ | |
| 5. $\sqrt[3]{\frac{5}{12}}$ | 12. $20\sqrt{\frac{a}{27b^2}}$ | |
| 6. $\sqrt{\frac{2}{5}}$ | | |
| 7. $\frac{4}{\sqrt{6}}$ | | |

To reduce a mixed surd to an entire surd

EXERCISES

1. Express $2a\sqrt{5b}$ as an entire surd.

SOLUTION

$$2a\sqrt{5b} = \sqrt{4a^2}\sqrt{5b} = \sqrt{4a^2 \times 5b} = \sqrt{20a^2b}$$

RULE. Raise the coefficient to a power corresponding to the index of the given radical and introduce the result under the radical sign as a factor.

Express as entire surds:

- | | | | |
|-------------------|-----------------------------|--------------------------------|---|
| 2. $2\sqrt{2}$ | 6. $4\sqrt{5}$ | 9. $\frac{3x}{4}\sqrt{x^3}$ | 11. $\frac{4a}{5b}\sqrt{4\frac{3}{8}}$ |
| 3. $3\sqrt{5}$ | 7. $\frac{1}{2}\sqrt{8}$ | | |
| 4. $5\sqrt{2}$ | 8. $\frac{a}{2}\sqrt[3]{2}$ | 10. $\frac{1}{2b}\sqrt[4]{bc}$ | 12. $\frac{3}{2}\sqrt[3]{\frac{3}{2}a^2}$ |
| 5. $3\sqrt[3]{3}$ | | | 13. $\frac{2}{3}\sqrt[3]{1\frac{1}{8}}$ |

To reduce radicals to the same order

EXERCISES

1. Reduce $\sqrt[4]{3}$, $\sqrt{2}$, and $\sqrt[3]{4}$ to radicals of the same order.

SOLUTION

$$\begin{aligned}\sqrt[4]{3} &= 3^{\frac{1}{4}} = 3^{\frac{3}{12}} = \sqrt[12]{3^3} = \sqrt[12]{27} \\ \sqrt{2} &= 2^{\frac{1}{2}} = 2^{\frac{6}{12}} = \sqrt[12]{2^6} = \sqrt[12]{64} \\ \sqrt[3]{4} &= 4^{\frac{1}{3}} = 4^{\frac{4}{12}} = \sqrt[12]{4^4} = \sqrt[12]{256}\end{aligned}$$

RULE. Express the given radicals with fractional exponents having the lowest common denominator.

Raise each number to the power indicated by the numerator of its fractional exponent, and indicate the root expressed by the common denominator.

Reduce to radicals of the same order:

- | | |
|--|--|
| 2. $\sqrt{2}$ and $\sqrt[4]{3}$ | 7. $\sqrt[10]{13}$, $\sqrt{5}$, and $\sqrt[5]{4}$ |
| 3. $\sqrt{5}$ and $\sqrt[3]{6}$ | 8. $\sqrt{3}$, $\sqrt[3]{5}$, and $\sqrt[4]{\frac{1}{2}}$ |
| 4. $\sqrt[4]{7}$ and $\sqrt{10}$ | 9. $\sqrt{ab}$, $\sqrt[3]{ab^2}$, and $\sqrt[4]{2}$ |
| 5. $\sqrt[6]{10}$, $\sqrt{2}$, and $\sqrt[3]{5}$ | 10. $\sqrt{a}$, $\sqrt[3]{b}$, $\sqrt[4]{x}$, and $\sqrt[6]{y}$ |
| 6. $\sqrt[6]{4}$, $\sqrt[4]{2}$, and $\sqrt{3}$ | 11. $\sqrt[3]{a+b}$ and $\sqrt{x+y}$ |

12. Which is greater, $\sqrt[5]{5}$ or $\sqrt{2}$? $\sqrt[3]{4}$ or $\sqrt{3}$?
 13. Which is greater, $\sqrt[3]{3}$ or $\sqrt[4]{4}$? $3\sqrt{2}$ or $3\sqrt[3]{4}$?

Arrange in order of ascending value :

14. $\sqrt[3]{3}$, $\sqrt{2}$, $\sqrt[6]{7}$ 17. $\sqrt{2}$, $\sqrt[3]{5}$, $\sqrt{2\frac{1}{2}}$, $\sqrt[3]{4}$
 15. $\sqrt{2}$, $\sqrt[3]{4}$, $\sqrt[4]{5}$ 18. $\sqrt{7}$, $\sqrt[4]{48}$, $\sqrt[3]{4}$, $\sqrt[6]{63}$
 16. $\sqrt{2}$, $\sqrt[5]{3}$, $\sqrt[15]{30}$ 19. $\sqrt[3]{4}$, $\sqrt[4]{5}$, $\sqrt[6]{13}$, $\sqrt[12]{150}$

ADDITION AND SUBTRACTION OF RADICALS

Radicals which when reduced are of the same order (see page 265) and have the same radicand are called **similar radicals**.

Thus, $2\sqrt{3}$ and $7\sqrt{3}$ are similar radicals.

PRINCIPLE. *Only similar radicals can be united into one term by addition or subtraction.*

EXERCISES

1. Find the sum of $\sqrt{50}$, $\sqrt{8}$, and $6\sqrt{\frac{1}{2}}$.

SOLUTION

$$\sqrt{50} = 5\sqrt{2}$$

$$\sqrt{8} = 2\sqrt{2}$$

$$6\sqrt{\frac{1}{2}} = 3\sqrt{2}$$

$$\text{Sum} = 10\sqrt{2}$$

EXPLANATION. Since the radicals when reduced are of the same order and have the same radicand, they are similar, and their sum is obtained by prefixing the sum of the coefficients to the common radical factor.

Find the sum of :

2. $\sqrt{50}$, $\sqrt{18}$, $\sqrt{98}$

3. $\sqrt{27}$, $\sqrt{12}$, $\sqrt{75}$

4. $\sqrt{20}$, $\sqrt{80}$, $\sqrt{45}$

5. $\sqrt{8a^3b}$, $\sqrt{18ab^3}$

6. $\sqrt{75a^4b}$, $\sqrt{48b^3c^2}$

7. $\sqrt{28}$, $\sqrt{63}$, $\sqrt{700}$

8. $\sqrt[3]{250}$, $\sqrt[3]{16}$, $\sqrt[3]{54}$

9. $\sqrt[3]{128}$, $\sqrt[3]{686}$, $\sqrt[3]{\frac{1}{4}}$

10. $\sqrt[3]{135}$, $\sqrt[3]{320}$, $\sqrt[3]{625}$

11. $\sqrt[3]{500}$, $\sqrt[3]{108}$, $\sqrt[3]{-32}$

Find the sum of:

$$(12.) \sqrt{\frac{25 a^3}{8}}, \frac{1}{4} \sqrt{98 a^3}$$

$$13. 4 \sqrt{\frac{x^2}{2}}, \frac{1}{12} \sqrt{288 x^2}$$

$$(14.) 3 \sqrt{2(a-2)^2}, 3 \sqrt{18 x^2}$$

$$15. \sqrt{\frac{9 a^2}{625}}, 6 \sqrt{\frac{a^4}{3}}$$

$$(16.) \sqrt{18 a}, 2 \sqrt{\frac{a}{2}}$$

$$17. 5 \sqrt{50 a}, 7 \sqrt{\frac{49 a}{2}}$$

$$(18.) \sqrt{\frac{1}{2}}, \sqrt{12 \frac{1}{2}}, \sqrt{\frac{1}{8}}, \sqrt{1 \frac{1}{8}}$$

$$19. \sqrt{\frac{1}{3}}, \sqrt{75}, \frac{2}{3} \sqrt{3}, \sqrt{12}$$

$$(20.) \sqrt{\frac{3}{4}}, \frac{1}{3} \sqrt{3}, \frac{7}{18} \sqrt{27}, \sqrt{147}$$

$$21. \sqrt[3]{40}, \sqrt{28}, \sqrt[3]{5}, \sqrt{175}$$

$$(22.) \frac{2}{7} \sqrt{147 a^2 b} - 9 \sqrt{\frac{49 b}{3}} + 4 a \sqrt{\frac{3 b}{2}}$$

$$23. \frac{1}{3} \sqrt{72 a^2 b} + 7 a \sqrt{\frac{9 b}{8}} - \sqrt{2 b(5 b - 2)^2}$$

Unite:

$$(24.) \sqrt{245} - \sqrt{405} + \sqrt{45}$$

$$25. \sqrt{12} + 3 \sqrt{75} - 2 \sqrt{27}$$

$$(26.) 5 \sqrt{72} + 3 \sqrt{18} - \sqrt{50}$$

$$27. \sqrt[3]{128} + \sqrt[3]{686} - \sqrt[3]{54}$$

$$28. \sqrt{112} - \sqrt{343} + \sqrt{448}$$

$$29. \sqrt{(a+b)^2 c} - \sqrt{(a-b)^2 c}$$

$$30. \sqrt[3]{abx} - \sqrt[3]{125 abx} + \sqrt[3]{512 abx}$$

$$31. 4 \sqrt{3 a} + 2 \sqrt{12 a} + \sqrt{75 a}$$

$$32. 2 \sqrt{2(a+b)} + \sqrt{18(a+b)} + \sqrt{32(a+b)}$$

$$33. \sqrt{\frac{a}{x^2}} + \sqrt{\frac{a}{y^2}} - \sqrt{\frac{a}{z^2}}$$

$$35. \sqrt[3]{\frac{8}{5}} + \sqrt[3]{\frac{1}{5}} + \sqrt[3]{5 \frac{2}{5}}$$

$$34. \sqrt{\frac{a}{bc}} + \sqrt{\frac{b}{ac}} + \sqrt{\frac{c}{ab}}$$

$$36. \frac{x+y}{x-y} \sqrt{\frac{x-y}{x+y}} + \sqrt{9(x^2-y^2)}$$

MULTIPLICATION OF RADICALS

$$a^{\frac{1}{2}} \times a^{\frac{1}{3}} = a^{\frac{1}{2} + \frac{1}{3}} = a^{\frac{3}{6} + \frac{2}{6}} = a^{\frac{5}{6}}$$

$$\text{That is, } \sqrt{a} \times \sqrt[3]{a} = \sqrt[6]{a^3} \times \sqrt[6]{a^2} = \sqrt[6]{a^5}$$

Since fractional exponents to be united by addition must be expressed with a common denominator, radicals to be united by multiplication must be expressed with a common root index; that is, they must be radicals of the same order.

EXERCISES

1. Multiply (a) $\sqrt{7}$ by $\sqrt{5}$; (b) $5\sqrt{2}$ by $2\sqrt[3]{5}$.

SOLUTIONS

$$(a) \sqrt{7} \times \sqrt{5} = \sqrt{7 \times 5} = \sqrt{35}$$

$$(b) 5\sqrt{2} \times 2\sqrt[3]{5} = 10\sqrt[6]{8} \times 25 = 10\sqrt[6]{200}$$

RULE. Multiply the coefficients for the coefficient of the product and the radicands for the radical factor of the product. Reduce the result to the simplest form.

Multiply :

2. $\sqrt{2}$ by $\sqrt{8}$

6. $3\sqrt{20}$ by $2\sqrt{2}$

3. $\sqrt{2}$ by $\sqrt[3]{6}$

7. $2\sqrt[3]{3}$ by $3\sqrt{45}$

4. $\sqrt[3]{3}$ by $\sqrt{15}$

8. $2\sqrt{24}$ by $\sqrt[3]{18}$

5. $2\sqrt{5}$ by $3\sqrt{10}$

9. $5\sqrt[3]{5}$ by $\sqrt[3]{25}$

10. Find the value of $\sqrt{\frac{2}{3}} \times \sqrt{\frac{4}{5}} \times \sqrt{\frac{3}{4}}$.

11. Multiply $2\sqrt{2} + 3\sqrt{3}$ by $5\sqrt{2} - 2\sqrt{3}$.

SOLUTION

$$2\sqrt{2} + 3\sqrt{3}$$

$$5\sqrt{2} - 2\sqrt{3}$$

$$\hline 20 \quad + 15\sqrt{6}$$

$$- 4\sqrt{6} - 18$$

$$\hline 20 \quad + 11\sqrt{6} - 18 = 2 + 11\sqrt{6}$$

Multiply :

12. $\sqrt{5} + \sqrt{3}$ by $\sqrt{5} - \sqrt{3}$

17. $\sqrt{6} - \sqrt{5}$ by $\sqrt{6} - \sqrt{5}$

13. $\sqrt{7} + \sqrt{2}$ by $\sqrt{7} - \sqrt{2}$

18. $5 - \sqrt{5}$ by $1 + \sqrt{5}$

14. $(\sqrt{6} - 2\sqrt{5})^2$

19. $4\sqrt{7} + 1$ by $4\sqrt{7} - 1$

15. $(\sqrt[3]{2} + 3\sqrt[3]{3})^2$

20. $2\sqrt{6} + 3\sqrt{3}$ by $3\sqrt{3} - 5\sqrt{6}$

16. $(\sqrt{5} - 2\sqrt{3})(\sqrt{5} + 2\sqrt{3})$

21. $3a + \sqrt{5}$ by $2a - \sqrt{5}$

Find the indicated products:

22. $(x - \sqrt{xyz} + yz)(\sqrt{x} + \sqrt{yz})$ 28. $(6 + 2\sqrt{5})^2$
 23. $(x\sqrt{x} - y\sqrt{y})(\sqrt{x} + \sqrt{y})$ 29. $(3 - \sqrt{2a - 5})^2$
 24. $(\sqrt{x+1} + 2)^2(\sqrt{x+1} - 2)^2$ 30. $(3a\sqrt{b} - \sqrt{a})^2$
 25. $(\sqrt{2} + \sqrt{3} + \sqrt{\frac{2}{3}})(\sqrt{6} - \sqrt{2})$ 31. $(2\sqrt{5} + \sqrt{3})^2$
 26. $(3\sqrt{\frac{5}{3}} - 2\sqrt{\frac{3}{5}})(2\sqrt{\frac{3}{5}} + 3\sqrt{\frac{5}{3}})$ 32. $(2\sqrt{a} + \sqrt{3b})^2$
 27. $(a^2 - b^2)^{\frac{1}{2}}(a^2 + 3ab + 2b^2)^{\frac{1}{2}}$ 33. $(\sqrt{\frac{3}{2}} - \sqrt{\frac{2}{3}})^2$
 34. $(\sqrt{20a} + \sqrt[3]{4a^2})(\sqrt{20a} - \sqrt[3]{4a^2})$
 35. $(2\sqrt{3} - 3\sqrt{2} + \sqrt{5})(3\sqrt{3} - 2\sqrt{2} - \sqrt{5})$
 36. $(5\sqrt{\frac{1}{2}} - \sqrt{\frac{1}{8}} + \sqrt{\frac{4}{3}})(5\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{8}} - \sqrt{\frac{4}{3}})$
 37. $\left(2a - \sqrt{\frac{b}{a}}\right)^2(\sqrt{2a} + b)(\sqrt{2a} - b)$
 38. $(\sqrt{5} - \sqrt{2})(2\sqrt{5} + 3\sqrt{2})(4 - \sqrt{10})$
 39. $\left(a + \frac{b}{2} + \sqrt{\frac{b^2}{4} - c^2}\right)\left(a + \frac{b}{2} - \sqrt{\frac{b^2}{4} - c^2}\right)$
 40. $(a^2 + 3ab - 4b^2)^{\frac{1}{2}}(a^2 + 2ab - 8b^2)^{\frac{1}{2}}$
 41. $(2x^2 + 7xy - 15y^2)^{\frac{1}{2}}(4x^2 - 9y^2)^{\frac{1}{2}}$
 42. $(2x - 5y)^{\frac{1}{2}}(3x + 2y)^{\frac{1}{2}}(6x^2 + xy - 2y^2)^{\frac{1}{2}}$

DIVISION OF RADICALS

$$a^{\frac{1}{2}} \div a^{\frac{1}{3}} = a^{\frac{1}{2} - \frac{1}{3}} = a^{\frac{3}{6} - \frac{2}{6}} = a^{\frac{1}{6}}$$

That is, $\sqrt{a} \div \sqrt[3]{a} = \sqrt[6]{a^3} \div \sqrt[6]{a^2} = \sqrt[6]{a^3 \div a^2} = \sqrt[6]{a}$

In division, when one fractional exponent is subtracted from another, the exponents must be expressed with a common denominator. When one radical is divided by another, the radicals must be expressed with a common root index; that is, they must be radicals of the same order.

EXERCISES

1. Divide $10\sqrt{60}$ by $5\sqrt{12}$.

SOLUTION

$$10\sqrt{60} \div 5\sqrt{12} = \frac{10}{5}\sqrt{60} \div \sqrt{12} = 2\sqrt{5}$$

2. Divide $\sqrt[3]{2}$ by $\sqrt{2}$.

SOLUTION

$$\sqrt[3]{2} \div \sqrt{2} = \sqrt[3]{4} \div \sqrt[3]{8} = \sqrt[3]{\frac{4}{8}} = \sqrt[3]{\frac{1}{2}} = \sqrt[3]{\frac{32}{64}} = \frac{1}{2}\sqrt[3]{32}$$

RULE. To the quotient of the coefficients annex the quotient of the radicands written under the common radical sign. Simplify the resulting radicand, if possible.

Find the quotients:

- | | | | |
|--|--|--|---|
| 3. $\frac{\sqrt{50}}{\sqrt{8}}$ | 8. $\frac{2\sqrt[4]{100}}{4\sqrt{5}}$ | 13. $\frac{5a}{\sqrt[3]{6b}}$ | 18. $\frac{\sqrt{54ab^3}}{\sqrt{3a^3}}$ |
| 4. $\frac{\sqrt{72}}{2\sqrt{6}}$ | 9. $\frac{2\sqrt{5}}{\sqrt[3]{5}}$ | 14. $\frac{12}{\sqrt[3]{18a^2}}$ | 19. $\frac{\sqrt{125a^3}}{\sqrt{15a}}$ |
| 5. $\frac{4\sqrt{5}}{\sqrt{40}}$ | 10. $\frac{\sqrt{\frac{1}{4}}}{\sqrt[6]{\frac{1}{8}}}$ | 15. $\frac{\sqrt[3]{\frac{1}{4}}}{\sqrt{\frac{1}{8}}}$ | 20. $\frac{\sqrt{\frac{1}{2}}}{\sqrt{\frac{3}{8}}}$ |
| 6. $\frac{6\sqrt{7}}{\sqrt{126}}$ | 11. $\frac{\sqrt[3]{\frac{3}{4}}}{\frac{1}{2}\sqrt[6]{\frac{4}{5}}}$ | 16. $\frac{\sqrt{3}-2}{\sqrt{3}}$ | 21. $\frac{\sqrt[3]{8a^{2n+1}}}{\sqrt{3a}}$ |
| 7. $\frac{7\sqrt{75}}{5\sqrt{28}}$ | 12. $\frac{3\sqrt[3]{\frac{2}{3}}}{\sqrt{\frac{3}{4}}}$ | 17. $\frac{\sqrt{32}}{3\sqrt{2}}$ | 22. $\frac{3\sqrt{15}}{2\sqrt{10}}$ |
| 23. $\frac{\sqrt{a-b}}{\sqrt{a+b}}$ | 25. $\frac{\sqrt{30}-\sqrt{5}}{\sqrt{5}}$ | 27. $\frac{\sqrt{8}-\sqrt{6}}{\sqrt{2}}$ | |
| 24. $\frac{\sqrt[3]{54}}{\sqrt{2}}$ | 26. $\frac{6-3\sqrt{2}+\sqrt{12}}{\sqrt{3}}$ | 28. $\frac{\sqrt{a^3x}+\sqrt{ax^3}}{\sqrt{ax}}$ | |
| 29. $\frac{3\sqrt{15}}{\sqrt{27}} \times \frac{5\sqrt{32}}{6\sqrt{105}} \div \frac{5\sqrt{6}}{2\sqrt{63}}$ | 31. $\frac{3\sqrt{14}\sqrt{21}}{\sqrt{2}} \div \frac{8\sqrt{12}}{\sqrt{2}}$ | | |
| 30. $\frac{\sqrt{\frac{2}{3}(a^2-b^2)(a+b)}}{\sqrt{\frac{3}{2}(a-b)}}$ | 32. $\frac{(\sqrt{3}+5\sqrt{2})(2\sqrt{3}+3\sqrt{2})}{(2\sqrt{3}-\sqrt{2})(2\sqrt{3}+\sqrt{2})}$ | | |

REVIEW EXERCISES IN RADICALS

Simplify :

1. $2\sqrt{\frac{5a}{3b}} - \sqrt{60ab}$

7. $a\left(\sqrt{\frac{x}{a}} - x\sqrt{\frac{a}{x}}\right)^2$

2. $\frac{1}{3}\sqrt{90a^2b} - 4\sqrt{\frac{5b}{8}}$

8. $\sqrt{\frac{4}{75}} + 6\sqrt{\frac{1}{3}} - \sqrt{12}$

3. $\sqrt{(a+b)(a^2-b^2)}$

9. $b\sqrt{a} - a^2b^2\sqrt{\frac{1}{ab^2}} + \sqrt{a^3b^2}$

4. $(\sqrt{1-2x} + 1)^2$

10. $(5\sqrt{5} - 4)(5\sqrt{5} + 8)$

5. $(2\sqrt{x} - x\sqrt{2})^2$

11. $(3\sqrt{2} - 2\sqrt{3})(4\sqrt{2} + 3\sqrt{3})$

6. $(2 + \sqrt{x-1})^2$

12. $(7\sqrt{2} - \sqrt{5})(7\sqrt{2} + \sqrt{5})$

13. $\sqrt{9} + \sqrt{16} - \sqrt[3]{8} - \sqrt[4]{81} + \sqrt{144}$

14. $\sqrt[4]{16a^4} + \sqrt[3]{27a} - \sqrt{81a^2} - \sqrt[3]{216a^4}$

15. What is the value of $8x^2 - 6ax$ when $x = \frac{a - \sqrt{b}}{2}$?

16. If $x = 2\sqrt{5} - 1$, find the value of $x^2 + 2x - 4$.

17. Find the value of $\sqrt{a^2bc^3d}$ when $a = 7, b = 27, c = 25, d = 12$.

Find the quotients :

18. $(4\sqrt[5]{7} - 8\sqrt[5]{21} + 6\sqrt[5]{42}) \div 2\sqrt[5]{7}$

19. $\frac{(3\sqrt{5} - 2\sqrt{2})(2\sqrt{5} + 4\sqrt{2})}{2\sqrt{2} \times 2\sqrt{8}}$

RATIONALIZATION

We may obtain a decimal approximately equal to $\frac{1}{\sqrt{3}}$ by dividing 1 by 1.732 ... (see Table, page 251); but a great saving of labor may be effected by first changing the fraction to an equal fraction having a *rational* denominator, thus :

$$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = \frac{1.732}{3} = .577$$

The process of multiplying an expression containing a surd by any number that will make the product rational is called **rationalization**.

The factor by which a surd expression is multiplied to render the product rational is called the **rationalizing factor**.

The process of reducing a fraction having an irrational denominator to an equal fraction having a rational denominator is called **rationalizing the denominator**.

ILLUSTRATIVE EXAMPLES

Rationalize the denominator of (1) $\frac{5}{\sqrt{2}}$; (2) $\frac{1}{\sqrt[3]{a}}$;
(3) $\frac{1}{\sqrt[3]{a^2}}$.

SOLUTIONS

$$1. \frac{5}{\sqrt{2}} = \frac{5}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$$

$$2. \frac{1}{\sqrt[3]{a}} = \frac{1}{\sqrt[3]{a}} \times \frac{\sqrt[3]{a^2}}{\sqrt[3]{a^2}} = \frac{\sqrt[3]{a^2}}{\sqrt[3]{a^3}} = \frac{1}{a} \sqrt[3]{a^2}$$

$$3. \frac{1}{\sqrt[3]{a^2}} = \frac{1}{\sqrt[3]{a^2}} \times \frac{\sqrt[3]{a}}{\sqrt[3]{a}} = \frac{\sqrt[3]{a}}{\sqrt[3]{a^3}} = \frac{1}{a} \sqrt[3]{a}$$

EXERCISES

Rationalize the denominator of each of the following, and find the value to the nearest tenth (see Table, page 251):

$$1. \sqrt{\frac{2}{3}}$$

$$3. \frac{3}{\sqrt{6}}$$

$$5. \sqrt{\frac{9}{14}}$$

$$7. \frac{15}{\sqrt{50}}$$

$$9. \frac{1}{\sqrt[3]{16}}$$

$$2. \frac{2}{\sqrt{5}}$$

$$4. \frac{6}{\sqrt{8}}$$

$$6. \sqrt{\frac{7}{45}}$$

$$8. \frac{1}{\sqrt{125}}$$

$$10. \frac{1}{\sqrt[3]{5^2}}$$

Rationalize the denominator of each of the following, and express the result in the simplest radical form:

$$11. \frac{\sqrt{a}}{\sqrt{b}}$$

$$14. \frac{\sqrt{63x}}{\sqrt{8b^2}}$$

$$17. \frac{\sqrt[3]{a}}{\sqrt[3]{ax^2}}$$

$$20. \frac{\sqrt{a+b}}{\sqrt{a-b}}$$

$$12. \frac{1}{\sqrt{x^7}}$$

$$15. \frac{2\sqrt{a}}{\sqrt{by}}$$

$$18. \sqrt{\frac{x^2y}{3xy^2}}$$

$$21. \sqrt{\frac{x-2}{x+2}}$$

$$13. \frac{ax}{\sqrt{2a^3x}}$$

$$16. \frac{\sqrt[3]{6}}{\sqrt[3]{12}}$$

$$19. \frac{\sqrt{r-1}}{\sqrt{r+1}}$$

$$22. \sqrt{\frac{a-b}{a^2-b^2}}$$

(The work from this point to page 279 may be postponed until after Chapter XV without interrupting the sequence.)

A binomial of which one term or both terms are surds is called a **binomial surd**.

$2 + \sqrt{5}$, $\sqrt{2} + \sqrt{5}$, $\sqrt[3]{2} + 1$, and $\sqrt{3} - \sqrt[3]{2}$ are binomial surds.

A binomial surd whose surd or surds are of the second order is called a **binomial quadratic surd**.

$\sqrt{2} + \sqrt{5}$ and $a + \sqrt{b}$ are binomial quadratic surds.

Two binomial quadratic surds that differ only in the sign of one of the terms are called **conjugate surds**.

$3 + \sqrt{5}$ and $3 - \sqrt{5}$ are conjugate surds; also $\sqrt{3} + \sqrt{2}$ and $\sqrt{3} - \sqrt{2}$.

Since $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 = a - b$, the product of any two conjugate surds is rational. Hence

PRINCIPLE. *A binomial quadratic surd may be rationalized by multiplying it by its conjugate.*

EXERCISES

1. Rationalize the denominator of $\frac{\sqrt{7} - \sqrt{3}}{\sqrt{7} + \sqrt{3}}$.

SOLUTION

$$\frac{\sqrt{7} - \sqrt{3}}{\sqrt{7} + \sqrt{3}} = \frac{(\sqrt{7} - \sqrt{3})(\sqrt{7} - \sqrt{3})}{(\sqrt{7} + \sqrt{3})(\sqrt{7} - \sqrt{3})} = \frac{7 - 2\sqrt{21} + 3}{7 - 3} = \frac{5 - \sqrt{21}}{2}$$

Rationalize the denominator of:

2. $\frac{4}{3 + \sqrt{2}}$

6. $\frac{\sqrt{a} + \sqrt{x}}{\sqrt{a} - \sqrt{x}}$

3. $\frac{c}{\sqrt{a} - \sqrt{b}}$

7. $\frac{a + 2\sqrt{b}}{a - 2\sqrt{b}}$

4. $\frac{\sqrt{3} - \sqrt{2}}{\sqrt{3} + \sqrt{2}}$

8. $\frac{7}{\sqrt{3} + 2\sqrt{5}}$

5. $\frac{2 - \sqrt{2}}{5 - 3\sqrt{2}}$

9. $\frac{3a}{\sqrt{6a} + 3\sqrt{8b}}$

Rationalize the denominator of :

$$10. \frac{3\sqrt{3} - 2\sqrt{2}}{4\sqrt{2} + 6\sqrt{3}}$$

$$11. \frac{x - \sqrt{x^2 - 1}}{x + \sqrt{x^2 - 1}}$$

$$12. \frac{8ab}{2\sqrt{15a} + 4\sqrt{12b}}$$

$$13. \frac{\sqrt{a^2 + a + 1} - 1}{\sqrt{a^2 + a + 1} + 1}$$

$$14. \frac{\sqrt{a+b} - \sqrt{a-b}}{\sqrt{a+b} + \sqrt{a-b}}$$

$$15. \frac{\sqrt{x^2 - 2} - \sqrt{x^2 + 2}}{\sqrt{x^2 - 2} + \sqrt{x^2 + 2}}$$

Reduce to a simple expression having two significant figures :

$$16. \frac{4}{2 + \sqrt{3}}$$

$$18. \frac{5 - \sqrt{6}}{3 - \sqrt{6}}$$

$$17. \frac{8}{3 - \sqrt{5}}$$

$$19. \frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} - \sqrt{2}}$$

$$20. \text{Rationalize the denominator of } \frac{\sqrt{2} - \sqrt{3} - \sqrt{5}}{\sqrt{2} + \sqrt{3} + \sqrt{5}}.$$

SOLUTION

$$\begin{aligned} \frac{\sqrt{2} - \sqrt{3} - \sqrt{5}}{\sqrt{2} + \sqrt{3} + \sqrt{5}} &= \frac{(\sqrt{2} - \sqrt{5}) - \sqrt{3}}{(\sqrt{2} + \sqrt{3}) + \sqrt{5}} \times \frac{(\sqrt{2} - \sqrt{5}) + \sqrt{3}}{(\sqrt{2} + \sqrt{3}) - \sqrt{5}} \\ &= \frac{2 - 2\sqrt{10} + 5 - 3}{2 + 2\sqrt{6} + 3 - 5} = \frac{4 - 2\sqrt{10}}{2\sqrt{6}} \\ &= \frac{2 - \sqrt{10}}{\sqrt{6}} = \frac{2\sqrt{6} - 2\sqrt{15}}{6} = \frac{\sqrt{6} - \sqrt{15}}{3} \end{aligned}$$

Rationalize the denominator of :

$$21. \frac{\sqrt{2} - \sqrt{5} - \sqrt{7}}{\sqrt{2} + \sqrt{5} + \sqrt{7}}$$

$$23. \frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} + \sqrt{2} - \sqrt{6}}$$

$$22. \frac{1}{\sqrt{2} + \sqrt{3} + \sqrt{5}}$$

$$24. \frac{2\sqrt{2} - 3\sqrt{3} + 4\sqrt{5}}{\sqrt{2} + \sqrt{3} - \sqrt{5}}$$

POWERS AND ROOTS OF RADICALS

In finding powers and roots of radicals, it is frequently convenient to use fractional exponents.

EXERCISES

1. Find the cube of $3\sqrt{x}$.

SOLUTION

$$(3\sqrt{x})^3 = (3x^{\frac{1}{2}})^3 = 3^3 x^{\frac{3}{2}} = 27 x^{\frac{3}{2}} = 27 \sqrt{x^3} = 27 x\sqrt{x}$$

2. Find the square of $3\sqrt[4]{16}$.

SOLUTION

$$(3\sqrt[4]{16})^2 = (3 \cdot 16^{\frac{1}{4}})^2 = 3^2 \cdot 16^{\frac{1}{2}} = 9 \cdot 4 = 36$$

Square :

Cube :

Expand as indicated :

3. $2\sqrt{xy}$

7. $2\sqrt{2}$

11. $(3\sqrt{xy})^4$

4. $3\sqrt[3]{2a}$

8. $3\sqrt[3]{x^2}$

12. $(-2\sqrt[6]{2a})^3$

5. $b\sqrt[3]{3b^2}$

9. $\sqrt[4]{c^3d^3}$

13. $(-3a^{\frac{m}{2}}x^{\frac{n}{3}})^6$

6. $y^2\sqrt[4]{4c^2d}$

10. $\sqrt[6]{4n^3}$

14. $(-2\sqrt{a}\sqrt[3]{b})^5$

15. Find the square root of $\sqrt[3]{27x}$.

SOLUTION

$$\sqrt{\sqrt[3]{27x}} = [(27x)^{\frac{1}{3}}]^{\frac{1}{2}} = (27x)^{\frac{1}{6}} = \sqrt[6]{27x}$$

Find the square root of :

16. $\sqrt{ab}$

18. $\sqrt[4]{4a^2}$

20. $\sqrt[5]{a^4b^4}$

17. $\sqrt[3]{3c}$

19. $\sqrt[3]{x^n}$

21. $\sqrt[n]{c^n x^2}$

Find the cube root of :

22. $\sqrt[3]{3a}$

24. $\sqrt[3]{ab^3x}$

26. $-\sqrt{a^n b^n}$

23. $\sqrt[3]{7a^3}$

25. $-8\sqrt[3]{a^6}$

27. $-27\sqrt[5]{x^3 z^3}$

Simplify these indicated roots :

28. $\sqrt[3]{\sqrt{a^6 x^4}}$

30. $\sqrt[4]{\sqrt[n]{a^4 x^8}}$

32. $(\sqrt[4]{729})^{\frac{1}{3}}$

29. $(\sqrt[3]{4x^4 y^4})^{\frac{1}{2}}$

31. $(\sqrt{x^m y^m})^{\frac{1}{mn}}$

33. $\sqrt{\left(\frac{-125}{27}\right)^{\frac{2}{3}}}$

To find the square root of a binomial quadratic surd by inspection

The square of a binomial may be written in the form

$$(a + b)^2 = (a^2 + b^2) + 2ab$$

Thus, $(\sqrt{2} + \sqrt{6})^2 = (2 + 6) + 2\sqrt{12} = 8 + 2\sqrt{12}$.

Therefore the terms of the square root of $8 + 2\sqrt{12}$ may be obtained by separating $\sqrt{12}$ into two factors such that the sum of their squares is 8. They are $\sqrt{2}$ and $\sqrt{6}$.

$$\sqrt{8 + 2\sqrt{12}} = \sqrt{2} + \sqrt{6}$$

PRINCIPLE. *The terms of the square root of a binomial quadratic surd that is a square may be obtained by dividing the irrational term by 2 and then separating the quotient into two factors, the sum of whose squares is the rational term.*

EXERCISES

- Find the principal square root of $14 + 8\sqrt{3}$.

SOLUTION

$$14 + 8\sqrt{3} = 14 + 2(4\sqrt{3}) = 14 + 2\sqrt{48}$$

Since $\sqrt{48} = \sqrt{6} \times \sqrt{8}$ and $14 = 6 + 8$,

$$\sqrt{14 + 8\sqrt{3}} = \sqrt{6} + \sqrt{8} = \sqrt{6} + 2\sqrt{2}$$

Find the principal square root of:

- | | | |
|----------------------|-----------------------|---|
| 2. $11 - 6\sqrt{2}$ | 7. $11 + 2\sqrt{30}$ | 12. $1 + \frac{2}{3}\sqrt{2}$ |
| 3. $12 + 2\sqrt{35}$ | 8. $7 - 2\sqrt{10}$ | 13. $2 + \sqrt{3}$ |
| 4. $16 - 2\sqrt{60}$ | 9. $12 - 6\sqrt{3}$ | 14. $6 + \sqrt{35}$ |
| 5. $15 + 2\sqrt{26}$ | 10. $17 + 12\sqrt{2}$ | 15. $4 - \sqrt{15}$ |
| 6. $16 - 2\sqrt{55}$ | 11. $12 + 4\sqrt{5}$ | 16. $\frac{7}{2} - \frac{3}{2}\sqrt{5}$ |

Problems

- In a right triangle, if the length of one leg is 3 and the length of the other leg is $3\sqrt{2}$, find the length of the hypotenuse.

2. The formula for the volume of a right circular cone is $V = \frac{1}{3} \pi r^2 h$. If $r = 2\sqrt{3}$ and $h = 6$, find V in terms of π .

3. The formula for the volume of a right circular cylinder is $V = \pi r^2 h$. Find V in terms of π if $r = \frac{3}{4}\sqrt{2}$ and $h = 8$.

4. The volume of a right circular cone = 120π . If $h = 9$, find r . (Formula in exercise 2.)

5. The surface of a sphere is found by the formula $S = 4\pi r^2$. If $S = 200\pi$, find the value of r .

6. What is the area of a rectangle the dimensions of which are $6\sqrt{2}$ and $3\sqrt{6}$?

7. Solve the equation $\frac{x^2}{108} = \frac{2}{3}$. (Two answers.)

8. If a is the length of one side of a square, what is the length of the diagonal of the square?

RADICAL EQUATIONS

When the following equations have been freed of radicals, the resulting equations will be found to be linear equations. Other varieties of **radical**, or **irrational**, equations are treated later.

EXERCISES

1. Solve the equation $\sqrt{x-5} + \sqrt{x} = 5$.

SOLUTION

$$(1) \quad \sqrt{x-5} + \sqrt{x} = 5$$

$$(2) \quad \sqrt{x-5} = 5 - \sqrt{x}$$

$$(3) \quad x - 5 = 25 - 10\sqrt{x} + x \quad (2) \text{ squared}$$

$$(4) \quad 10\sqrt{x} = 30$$

$$(5) \quad \sqrt{x} = 3$$

$$(6) \quad x = 9 \quad (5) \text{ squared}$$

CHECK. $\sqrt{9-5} + \sqrt{9} = \sqrt{4} + \sqrt{9} = 2 + 3 = 5$; that is, $5 = 5$.

GENERAL DIRECTIONS. Arrange the equation so that the radical term, if there is but one, or the most complex radical term, if there is more than one, may constitute one member of the equation.

Then raise each member to a power corresponding to the order of that radical and simplify.

If the equation is not freed of radicals, proceed again as at first.

Solve, and check each result:

$$2. \sqrt{n} + 1 = 4$$

$$12. \sqrt[3]{4x - 16} = 2$$

$$3. \sqrt{x} - 2 = 10$$

$$13. 1 + 2\sqrt{x} = 7 - \sqrt{x}$$

$$4. 3 + 2\sqrt{n} = 15$$

$$14. \sqrt{x - 21} = \sqrt{x} - 1$$

$$5. 3\sqrt{2d} - 4 = 32$$

$$15. \sqrt{x^2 - 11} + 1 = x$$

$$6. \sqrt{d + 11} = 4$$

$$16. \sqrt{s - 16} = 8 - \sqrt{s}$$

$$7. \sqrt{n + 5} = 13$$

$$17. \sqrt{x - 15} + \sqrt{x} = 15$$

$$8. \sqrt{x + d^2} = c$$

$$18. \sqrt{2y} - \sqrt{2y - 15} = 1$$

$$9. \sqrt[3]{r - 2} = 2$$

$$19. \sqrt{x + 2} = \sqrt{x + 32}$$

$$(10.) \sqrt[3]{x + b^3} = a$$

$$20. \sqrt{m + 4} = 4 - \sqrt{m - 4}$$

$$11. \sqrt[3]{x} + b = a$$

$$(21.) \sqrt{b - 5} + \sqrt{b + 7} = 6$$

$$22. \frac{5}{\sqrt{3x + 2}} = \sqrt{3x + 2} + \sqrt{3x - 1}$$

SUGGESTION. Clear the equation of fractions.

$$23. \frac{\sqrt{x + 12}}{\sqrt{x + 5}} = \frac{\sqrt{x + 8}}{\sqrt{x + 3}}$$

SUGGESTION. Reduce each fraction to a mixed number and simplify before clearing of fractions.

$$24. \frac{\sqrt{v} - 6}{\sqrt{v} - 1} = \frac{\sqrt{v} - 8}{\sqrt{v} - 5}$$

$$25. \frac{\sqrt{2r} + 6}{\sqrt{2r} + 4} = \frac{\sqrt{2r} + 2}{\sqrt{2r} + 1}$$

$$(26.) \frac{\sqrt{11n} + \sqrt{2n + 3}}{\sqrt{11n} - \sqrt{2n + 3}} = \frac{8}{3}$$

The student will observe that radical equations are freed of radicals either by *rationalization* or by raising to a power.

$$\text{Thus,} \quad (1) \sqrt{2x} - 6 = 0 \quad (2) \sqrt{2x} + 6 = 0$$

$$\begin{array}{rcl} \text{Multiply by} & \frac{\sqrt{2x} + 6}{2x - 36} & \frac{\sqrt{2x} - 6}{2x - 36} \\ & 2x - 36 = 0 & 2x - 36 = 0 \\ & \therefore x = 18 & \therefore x = 18 \end{array}$$

If the positive, or principal, square root of $2x$ is taken, $x = 18$ satisfies (1) but not (2). If the negative square root of $2x$ is taken, $x = 18$ satisfies (2) but not (1).

It has been agreed, however, that the sign $\sqrt{}$ shall denote only principal roots, and because of this arbitrary *convention*, our conclusion must be that (1) has the root $x = 18$ and that (2) has no root or is *impossible*.

According to this view, when both members of (1) are multiplied by $\sqrt{2x + 6}$, no root is introduced because $\sqrt{2x + 6} = 0$ has no root; but when both members of (2), which has no root, are multiplied by $\sqrt{2x - 6}$, the root of $\sqrt{2x - 6} = 0$, which is $x = 18$, is introduced.

A root may be introduced in this way by *rationalization* or by the equivalent process of *squaring*. Such a root is called an **extraneous root**.

Thus,

(1)	$\sqrt{2x + 6} = 0$	
(2)	$\sqrt{2x} = -6$	(1) $- 6$
(3)	$2x = 36$	(2) squared
(4)	$\therefore x = 18$	(3) $\div 2$

Checking, we have $\sqrt{2 \cdot 18} + 6 = 6 + 6$, which does not equal ($\neq$) 0.

EXERCISES

1. Solve, if possible, the equation $\sqrt{x - 7} - \sqrt{x} = 7$.

SOLUTION

Squaring, simplifying, etc., we have

$$\sqrt{x} = -4$$

Squaring, we have

$$x = 16$$

Checking, we have

$$\sqrt{16 - 7} - \sqrt{16} = \sqrt{9} - \sqrt{16} = 3 - 4, \text{ which does not equal } 7$$

Hence the equation has no root, or is *impossible*.

Solve, and check to discover which of the following equations are impossible; then change these to true equations:

2. $\sqrt{2r} + \sqrt{2r - 3} = 1$ 4. $2\sqrt{x} + \sqrt{4x - 11} = 1$

3. $\sqrt{3s + 7} + \sqrt{3s} = 7$ 5. $\sqrt{4x} - \sqrt{x} = \sqrt{9x - 32}$

6. $\sqrt{4y + 5} - 2\sqrt{y - 1} = 9$

CHAPTER XIV

QUADRATIC EQUATIONS

First-degree and quadratic equations

The equation $x - 1 = 0$ is of the *first* degree and has *one* root, $x = 1$. Similarly, $x - 5 = 0$ is of the first degree and has one root, $x = 5$. Consequently, the product of these two linear equations, which is

$$(x - 1)(x - 5) = 0, \text{ or } x^2 - 6x + 5 = 0,$$

is of the *second* degree and has *two* roots, 1 and 5.

An equation which when simplified contains the square of the unknown number, but no higher power, is called an **equation of the second degree**, or a **quadratic equation**.

In the equations examined up to this point, if there was *one* unknown number, there was but *one* value of that unknown number, or *one* root. In order that we may see more clearly why we have two roots in these new equations, we shall study the graphs of certain types of quadratic equations and see how the graphs give us a method of solution.

GRAPHIC SOLUTION

Let it be required to solve graphically $x^2 - 6x + 5 = 0$.

To solve the equation graphically, we must first draw the graph of $x^2 - 6x + 5$. To do this, let $y = x^2 - 6x + 5$.

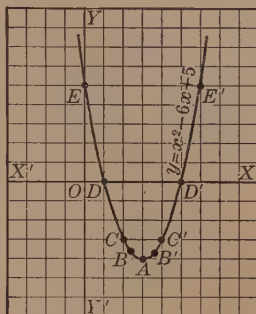
The graph of $y = x^2 - 6x + 5$ will represent *all* the corresponding real values of x and of $x^2 - 6x + 5$, and among them will be the values of x that make $x^2 - 6x + 5$ equal to zero, that is, the *roots* of the equation $x^2 - 6x + 5 = 0$.

When the coefficient of x^2 is $+1$, as in this instance, it is convenient to take for the first value of x a number equal to half the coefficient of x with its sign changed. Next, values of x differing from this value *by equal amounts* may be taken.

Thus, first substituting $x = 3$, we find that $y = -4$, locating the point $A = (3, -4)$. Next we give values to x differing from 3 by equal amounts, as $2\frac{1}{2}$ and $3\frac{1}{2}$, 2 and 4, 1 and 5, 0 and 6. We find that y has the same value for $x = 3\frac{1}{2}$ as for $x = 2\frac{1}{2}$, for $x = 4$ as for $x = 2$, etc. The table below gives a record of the points and their coördinates:

$$y = x^2 - 6x + 5$$

x	y	POINTS
3	-4	A
$2\frac{1}{2}, 3\frac{1}{2}$	$-3\frac{3}{4}$	B, B'
2, 4	-3	C, C'
1, 5	0	D, D'
0, 6	5	E, E'



Plotting the points A, B, B', C, C' , etc., whose coördinates are given in the preceding table, and drawing a smooth curve through them, we obtain the graph of $y = x^2 - 6x + 5$, as shown in the figure.

When $y = 0$, that is, when $x^2 - 6x + 5 = 0$, we see that the curve crosses the x -axis twice at the points $D(x = 1)$ and $D'(x = 5)$. Hence the roots of the equation are 1 and 5.

A curve like that obtained by plotting the graph of $x^2 - 6x + 5$ or of any quadratic expression of the form $ax^2 + bx + c$ is called a parabola.

EXERCISES

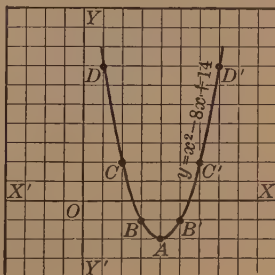
1. Solve graphically the equation $x^2 - 8x + 14 = 0$.

SOLUTION

Since the coefficient of x is -8 , first substitute 4 for x . Points and their coördinates are given in the table:

$$y = x^2 - 8x + 14$$

x	y	POINTS
4	-2	A
3, 5	-1	B, B'
2, 6	2	C, C'
1, 7	7	D, D'



Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 - 8x + 14$, which *crosses* the x -axis approximately at $x = 2.6$ and $x = 5.4$. Hence, to the nearest tenth, the roots of $x^2 - 8x + 14 = 0$ are 2.6 and 5.4.

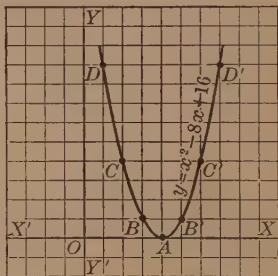
2. Solve graphically the equation $x^2 - 8x + 16 = 0$.

SOLUTION

Since the coefficient of x is -8 , first substitute 4 for x . Points and their coördinates are given in the table:

$$y = x^2 - 8x + 16$$

x	y	POINTS
4	0	A
3, 5	1	B, B'
2, 6	4	C, C'
1, 7	9	D, D'



Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 - 8x + 16$, which *touches* the x -axis at $x = 4$.

This fact is interpreted graphically to mean that the roots of the equation $x^2 - 8x + 16 = 0$ are *equal*, both being represented by the x -distance of the point of contact. Hence the roots are 4 and 4.

NOTE. The student may later show that 4 and 4 are the roots by solving the equation algebraically.

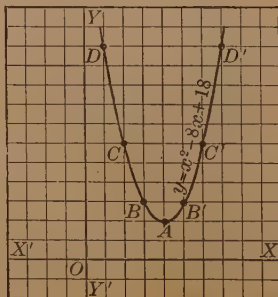
3. Solve graphically the equation $x^2 - 8x + 18 = 0$.

SOLUTION

Since the coefficient of x is -8 , first substitute 4 for x . Points and their coördinates are given in the table:

$$y = x^2 - 8x + 18$$

x	y	POINTS
4	2	A
3, 5	3	B, B'
2, 6	6	C, C'
1, 7	11	D, D'



Plotting these points and drawing a smooth curve through them, we have the graph of $y = x^2 - 8x + 18$, which neither crosses nor touches the x -axis. This fact is interpreted graphically to mean that the roots of the equation $x^2 - 8x + 18 = 0$ are *imaginary*, since there are no *real* roots of the equation.

In each of the preceding graphs, the point A , whose y -distance is the least algebraically of any point in the graph, is called the **minimum point**.

When the coefficient of x^2 is $+1$, it is evident from the preceding graphs that:

PRINCIPLES. 1. *If the minimum point lies below the x -axis, the roots are real and unequal. Hence there are two roots.*

2. *If the minimum point lies on the x -axis, the roots are real and equal. Hence there is but one root.*

3. *If the minimum point lies above the x -axis, there are no real roots. Hence the roots are said to be imaginary.*

Solve graphically, giving real roots to the nearest tenth:

(4.) $x^2 - 4x + 3 = 0$

9. $x^2 - 2x - 2 = 0$

5. $x^2 - 6x + 7 = 0$

(10) $x^2 = 6x - 9$

6. $x^2 - 4x = -2$

11. $x^2 + 4x + 2 = 0$

7. $x^2 + 2(x + 1) = 0$

12. $x^2 - 2x + 6 = 0$

8. $x^2 - 4x + 6 = 0$

13. $x^2 - 4x - 1 = 0$

14. Solve graphically $4x - 2x^2 + 1 = 0$.

SUGGESTION. When both members of the given equation are divided by -2 , the coefficient of x^2 , the equation becomes

$$x^2 - 2x - \frac{1}{2} = 0$$

The roots may be found by plotting the graph of $y = x^2 - 2x - \frac{1}{2}$.

Solve graphically, giving real roots to the nearest tenth:

(15.) $2x^2 + 8x + 7 = 0$

17. $12x - 4x^2 - 1 = 0$

16. $2x^2 - 12x + 15 = 0$

(18.) $11 + 8x + 2x^2 = 0$

ALGEBRAIC SOLUTION

An equation that contains only the second power of the unknown number is called an **incomplete quadratic equation**; as, $2x^2 = 8$ and $5x^2 = 16 + 2x^2$.

The equation $x^2 = 16$ has two roots, for it may be reduced to the form $(x - 4)(x + 4) = 0$, which is equivalent to the two linear equations $x - 4 = 0$ and $x + 4 = 0$, each of which has one root, namely, $+4$ and -4 .

If the product of two or more factors is equal to zero, the equation is satisfied if any of them is equal to zero.

PRINCIPLE. *Every incomplete quadratic equation has two roots numerically equal, but opposite in sign.*

To solve incomplete quadratic equations

EXERCISES

1. Find the value of x in $10x^2 = 99 - x^2$.

FIRST SOLUTION

- | | | |
|-----|-------------------------|--------------|
| (1) | $10x^2 = 99 - x^2$ | |
| (2) | $11x^2 - 99 = 0$ | |
| (3) | $x^2 - 9 = 0$ | |
| (4) | $(x + 3)(x - 3) = 0$ | (3) factored |
| (5) | $x = -3 \text{ or } +3$ | From (4) |

SECOND SOLUTION

Solve by extracting the square root of each member.

- | | | |
|-----|--------------------|--------------------|
| (1) | $10x^2 = 99 - x^2$ | |
| (2) | $11x^2 = 99$ | |
| (3) | $x^2 = 9$ | |
| (4) | $x = \pm 3$ | Square root of (3) |

NOTE. In taking the square root of (3) we may have the sign $\pm$ before the square root of each member, that is, $\pm(x) = \pm 3$. This would mean, first, $+x = +3$; second, $+x = -3$; third, $-x = +3$; and fourth, $-x = -3$. Since, however, the third equation may be derived from the second and the fourth from the first, by multiplying both members by -1 , the first two equations express all values of x .

CHECK. (1) We may substitute each value in the first equation as usual or (2) we may check the result by solving the equation graphically.

2. Solve the equation $3x^2 = 24$, and check the results.

SOLUTION

$$(1) \quad 3x^2 = 24$$

$$(2) \quad x^2 = 8$$

$$(3) \quad x = \pm \sqrt{8} \quad \text{Square root of (2)}$$

$$(4) \quad x = -2\sqrt{2} \text{ or } +2\sqrt{2} \quad \text{From (3)}$$

CHECK. Substitute these values in (1):

$$3(\pm 2\sqrt{2})^2 \quad | \quad 24$$

$$3 \times 8 \quad | \quad 24$$

$$24 = 24$$

Solve, and check each result:

$$3. \quad 3x^2 - 5 = 22$$

$$8. \quad 2x^2 - 25 = 73$$

$$4. \quad 2x^2 + 3x^2 = 80$$

$$9. \quad 7x^2 = 4x^2 + 24$$

$$5. \quad 4x^2 = \frac{1}{9}$$

$$10. \quad 4n^2 + 9 = 5n^2 - 7$$

$$6. \quad \frac{3}{4}x^2 - 5 = 22$$

$$11. \quad (x+2)^2 - 4(x+2) = 4$$

$$7. \quad 5x^2 - 75 = 2x^2$$

$$12. \quad (3u-8)(3u+8) = 17$$

$$(13.) \quad (x+1)^3 - (x-1)^3 = 38$$

$$14. \quad (x+1)^2 = x(3x+2) - 3$$

$$15. \quad 5(s+2) = 3s^2 + s(5-s)$$

$$16. \quad (2r+1)(2r+3) = 8(r+3)$$

$$17. \quad (x+2)^2 - 4(x+2) - 2 = 2$$

$$18. \quad 2(3-2x) + 20 = (x-1)^2 - 2x$$

$$19. \quad \frac{x-8}{6} = \frac{6}{x+8}$$

$$25. \quad \frac{n+3}{n+2} + \frac{n-3}{n-2} = 1\frac{7}{8}$$

$$20. \quad \frac{1}{1-x} + \frac{1}{x+1} = \frac{8}{3}$$

$$26. \quad \frac{a-2}{a+1} + \frac{a+2}{a-1} = -1$$

$$21. \quad \frac{a}{x} + \frac{x}{a} = \frac{ab}{x}$$

$$27. \quad \frac{x-2}{x+2} - \frac{x+2}{2-x} = \frac{40}{x^2-4}$$

$$22. \quad \frac{x}{a+b} - \frac{a-b}{x} = 0$$

$$28. \quad \frac{t}{12} + \frac{t^2-15}{5t} = \frac{t}{5}$$

$$23. \quad \frac{x}{12} + \frac{x^2-15}{5x} = \frac{x}{5}$$

$$29. \quad \sqrt{x^2-9} = 4$$

$$24. \quad (x+4)^2 = 8x + 25$$

$$30. \quad 6ax^2 - 54a^5 = 0$$

Problems

1. What negative number is equal to its reciprocal?
2. The length of a 10-acre field is 4 times its width. What are its dimensions?
3. How many rods of fence will inclose a square garden whose area is $2\frac{1}{2}$ acres?
4. A 4-foot length of curbing contains 3840 cu. in. If its width is 5 times its thickness, find these dimensions.
5. The width of a large American flag is $\frac{3}{5}$ its length. The area of the flag is 1500 square feet. What are the dimensions of the flag?
6. John's father sold his pumpkins for \$200. The number of tons sold was 2 times the number of dollars he received per ton. Find the number of tons sold and the price per ton.
7. The width of a rectangular field is $\frac{3}{4}$ its length. If the length of a path extending diagonally across the field is 40 rods, what are the dimensions of the field?
8. A shipment of railroad ties measuring 400,000 board feet contained as many carloads as there were board feet in a tie. If each car held 250 ties, find the total number of ties and the number of board feet in one tie.
9. Ethel saved a certain amount of money each week for a camera that cost \$8. If the number of weeks during which she was saving this amount was $\frac{1}{2}$ the number of cents a week she saved, find the amount saved each week and the time necessary for her to save the full amount.

Formulas

NOTE. In the solution of formulas which involve the square root the negative result is rejected when it is impracticable.

EXERCISES

Solve the following formulas:

① $s = \frac{1}{2}gt^2$, for t (falling bodies)

2. $E = \frac{1}{2}Mv^2$, for v (kinetic energy)

3. $t = \pi \sqrt{\frac{l}{g}}$, for l (pendulum)

4. $F = \frac{mv^2}{R}$, for v (centripetal force)

5. $G = \frac{mm'}{d^2}$, for d (gravitation)

6. When $g = 32.16$, the formula in exercise 1 gives the number of feet (s) through which a body will fall in t seconds, starting from rest. How long will it take a brick to fall to the sidewalk from the top of a building 100.5 feet high?

7. To lighten a balloon at the height of 2500 feet, a bag of sand was thrown out. Find the time, to the nearest tenth of a second, required for it to reach the earth.

Solve the following geometric formulas:

8. $c^2 = a^2 + b^2$, for b

9. $c^2 = 2(a^2 + b^2) - 4m^2$, for m

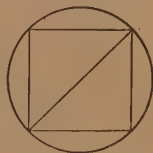
10. $A = .7854 d^2$, for d

11. $V = \frac{1}{3} \pi r^2 h$, for r

12. Using the formula in exercise 8, find, to the nearest tenth, the hypotenuse, c , of a right triangle whose other two sides are $a = 8.5$ and $b = 5.3$.

13. By means of the same formula, find, to the nearest tenth, side a of a right triangle whose hypotenuse, c , is 15.4 and whose side b is 9.7.

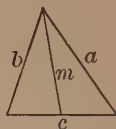
14. From the same formula and the accompanying figure, find, to the nearest tenth, the side, a , of a square inscribed in a circle whose diameter, d , is 10.



15. Using the same formula and the accompanying figure, deduce a formula for the altitude, a , of an equilateral triangle in terms of its side, b .



16. From the formula in exercise 9, find the length of the median, m , to side c of the triangle in the accompanying figure, if $a = 11$, $b = 8$, and $c = 9$.



17. Substituting in the formula in exercise 10, find, to the nearest tenth of a foot, the diameter, d , of a circle whose area, A , is 1000 square feet.

18. Using the formula in exercise 11, find, to the nearest centimeter, the radius, r , of the base of a conical vessel 20 centimeters high ($h = 20$) that will hold a liter of water ($V = 1$ liter = 1000 cubic centimeters; $\pi = 3\frac{1}{7}$).

To solve complete quadratic equations by factoring

A quadratic equation that contains both the second and the first power of one unknown number is called a **complete quadratic equation**; as, $x^2 + 3x = 10$ and $4x^2 - x = 3$.

ILLUSTRATIVE EXAMPLE

Solve $3x^2 = 10x - 3$.

SOLUTION

$$(1) \quad 3x^2 = 10x - 3$$

$$(2) \quad 3x^2 - 10x + 3 = 0$$

$$(3) \quad (x - 3)(3x - 1) = 0$$

$$(4) \quad x - 3 = 0 \text{ or } 3x - 1 = 0$$

$$(5) \quad x = 3 \text{ or } \frac{1}{3}$$

(2) factored

From (3)

From (4)

$$\begin{array}{l|l} \text{CHECK.} & 3(3)^2 \quad | \quad 10(3) - 3 \\ & 3 \cdot 9 \quad | \quad 10(3) - 3 \\ & 27 \quad | \quad 30 - 3 \\ & 27 = 27 \end{array}$$

$$\begin{array}{l|l} & 3(\frac{1}{3})^2 \quad | \quad 10(\frac{1}{3}) - 3 \\ & 3(\frac{1}{9}) \quad | \quad \frac{10}{3} - 3 \\ & \frac{3}{9} \quad | \quad 3\frac{1}{3} - 3 \\ & \frac{1}{3} = \frac{1}{3} \end{array}$$

NOTE. It is necessary to substitute both values found for the unknown number to be sure that the result is correct.

RULE. Reduce the equation to the form $ax^2 + bx + c = 0$, factor the first member, and equate each factor to zero, thus obtaining two linear, or simple, equations together equivalent to the given quadratic equation.

EXERCISES

Solve by factoring, and check results :

- | | |
|--------------------------|----------------------------|
| 1. $x^2 - 5x + 6 = 0$ | 11. $x^2 - 11(x + 3) = 9$ |
| 2. $x^2 + 10x + 21 = 0$ | 12. $55 + x(x + 16) = 0$ |
| 3. $x^2 + 12x - 28 = 0$ | 13. $2x^2 - 7x + 3 = 0$ |
| 4. $x^2 - 20x + 51 = 0$ | 14. $2z^2 - z - 3 = 0$ |
| 5. $x^2 - 5x = 24$ | 15. $3v^2 - 2v - 8 = 0$ |
| 6. $x^2 - 1 = 3(x + 1)$ | 16. $10r^2 - 27r + 5 = 0$ |
| 7. $x^2 + 10x = 39$ | 17. $6(s^2 + 1) = 13s$ |
| 8. $60 + x^2 = 17x$ | 18. $2t(t + 3) + 4 = 0$ |
| 9. $x(x - 1) = 42$ | 19. $3(x^2 - 2) - 7x = 0$ |
| 10. $x^2 - 3 = 2(x + 6)$ | 20. $10(2 - 3x + x^2) = 0$ |

To solve complete quadratic equations by completing the square

Since $(x + a)^2 = x^2 + 2ax + a^2$, the general form of the square of a binomial is $x^2 + 2ax + a^2$. Hence an expression like $x^2 + 2ax$ may be made a square by adding the term a^2 , which is the square of half the coefficient of x .

ILLUSTRATIVE EXAMPLES

1. Solve the equation $x^2 + 6x = -5$.

SOLUTION

- | | | |
|-----|-------------------------|-----------------------------|
| (1) | $x^2 + 6x = -5$ | |
| (2) | $x^2 + 6x + 9 = 4$ | (1) + 9 |
| (3) | $(x + 3)^2 = (2)^2$ | Each member of (2) factored |
| (4) | $x + 3 = \pm 2$ | Square root of (3) |
| (5) | $x = -3 \pm 2$ | |
| (6) | $x = -1 \text{ or } -5$ | Terms in (5) united |

CHECK.

$$\begin{array}{r|l} (-5)^2 + 6(-5) & -5 \\ 25 - 30 & -5 \\ -5 & = -5 \end{array}$$

$$\begin{array}{r|l} (-1)^2 + 6(-1) & -5 \\ 1 - 6 & -5 \\ -5 & = -5 \end{array}$$

2. Solve the equation $x^2 - 5x - 15 = 0$.

SOLUTION

$$(1) \quad x^2 - 5x - 15 = 0$$

$$(2) \quad x^2 - 5x + \left(\frac{5}{2}\right)^2 = 15 + \left(\frac{5}{2}\right)^2 \quad (1) + 15 + \left(\frac{5}{2}\right)^2$$

$$(3) \quad x^2 - 5x + \frac{25}{4} = 15 + \frac{25}{4} = \frac{85}{4}$$

$$(4) \quad x - \frac{5}{2} = \pm \frac{\sqrt{85}}{2} \quad \text{Square root of (3)}$$

$$(5) \quad x = \frac{5}{2} \pm \frac{\sqrt{85}}{2}$$

$$(6) \quad x = \frac{5 + \sqrt{85}}{2} \text{ or } \frac{5 - \sqrt{85}}{2}$$

3. Solve the equation $5x^2 - 3x - 2 = 0$.

SOLUTION

$$(1) \quad 5x^2 - 3x - 2 = 0$$

$$(2) \quad 5x^2 - 3x = 2$$

$$(3) \quad x^2 - \frac{3x}{5} = \frac{2}{5}$$

$$(4) \quad x^2 - \frac{3x}{5} + \frac{9}{100} = \frac{2}{5} + \frac{9}{100} = \frac{49}{100} \quad (3) + \left(\frac{3}{10}\right)^2$$

$$(5) \quad x - \frac{3}{10} = \pm \frac{7}{10} \quad \text{Square root of (4)}$$

$$(6) \quad x = \frac{3}{10} \pm \frac{7}{10}$$

$$(7) \quad x = 1 \text{ or } -\frac{2}{5}$$

Steps in the solution of quadratic equations in the form of $x^2 + bx + c = 0$ by completing the square

1. Arrange the equation so that the terms containing x^2 and x are in one member and the known terms in the other member.

2. Make the coefficient of x^2 unity.

3. Complete the square by adding to each member the square of half the coefficient of x .

4. Find the square root of both members.

5. Solve the two simple equations thus formed.

EXERCISES

Solve, and check results :

1. $x^2 - 4x = 5$

3. $8x = x^2 - 9$

2. $x^2 - 6x = 7$

4. $x^2 + 2x = 15$

Solve. Check the results in examples 5 to 17:

5. $x^2 - 2x = 24$

18. $3x^2 - 6x = 3$

6. $x^2 + 8x = -15$

19. $x^2 + 8.5x = -17.5$

7. $63 = x^2 + 2x$

20. $v^2 + 3v = 1$

8. $x^2 - 12x = -11$

21. $2x(x - 2) = 8$

9. $x^2 - 40 = 6x$

22. $r^2 + 4r - 7 = 0$

10. $y^2 = 10 - 3y$

23. $l^2 - 11l + 28 = 0$

11. $v^2 + 5v = 14$

24. $4x^2 + 35 = 24x$

12. $n(n - 1) = 2$

25. $12x^2 + 2x = 5$

13. $2x^2 = x + 5$

26. $3x^2 - 7x + 4 = 0$

14. $3x^2 + 5 = 8x$

27. $4x^2 - 6x = 3$

15. $6x^2 + 2x = 7$

28. $9x^2 = 5x + 4$

16. $3x^2 - 5x = 8$

29. $16x^2 - 9 = 7x$

17. $2x^2 - 3x = 5$

30. $9x^2 = 2 + 6x$

To solve quadratic equations by the quadratic formula

ILLUSTRATIVE EXAMPLES

1. Solve the general quadratic equation $ax^2 + bx + c = 0$.

SOLUTION

(1) $ax^2 + bx + c = 0$

(2) $ax^2 + bx = -c$

(3) $x^2 + \frac{bx}{a} = -\frac{c}{a}$

(4) $x^2 + \frac{bx}{a} + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$, or $= \frac{b^2 - 4ac}{4a^2}$

(5) $x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$, or $\pm \frac{\sqrt{b^2 - 4ac}}{2a}$

(6) $x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$

(7) $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The quadratic formula in (7) above may be used in obtaining roots of any quadratic equation by substituting the numerical values of a , b , and c found in the given equation after it is reduced to the form $ax^2 + bx + c = 0$.

2. Solve by the quadratic formula $3x^2 + 4x = 4$.

SOLUTION

$$(1) \quad 3x^2 + 4x = 4$$

$$(2) \quad 3x^2 + 4x - 4 = 0$$

$$(3) \quad x = \frac{-4 \pm \sqrt{16 - 4 \cdot 3(-4)}}{2 \cdot 3}$$

$$(4) \quad = \frac{-4 \pm \sqrt{16+48}}{6} = \frac{-4 \pm 8}{6}$$

$$(5) \quad = \frac{2}{3} \text{ or } -2$$

$$\text{CHECK. } \left. \begin{array}{l} 3(\frac{2}{3})^2 + 4(\frac{2}{3}) \\ \frac{12}{9} + \frac{8}{3} \\ \frac{12}{3} \\ 4 = 4 \end{array} \right\} \begin{array}{l} 4 \\ 4 \\ 4 \\ 4 = 4 \end{array}$$

$$\text{Formula: } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{Here } a = 3, b = 4, c = -4.$$

$$\text{Hence } b^2 = 16 \text{ and}$$

$$-4ac = -4 \times 3 \times -4 = 48$$

$$\left. \begin{array}{l} 3(-2)^2 + 4(-2) \\ 12 - 8 \\ 4 = 4 \end{array} \right\} \begin{array}{l} 4 \\ 4 \\ 4 = 4 \end{array}$$

EXERCISES

Solve by using the quadratic formula :

$$1. \quad 2x^2 - 11x + 12 = 0$$

$$2. \quad 5x^2 - 14x = -8$$

$$3. \quad 2x^2 + 5x = 7$$

$$4. \quad 2x^2 + 7x = -6$$

$$5. \quad 3x^2 - 7x = -2$$

$$6. \quad 4x^2 - x - 3 = 0$$

$$7. \quad 2a^2 - 5a = 63$$

$$8. \quad 6a^2 + 26a + 24 = 0$$

$$9. \quad 2x^2 + x - 6 = 0$$

$$10. \quad 2x^2 - 5x + 2 = 0$$

$$11. \quad 15x^2 - x - 2 = 0$$

$$12. \quad 12x^2 - 5x - 2 = 0$$

$$13. \quad 11x^2 - 10x - 1 = 0$$

$$14. \quad 5x^2 + 8x = -3$$

$$15. \quad 5x^2 - 7x = -2$$

$$16. \quad 6x^2 + 5x = -1$$

$$17. \quad 2 + 5x + 2x^2 = 0$$

$$18. \quad 6x^2 + 2 = 7x$$

$$19. \quad 4x^2 + 4x = 3$$

$$20. \quad 3x + 2x^2 = 9$$

$$21. \quad 6m^2 + 23m + 20 = 0$$

$$22. \quad 30p^2 + 59p + 15 = 0$$

$$23. \quad x^2 + 10 = 6x$$

$$24. \quad x^2 = -4(x + 3)$$

$$25. \quad x^2 = 4(2x - 3)$$

$$26. \quad 5x^2 + 18 = 6x$$

$$27. \quad x(3x + 4) = -2$$

$$28. \quad x^2 + 4 = 2x$$

Miscellaneous quadratic equations to be solved by any method

GENERAL DIRECTIONS. 1. Reduce the equation to the general form $ax^2 + bx + c = 0$.

2. If the factors are readily seen, solve by factoring.

3. If the factors are not readily seen, solve by completing the square or by the formula.

EXERCISES

Solve, and check:

$$1. x^2 + 5 = 6x$$

$$2. x^2 = 3x + 10$$

$$3. 5x + 3x^2 = 2$$

$$4. 2x^2 - 7x = 2$$

$$5. x^2 - 12x = 28$$

$$6. 3x^2 - 10x = 8$$

$$7. x^2 - 13x - 30 = 0$$

$$8. x^2 + 4(x - 3) = 0$$

$$9. 6 + 11x + 3x^2 = 0$$

$$10. (x + 5)^2 = 10x + 74$$

$$11. 4x^2 - 3x - 2 = 0$$

$$12. (x + 7)(x - 9) = 1 - 2x$$

$$13. x + \frac{1}{x} - \frac{5}{2} = 0$$

$$14. \frac{x}{9(x - 1)} = \frac{x - 2}{6}$$

$$15. \frac{x}{x - 2} + \frac{x - 2}{x} = \frac{5}{2}$$

$$16. \frac{3x}{5} = \frac{x^2}{2} + 2$$

$$17. \frac{3x^2 + 5}{10} = x - 1$$

$$18. \frac{12}{x} - 2 = \frac{12}{x + 1}$$

$$19. 2x + \frac{3}{5} = \frac{1}{5x}$$

$$20. \frac{x}{x - 5} - \frac{x - 5}{x} = \frac{3}{2}$$

$$21. \frac{x + 4}{x - 2} + 3 = \frac{(x + 3)^2}{x^2 - 9}$$

$$22. \frac{x^2}{x - 2} = \frac{4}{x - 2} + 5$$

$$23. \frac{x}{x + 2} + \frac{1}{2} = \frac{x + 2}{2x}$$

$$24. \frac{5x}{x + 7} + \frac{x + 6}{x + 3} = 3$$

$$25. \frac{x + 2}{x - 7} - \frac{x + 5}{x - 5} = 1$$

$$26. \frac{2x - 3}{x^2 - 3x} = 2 - \frac{3}{x^2 - 3x}$$

$$27. \frac{x - 2}{x + 2} - \frac{x + 2}{2 - x} = \frac{40}{x^2 - 4}$$

$$28. \frac{x^2}{9} + \frac{x^2 - 2x}{3x - 6} = \frac{35}{4}$$

$$29. \frac{x^2}{x - 2} - \frac{3x - 5}{2} = \frac{x + 2}{5}$$

$$30. \frac{y + 7}{y + 5} + \frac{y + 12}{y + 6} = 7$$

$$31. \frac{x - 3}{x + 4} + \frac{x + 2}{x - 2} = \frac{23}{10}$$

$$32. \frac{3x - 1}{x - 1} + \frac{2x + 1}{x + 1} = \frac{2x - 4}{x - 2}$$

Find the roots correct to the nearest hundredth :

33. $x^2 - 4x - 1 = 0$

37. $u^2 + 5u + 5.5 = 0$

34. $v^2 + 6v + 7 = 0$

38. $t^2 - 12t + 16.5 = 0$

35. $x^2 - 2x = 2$

39. $y^2 + 4y = -2$

36. $v^2 + 4v + .4 = 0$

40. $2x^2 - 5x + 1.2 = 0$

Problems

1. The sum of two numbers is 8, and their product is 15.
Find the numbers.

SOLUTION

Let n = one number

Then $8 - n$ = the other number

(1) $(8 - n)n = 15$

(2) $n = 3$ or 5

(3) $8 - n = 5$ or 3

Therefore the numbers are 3 and 5.

2. A fleet of destroyers carries 4000 men, and the number of men on each vessel is 40 less than 8 times the number of destroyers in the fleet. Find the number of destroyers and the number of men on each one.

SOLUTION

Let n = the number of destroyers in the fleet

Then $8n - 40$ = the number of men on each destroyer.

$n(8n - 40)$ = the total number of men with the fleet

(1) $n(8n - 40) = 4000$

(2) $n(8n - 40) - 4000 = 0$

(3) $8n^2 - 40n - 4000 = 0$

(4) $n^2 - 5n - 500 = 0$

(5) $(n - 25)(n + 20) = 0$

(4) factored

(6) $n - 25 = 0$

From (5)

(7) $n = 25$

From (6)

or (8) $n + 20 = 0$

From (5)

(9) $n = -20$

From (8)

(10) $8n - 40 = 160$ or -200 From (7) or (9)

The second values of n and $8n - 40$ are evidently inadmissible, since neither the number of ships nor the number of men on a ship can be negative. Hence there are 25 destroyers and 160 men on each.

CHECK. (1) The number of men on each, 160, is 40 less than 8 times 25, the number of destroyers. (2) 25 destroyers with 160 men each equals 4000 men, the number carried on the fleet.

3. One number is $\frac{2}{3}$ of another number and their product plus their sum is 69. Find the numbers.

4. If 25 is added to the square of a certain number, the sum is equal to the square of 13. What is the number?

5. If the square root of $\frac{2}{3}$ of a number is subtracted from the number the remainder is 140. Find the number.

6. The length of a rectangle is 15 feet greater than its width. If each dimension is decreased 2 feet, the area will be 324 square feet. Find the dimensions.

7. How many rods of fence will inclose a square garden whose area is $5\frac{5}{8}$ acres?

8. A certain number multiplied by $\frac{1}{4}$ of itself is equal to 16 more than 3 times the number. Find the number.

9. The sum of two numbers is 10, and their product is 21. What are the numbers?

SUGGESTION. Represent the numbers by $5 + n$ and $5 - n$.

10. The sum of two numbers is 16, and their product is 55. What are the numbers?

11. The difference between two numbers is 4, and the sum of their squares is 208. Find the numbers.

SUGGESTION. Represent the numbers by $n + 2$ and $n - 2$.

12. Find two consecutive integers whose product is 272.

13. Find two consecutive integers the sum of whose squares is 113.

14. Find two consecutive odd integers the sum of whose squares is $2(a^2 + 1)$.

15. The area of a floor is 180 square feet. If its length is 12 feet more than 3 times its width, what are its dimensions?

16. If the length of a square ventilator were 4 feet more and its width 6 feet less, its area would be 200 square feet. Find its dimensions.

17. A shipment of glass consisted of 80 sheets. If the number of sheets in each box was 2 more than the number of boxes, how many sheets of glass were there in each box?

18. In a paper of pins, the number of pins in each row is 6 more than twice the number of rows. If the paper contains 360 pins, how many pins are there in each row?

19. The height of a box is twice its length. If the area of the bottom is $6\frac{1}{4}$ square inches and its perimeter is 10 inches, what are the dimensions of the box? What is its volume?

20. The area of a sheet of paper is 208 square centimeters. If it were twice as wide, its width would exceed its length by 10 centimeters. Find its length; its width.

21. The sum of a certain number and its reciprocal is $\frac{13}{6}$. Find the number.

22. Find two consecutive integers the sum of the reciprocals of which is $\frac{5}{6}$.

23. If c times the reciprocal of a number is subtracted from the number, the result is $c - 1$. Find the number.

24. A poultry car accommodates 6720 fowls. The number of crates lacks 8 of being twice the number of fowls in each crate. How many crates are there? How many fowls are there in each crate?

25. If a dredge could lift 50 tons more of sand per minute, it would take 10 minutes less time to lift 10,000 tons. How many tons can the dredge lift per minute?

26. Two Alaskan dogs hauled freight 20 miles. If it had taken them 2 hours longer, the distance traveled per hour would have been $\frac{5}{6}$ of a mile less. How long did it take them?

27. A machine makes 6250 square feet of paper a minute. The number of feet in the length of the paper is 50 more than 3 times the number of inches in the width. Find the length of the paper.

28. Some boys agreed to pay \$12 for the use of a launch. As 2 of them failed to pay, the share of each of the others was \$.50 more. How many boys were there in the party?

29. The area of the floor of a gymnasium is 2650 square feet. The width is 3 feet less than the length. What are the dimensions of the floor?

30. A picture that is 18 inches by 12 inches has a frame of uniform width whose area is equal to that of the picture. Find the width of the frame.

31. A tank can be filled by two pipes in $24\frac{3}{4}$ minutes. If it takes the smaller pipe 10 minutes longer to fill the tank than it does the larger pipe, in what time can the tank be filled by each pipe?

SUGGESTION. Let m = the number of minutes required by the larger pipe and $m + 10$ = the number required by the smaller pipe.

Then $\frac{1}{m} + \frac{1}{m+10} = \frac{1}{24\frac{3}{4}}$

32. A tank can be filled by two pipes in 35 minutes. If the larger pipe alone can fill it in 24 minutes less time than the smaller pipe, in what time can each fill the tank?

33. John and Arthur together can do a piece of work in 3 days. If it takes John working alone $1\frac{3}{4}$ days longer than it does Arthur, in how many days can each do the work alone?

34. A cistern can be emptied by two pipes in $3\frac{1}{3}$ hours. The larger pipe alone can empty it in $1\frac{1}{2}$ hours less time than the smaller. In what time can each pipe empty the cistern?

35. A farmer bought a horse for d dollars and sold it for \$125, thus making a profit of d per cent. Find d .

36. A jeweler sold a clock for \$24, thus gaining a per cent equal to the number of dollars the clock cost. How much did the clock cost?

37. If a man puts \$2000 at interest, compounded annually, and at the end of 2 years finds that it amounts to \$2121.80, what rate of interest is he receiving?

38. Find the price of eggs per dozen when 2 less for 30 cents raises the price 2 cents a dozen.

39. Each page of a book of 400 pages was 10 inches by 6 inches. In later editions, the publishers saved 3000 square inches of paper by cutting down the margin equally on every side. By what width was the margin reduced?

40. Find the base and the altitude of a triangle whose area is 132 square inches, if the base exceeds the altitude by 10 inches.

41. The units' digit of a two-digit number is 3 greater than the tens' digit. The product of the number by the sum of its digits is 754. Find the number.

42. The denominator of a certain fraction exceeds its numerator by 1, and if the fraction is multiplied by the sum of its terms, the result is $3\frac{1}{3}$. Find the fraction.

43. A lawn 75 feet long and 60 feet wide is surrounded by a path of uniform width. If the area of the path is 2416 square feet, what is its width?

44. A man packed 2000 pounds of cherries in boxes. If each box had contained 6 pounds more, he would have used 75 boxes less. How many boxes did he use and how many pounds of cherries did each contain?

45. Three-inch squares were cut out of each corner of a square sheet of tin and a box was constructed by turning up the sides. Find the dimensions of the original sheet of tin if the volume of the box was 147 cubic inches.

46. The area of a rectangle is 18 square inches less than twice the area of a square. The width of the rectangle equals the width of the square, but the length of the rectangle exceeds that of the square by 7 inches. Find the side of the square.

47. An electric sign is 10 feet longer than it is wide and its area is 6375 square feet. Find its dimensions.

LITERAL EQUATIONS

Solve the following formulas (which are literal quadratic equations) and check the results by substituting simple numerical values for the literal known numbers:

1. $S = 4\pi r^2$, for r

4. $V = \frac{1}{12}\pi d^2 h$, for d

2. $P = \frac{nd^2}{2.5}$, for d

5. $b = \sqrt{\frac{4a^2r^2}{4r^2 - a^2}}$, for r

6. $S = a + vt - 16t^2$, for t

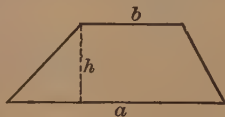
3. $F = \frac{1.5 WS^2}{R}$, for S

7. $h = r - \sqrt{r^2 - \left(\frac{z}{2}w\right)^2}$, for w

8. Arrange the formula $r = \sqrt{\frac{\pi v^3}{2.5}}$ in a form more convenient for computation. Solve the formula for v .

9. The formula $A = bh$ gives the area A of a parallelogram in terms of its base b and its altitude h . If the area of a parallelogram is 96 square feet and its base is 4 feet more than twice its height, what is its height? its base?

10. The area A of a trapezoid is expressed by the formula $A = \frac{1}{2}h(a + b)$. If the lower base a of a trapezoid is 5 feet longer than the upper base b , the altitude h is 1 foot shorter than b , and the area is 92 square feet, what are its dimensions?



11. Write the formula for the hypotenuse (h) of a right triangle in terms of the other two sides (a and b).

12. From the formula in exercise 11 and the figure at the right, deduce a formula for the diagonal (d) of a square whose side is s .



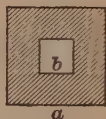
13. Find, to the nearest tenth of a foot, the diagonal of a square whose side is 8 feet; 3.2 feet.

14. If a baseball diamond is 90 feet square, what is the distance, to the nearest tenth of a foot, from first to third base?

15. Write the formula for the diagonal of a rectangle whose length is a and width is b . Solve for a .

16. The diagonal of a rectangle is 10 feet long. The rectangle is 2 feet longer than it is wide. Find its dimensions.

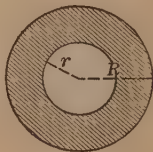
17. Express by an equality the area (A) of a square whose side is a ; the area (A) of a hollow square a units on the outside and b units on the inside.



18. The area of a hollow square is 40 square inches. If the outside dimension is twice the inside dimension plus 1 inch, what is the inside dimension?

19. The area of a flat ring is the difference between the areas of two circles of radii R and r , respectively, or $A = \pi(R^2 - r^2)$. Solve for r .

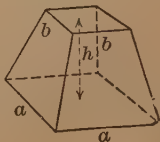
20. When the area of a ring is 1320 square feet and R equals $r + 2$, what is the value of r ? (Use $\pi = 3\frac{1}{7}$.)



21. The pressure P of the wind against a surface, in pounds per square foot, is computed from the formula $P = .005 V^2$, in which V is the velocity of the wind in miles per hour. Solve for V .

22. What is the velocity of the wind when it exerts a total pressure of 2.7 tons on a signboard that is 30 feet wide and 10 feet high?

23. The formula for the volume of the frustum of a square pyramid is $V = \frac{1}{3}h(a^2 + ab + b^2)$. Find a and b when $V = 98$, $h = 6$, and $a = b + 2$.



24. If L is the length of a pendulum that oscillates once in T seconds and l is the length of one that oscillates once in t seconds, then $\frac{L}{l} = \frac{T^2}{t^2}$. Solve for t .

25. If a pendulum that is 39.1 inches long oscillates once a second, how often does a pendulum that is 351.9 inches long oscillate?

26. A line is said to be divided in extreme and mean ratio when the longer part is a mean proportional between the whole line and the shorter part. Write the proportion for a line a whose longer part is b and whose shorter part is c . Solve for b .

27. Divide a line that is $4\frac{1}{2}$ feet long in extreme and mean ratio.

EXERCISES

Solve for x by the method best adapted, and check the results by substituting simple numerical values for the literal known numbers:

1. $x^2 - b = 0$
2. $6ax^2 - 54a = 0$
3. $x^2 - cd = cx - dx$
4. $x^2 - 4bx - 12b^2 = 0$
5. $x^2 + 2bx = b^2$
6. $x^2 + 3ax = 10a^2$
7. $x^2 - ax + bx + cx = 0$
8. $(a-x)^2 = (3x+a)(x-a)$
9. $abx^2 + a^2x - b^2x = ab$
10. $x^2 - 4bx - 7b^2 = 0$
11. $ax^2 = (a-b)(a^2-b^2) - bx^2$
12. $5cx - 2x^2 - 2c^2 = 0$
13. $16x^2 + 3a^2 - 16ax = 0$
14. $(c^2 + 1)x = cx^2 + c$
15. $a^2x^2 + 2ax^2 = (a^2 - 1)^2 - x^2$
16. $4x^2 + 12ax - 7a^2 = 0$
17. $5x^2 - 10bx - 7b^2 = 0$
18. $6ax^2 + abx = 2(6x + b)$
19. $x^2 - (b-a)c = x(a-b+c)$
20. $(b-c)x^2 + (c-a)x = b-a$
21. $\frac{a}{x} + \frac{x}{a} = \frac{ab}{x}$
22. $\frac{x}{a+b} - \frac{a-b}{x} = 0$
23. $\frac{a^2x^2}{b^2} + \frac{b^2}{c^2} = \frac{2ax}{c}$
24. $\frac{2x(b-x)}{3b-2x} = \frac{b}{4}$

Solve for x , and check:

$$25. \frac{a}{3} + \frac{5x}{4} - \frac{x^2}{3a} = 0$$

$$26. \frac{x^2 + 1}{n^2x - 2n} - \frac{1}{2 - nx} = \frac{x}{n}$$

$$27. \frac{x}{a + 1} + \frac{2a - 1}{x} = \frac{3ax}{x(a + 1)}$$

$$28. \frac{b}{x - a} + \frac{a}{x - b} - 2 = 0$$

$$29. \frac{2x - a}{b} + 3 = \frac{4a}{2x - b}$$

$$30. \frac{x + a}{x + b} + \frac{x - a}{x - b} = \frac{a^2 + b^2}{x^2 - b^2}$$

$$31. \frac{bx}{a - x} + b = \frac{a(x + 2b)}{a + b}$$

RADICAL EQUATIONS

(The work from this point to page 311 may be postponed until after Chapter XV without interrupting the sequence.)

The radical equations given here lead to *quadratic equations*, but the methods of freeing them of radicals are the same as in the cases already considered on pages 277 to 279, which lead to *linear equations*.

We have learned (page 279) that the processes of *rationalization* and *raising to powers* used in freeing radical equations of radicals are likely to introduce roots that do not check in accordance with the convention adopted. It is important, therefore, to test the roots found in the solution of equations to see whether any are **extraneous**, as well as to examine the processes employed in reducing equations to see whether any roots have been removed. No factor involving the unknown quantity should be removed from an equation without setting that factor equal to zero and solving the resulting equation.

EXERCISES

Solve and check, rejecting roots that do not satisfy the given equation, and accounting for roots that otherwise might be lost:

1. $2x - 3\sqrt{x} = 2$

5. $3x - \sqrt{x+3} - 1 = 0$

2. $x + 2\sqrt{x} = 3\sqrt{x}$

6. $\sqrt{25-6x} + \sqrt{25+6x} = 8$

3. $3\sqrt{x} - \sqrt{x-24} = 16$

7. $\sqrt{1-2x} - 2 = \sqrt{1-x}$

4. $\sqrt{2x} - \sqrt{x+1} = \frac{3}{\sqrt{x+1}}$

8. $\sqrt{5-x} = \sqrt{3+x} \sqrt{x-5}$

9. $\sqrt{x^2 - b^2} = \sqrt{x+b} \sqrt{a+b}$

10. $\sqrt{x+3} + \sqrt{4x+1} - \sqrt{10x+4} = 0$

Find the roots to the nearest hundredth:

11. $\sqrt{x^2 + 9} = \sqrt{2x+3} \sqrt{2x-3}$

12. $\sqrt{2x+50} = \sqrt{x} \sqrt{x+2}$

13. $\sqrt{x^2 + 3} = \sqrt{2x+1} \sqrt{2x-1}$

14. $\sqrt{x^2 + 5} = \sqrt{2x+3} \sqrt{x-2}$

15. $\sqrt{x} + 6 = \sqrt{3x-2}$

16. $\sqrt{2x} = \sqrt{x+2} + 1$

Solve for x , and check as directed above:

17. $\sqrt{3x-5} + \sqrt{x-9} = \sqrt{4x-4}$

18. $\frac{\sqrt{x+2a} - \sqrt{x-2a}}{\sqrt{x-2a} + \sqrt{x+2a}} = \frac{x}{2a}$

19. $x + \sqrt{x^2 - a^2} = \frac{a^2}{\sqrt{x^2 - a^2}}$

20. $\sqrt{x+a^2} - \sqrt{x-2a^2} = \sqrt{2x-5a^2}$

21. $3\sqrt{x+8} - \sqrt{x-8} = 2\sqrt{2x+2}$

22. $\frac{2\sqrt{x+1}}{2\sqrt{x}-1} = \frac{\sqrt{x+3} + \sqrt{x}}{\sqrt{x+3} - \sqrt{x}}$

23. $\frac{2x + \sqrt{4x^2 - 1}}{2x - \sqrt{4x^2 - 1}} = 4$

24. $\sqrt{\frac{x-a}{x+a}} + \sqrt{\frac{x+a}{x-a}} = a^2$

EQUATIONS IN THE QUADRATIC FORM

An equation that contains but two powers of an unknown number or expression, the exponent of one power being twice that of the other, as $ax^{2n} + bx^n + c = 0$, in which n represents any number, is in the **quadratic form**.

EXERCISES

Solve the following equations :

1. $x^4 + x^2 - 20 = 0$

2. $x^{\frac{1}{2}} - x^{\frac{1}{4}} = 6$

SOLUTION

$$x^4 + x^2 - 20 = 0$$

$$(x^2 - 4)(x^2 + 5) = 0$$

$$\therefore x^2 - 4 = 0 \text{ or } x^2 + 5 = 0$$

$$\therefore x = \pm 2 \text{ or } \pm \sqrt{-5}$$

Any one of these values substituted for x in the given equation satisfies the equation, and is therefore a root of it.

SOLUTION

$$x^{\frac{1}{2}} - x^{\frac{1}{4}} = 6$$

$$x^{\frac{1}{2}} - x^{\frac{1}{4}} + \left(\frac{1}{4}\right)^2 = \frac{25}{4}$$

$$x^{\frac{1}{4}} - \frac{1}{2} = \pm \frac{5}{2}$$

$$\therefore x^{\frac{1}{4}} = 3 \text{ or } -2$$

$$\therefore x = 81 \text{ or } 16$$

Since $x = 16$ does not check, 16 is not a root and should be rejected.

3. $x^4 + 17x^2 - 84 = 0$

8. $x^{\frac{1}{3}} - 3x^{\frac{1}{6}} = -2$

4. $3x^4 + 5x^2 - 8 = 0$

9. $x^{\frac{1}{2}} + 3x^{\frac{1}{4}} - 28 = 0$

5. $5x^4 + 6x^2 - 11 = 0$

10. $x + 3\sqrt{x} = 4$

6. $(x - 2)^2 + 3(x - 2) = 10$

11. $x^{\frac{2}{3}} - 4x^{\frac{1}{3}} = 12$

7. $(x^2 + 1)^2 + 4(x^2 + 1) = 45$

12. $x^{\frac{4}{3}} = 17x^{\frac{2}{3}} - 16$

13. $x - 4x^{\frac{2}{3}} + 3x^{\frac{1}{3}} = 0$

SOLUTION

Factor :

$$x^{\frac{1}{3}}(x^{\frac{1}{3}} - 1)(x^{\frac{1}{3}} - 3) = 0$$

$$\text{that is, } x^{\frac{1}{3}} = 0, x^{\frac{1}{3}} - 1 = 0, \text{ or } x^{\frac{1}{3}} - 3 = 0$$

$$\text{whence } x^{\frac{1}{3}} = 0, 1, \text{ or } 3$$

Cube each member of the equation : $x = 0, 1, \text{ or } 27$

Each of these values of x satisfies the given equation and is a root of it.

Solve, and check results :

14. $x^{\frac{3}{2}} - 4x - 5x^{\frac{1}{2}} = 0$

17. $5x = x\sqrt{x} + 6\sqrt{x}$

15. $x - 3x^{\frac{3}{4}} + 2x^{\frac{1}{2}} = 0$

18. $3x = x\sqrt[3]{x} + 2\sqrt[3]{x^2}$

16. $x + 2x^{\frac{5}{3}} - 3x^{\frac{4}{3}} = 0$

19. $2x + \sqrt{x} = 15x\sqrt{x}$

20. $x^2 - 3x + 2\sqrt{x^2 - 3x + 6} = 18$

SOLUTION

$$x^2 - 3x + 2\sqrt{x^2 - 3x + 6} = 18$$

$$x^2 - 3x + 6 + 2\sqrt{x^2 - 3x + 6} = 24$$

Let $\sqrt{x^2 - 3x + 6} = p$ and $x^2 - 3x + 6 = p^2$

$$p^2 + 2p = 24$$

$$p = 4 \text{ or } -6$$

$$\sqrt{x^2 - 3x + 6} = 4 \text{ or } -6$$

$$x^2 - 3x + 6 = 16$$

$$x = 5 \text{ or } -2$$

Since, according to page 279, the radical $\sqrt{x^2 - 3x + 6}$ cannot equal a negative number, $\sqrt{x^2 - 3x + 6} = -6$ is an impossible equation. Hence the only roots of the given equation are 5 and -2 .

21. $x - 2\sqrt{x + 1} = 7$

22. $\sqrt{x + 14} - \sqrt[4]{x + 14} = 2$

23. $x^2 + 2x + 10 - \sqrt{x^2 + 2x + 10} = 20$

24. $3x^2 - 16x = 7 - 3\sqrt{3x^2 - 16x + 21}$

25. $x^2 - x + 4\sqrt{x^2 - x - 8} = 20$

26. $3x^2 - x = 22 + 6\sqrt{3x^2 - x - 6}$

GRAPHS OF QUADRATIC EQUATIONS — TWO UNKNOWN NUMBERS

EXERCISES

1. Construct the graph of the equation $x^2 + y^2 = 25$.

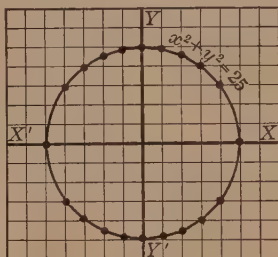
SOLUTION

Solve for y : $y = \pm\sqrt{25 - x^2}$

Any value numerically greater than 5 substituted for x will give the square root of a negative number for y —an *imaginary number* (page 351), which cannot be plotted. Hence we substitute only values of x between and including -5 and $+5$. The corresponding values of x and y , or of x and $\pm\sqrt{25 - x^2}$, are given in the table on page 306.

It will be observed that each value, except ± 5 , substituted for x in the equation $y = \pm \sqrt{25 - x^2}$ gives two values of y , and that values of x numerically equal give the same values of y ; thus, when $x = 2$, $y = \pm 4.6$, and also when $x = -2$, $y = \pm 4.6$.

x	y
0	± 5
± 1	± 4.9
± 2	± 4.6
± 3	± 4
± 4	± 3
± 5	0



The values given in the table serve to locate twenty points of the graph of $x^2 + y^2 = 25$. Plotting these points and drawing a smooth curve through them, we see that the graph is apparently a *circle*. It may be proved by geometry that this graph is a circle whose radius is 5.

The graph of any equation of the form $x^2 + y^2 = r^2$ is a **circle** whose radius is r and whose center is at the origin.

✓ 2. Construct the graph of the equation $x^2 + y^2 = 49$.

3. Construct the graph of the equation :

$$(x - 2)^2 + (y - 3)^2 = 9$$

SUGGESTION. Solving for y , we have $y = 3 \pm \sqrt{9 - (x - 2)^2}$.

Since any value less than -1 or greater than $+5$ substituted for x makes the value of y imaginary, the graph lies between $x = -1$ and $x = +5$.

The graph of any equation of the form $(x - a)^2 + (y - b)^2 = r^2$ is a **circle** whose radius is r and whose center is at the point (a, b) .

4. Construct the graph of the equation $y^2 = 3x + 9$.

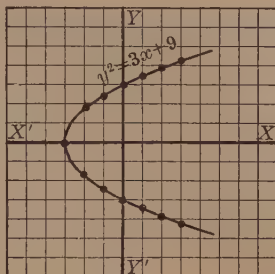
SOLUTION

Solve for y : $y = \pm \sqrt{3x + 9}$

It will be observed that any value smaller than -3 substituted for x will make y imaginary; consequently no point of the graph lies to the

left of $x = -3$. Beginning with $x = -3$, we substitute values for x and determine the corresponding values of y , as recorded in the table.

x	y
-3	0
-2	± 1.7
-1	± 2.4
0	± 3
1	± 3.5
2	± 3.9
3	± 4.2



Plotting these points and drawing a smooth curve through them, we find that the graph obtained is a curve which is called a **parabola**.

The graph of any equation of the form $y^2 = ax + c$ is a **parabola**.

5. Construct the graph of $y^2 = 5x + 8$.

6. Construct the graph of $y^2 = 4x$.

7. Construct the graph of $x^2 = 6y$.

8. Construct the graph of the equation $9x^2 + 25y^2 = 225$.

SOLUTION

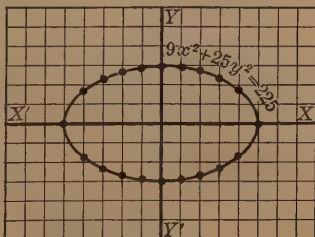
Solve for y :

$$y = \pm \frac{3}{5} \sqrt{25 - x^2}$$

Since any value numerically greater than 5 substituted for x will make the value of y imaginary, no point of the graph lies farther to the right or to the left of the origin than 5 units; consequently we substitute for x only values between and including -5 and $+5$.

Corresponding values of x and y are given in the following table.

x	y
0	± 3
± 1	± 2.9
± 2	± 2.7
± 3	± 2.4
± 4	± 1.8
± 5	0



Plotting these twenty points and drawing a smooth curve through them, we have the graph of $9x^2 + 25y^2 = 225$, which is called an **ellipse**.

The graph of any equation of the form $b^2x^2 + a^2y^2 = a^2b^2$ is an **ellipse**.

The ellipse is an important curve in science, as is shown by the fact that the orbits of the earth and the other planets in their rotation about the sun are *ellipses*. The paths of all projectiles, such as one from a cannon or a stream of water from a hose, are arcs of curves of the *parabola* type shown on page 307.

9. Construct the graph of the equation $9x^2 + 16y^2 = 144$.

10. Construct the graph of the equation $4x^2 - 9y^2 = 36$.

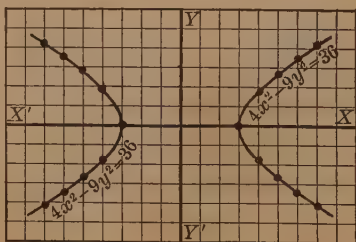
SOLUTION

Solve for y :

$$y = \pm \frac{2}{3} \sqrt{x^2 - 9}$$

Since any value numerically less than 3 substituted for x will make the value of y imaginary, no point of the graph lies between $x = +3$ and $x = -3$; consequently we substitute for x only ± 3 and values numerically greater than 3. Corresponding values of x and y are given in the table.

x	y
± 3	0
± 4	± 1.8
± 5	± 2.7
± 6	± 3.5
± 7	± 4.2



Plotting these eighteen points, we find that half of them are on one side of the y -axis and half on the other side, and since there are no points of the curve between $x = +3$ and $x = -3$, the graph has two separate branches, that is, it is *discontinuous*.

Drawing a smooth curve through each group of points, we see that the two branches thus constructed constitute the graph of the equation $4x^2 - 9y^2 = 36$, which is an **hyperbola**.

The graph of any equation of the form $b^2x^2 - a^2y^2 = a^2b^2$ is an **hyperbola**.

An hyperbola has two **branches** and is called a **discontinuous curve**.

11. Construct the graph of the equation:

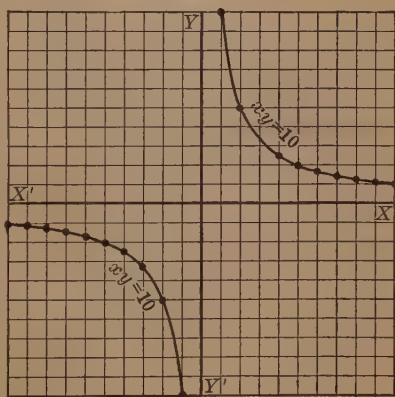
$$9x^2 - 16y^2 = 144$$

12. Construct the graph of the equation $xy = 10$.

SOLUTION

Substituting values for x and solving for y , we find the corresponding values of x and y as given in the table.

x	y	x	y
1	10	- 1	- 10
2	5	- 2	- 5
3	$3\frac{1}{3}$	- 3	- $3\frac{1}{3}$
4	$2\frac{1}{2}$	- 4	- $2\frac{1}{2}$
5	2	- 5	- 2
6	$1\frac{2}{3}$	- 6	- $1\frac{2}{3}$
7	$1\frac{1}{7}$	- 7	- $1\frac{1}{7}$
8	$1\frac{1}{4}$	- 8	- $1\frac{1}{4}$
9	$1\frac{1}{9}$	- 9	- $1\frac{1}{9}$
10	1	- 10	- 1



Plotting these points and drawing a smooth curve through each group of points, we see that the two branches of the curve found constitute the graph of the equation $xy = 10$, which is an **hyperbola**.

The graph of any equation of the form $xy = c$ is an **hyperbola**.

The graph of a second-degree equation in two unknown quantities may be a *circle*, a *parabola*, an *ellipse*, or an *hyperbola*.

13. Construct the graph of the equation $xy = 12$.

14. Construct the graph of the equation $xy = - 12$.

SIMULTANEOUS EQUATIONS INVOLVING QUADRATIC EQUATIONS—GRAPHIC SOLUTION

The graphic method of solving simultaneous equations that involve quadratic equations is precisely the same as for simultaneous linear equations, namely :

Construct the graph of each equation, both being referred to the same axes, and determine the coördinates of the points where the graphs intersect. If they do not intersect, interpret this fact.

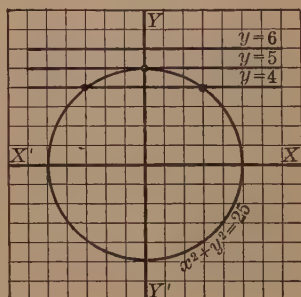
EXERCISES

1. Solve graphically $\begin{cases} x^2 + y^2 = 25 \\ y = 4 \end{cases}$

SOLUTION

The graphs (a circle and a straight line) are found to intersect at the points $x = 3, y = 4$; $x = -3, y = 4$

Since the graphs have *only* these two points in common, their coördinates are the only values of x and y that satisfy both equations, and are the roots sought. The pairs of values found are *real* and *different*, or *unequal*.



2. Solve graphically $\begin{cases} x^2 + y^2 = 25 \\ y = 5 \end{cases}$

SUGGESTION. Imagine the straight line $y = 4$, in the figure for exercise 1, moved upward until it coincides with the line $y = 5$. The real, unequal roots represented by the coördinates of the points of intersection approach equality, and when the line becomes the tangent line $y = 5$, they coincide. Hence the given system of equations has *two real, equal roots*: $x = 0, y = 5$, and $x = 0, y = 5$.

3. Find the nature of the roots of $\begin{cases} x^2 + y^2 = 25 \\ y = 6 \end{cases}$

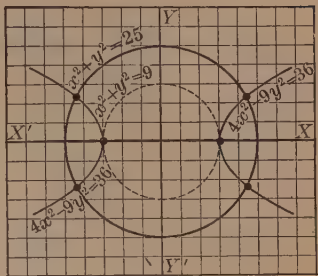
SUGGESTION. Imagine the straight line $y = 4$, in the figure for exercise 1, moved upward until it coincides with the line $y = 6$. The graphs will cease to have any points in common, showing that the given equations have *no common real values* of x and y . It is shown by algebraic solution of the equations that there are two roots and that both are *imaginary*.

4. Solve graphically :

$$\begin{cases} 4x^2 - 9y^2 = 36 \\ x^2 + y^2 = 25 \end{cases}$$

SUGGESTION. The graphs show that both of the given equations are satisfied by *four* different pairs of real values of x and y :

$$\begin{cases} x = 4.5; & 4.5; & -4.5; & -4.5 \\ y = 2.2; & -2.2; & -2.2; & 2.2 \end{cases}$$



5. What would be the nature of the roots in exercise 4, if the second equation were $x^2 + y^2 = 9$?

A system of two independent simultaneous equations in x and y , one linear and the other quadratic, has two roots.

The roots are real and unequal if the graphs intersect, real and equal if the graphs are tangent to each other, and imaginary if the graphs have no points in common.

If the two equations are both quadratic, there will usually be four sets of roots, but never more than four sets.

Find by graphic methods, to the nearest tenth, the real roots of the following, and the number of imaginary roots, if there are any. Discuss the graphs and the roots :

6. $\begin{cases} x^2 + y^2 = 36 \\ 2x - 3y = 6 \end{cases}$

7. $\begin{cases} 2x + 5y = 10 \\ 5x^2 + 2y^2 = 125 \end{cases}$

8. $\begin{cases} x - 3y = 2 \\ 9x^2 - 16y^2 = 144 \end{cases}$

9. $\begin{cases} xy = -1 \\ x - y = 2 \end{cases}$

10. $\begin{cases} x^2 - 4y^2 = 4 \\ x + y = 2 \end{cases}$

11. $\begin{cases} xy = -2 \\ x^2 + 4y^2 = 17 \end{cases}$

12. $\begin{cases} y^2 = 4x + 8 \\ x^2 + y^2 = 9 \end{cases}$

13. $\begin{cases} x^2 + y^2 = 16 \\ 9x^2 + 16y^2 = 144 \end{cases}$

14. $\begin{cases} x^2 + y^2 = 49 \\ x^2 - y^2 = 64 \end{cases}$

15. $\begin{cases} x^2 - y^2 = 49 \\ x^2 + y^2 = 64 \end{cases}$

16. $\begin{cases} 3x - y = 12 \\ x^2 - y^2 = 16 \end{cases}$

17. $\begin{cases} xy = 20 \\ x^2 + y^2 = 41 \end{cases}$

SIMULTANEOUS EQUATIONS INVOLVING QUADRATIC EQUATIONS — ALGEBRAIC SOLUTION

When one equation is of the first and the other of higher degree

Equations of this class may be solved by substitution.

EXERCISES

1. Solve the equations:

$$\begin{aligned} (1) \quad & \begin{cases} x + y = 5 \\ x^2 + 2y^2 = 17 \end{cases} \end{aligned}$$

SOLUTION

$$\begin{aligned} (3) \quad & y = 5 - x && \text{From (1)} \\ (4) \quad & x^2 + 2(5 - x)^2 = 17 && (3) \text{ substituted in (2)} \\ (5) \quad & x = 3 \text{ or } \frac{1}{3} && (4) \text{ solved} \\ (6) \quad & y = 2 \text{ or } \frac{4}{3} && 3 \text{ and } \frac{1}{3} \text{ substituted for } x \text{ in (3)} \end{aligned}$$

Hence x and y each have two corresponding values associated as follows:

$$\begin{cases} x = 3 \text{ or } \frac{1}{3} \\ y = 2 \text{ or } \frac{4}{3} \end{cases}$$

Solve the following equations:

$$2. \quad \begin{cases} x = 3y \\ x^2 + y^2 = 40 \end{cases}$$

$$9. \quad \begin{cases} x^2 + y^2 = 34 \\ x + y = 8 \end{cases}$$

$$3. \quad \begin{cases} x + y = 3 \\ x^2 + 2xy = 8 \end{cases}$$

$$10. \quad \begin{cases} x^2 + xy = 12 \\ x - y = 2 \end{cases}$$

$$4. \quad \begin{cases} x^2 - 2y^2 = 7 \\ x - y = 8 \end{cases}$$

$$11. \quad \begin{cases} x + y = 7 \\ 3x^2 + y^2 = 43 \end{cases}$$

$$5. \quad \begin{cases} x^2 + y^2 = 25 \\ 3x + 4y = 0 \end{cases}$$

$$12. \quad \begin{cases} 2x + y = 5 \\ 3x^2 - 7y^2 = 5 \end{cases}$$

$$6. \quad \begin{cases} 10x + y = 3xy \\ y - x = 2 \end{cases}$$

$$13. \quad \begin{cases} x^2 + xy + y^2 = 12 \\ 2x + 2 = 3y \end{cases}$$

$$7. \quad \begin{cases} m^2 - 3n^2 = 13 \\ m - 2n = 1 \end{cases}$$

$$14. \quad \begin{cases} 5x^2 + 3y^2 = 16 \\ 3x - 3y = 8 \end{cases}$$

$$8. \quad \begin{cases} 3y^2 - z^2 = 8 \\ 2y = 2 - z \end{cases}$$

$$15. \quad \begin{cases} 3x^2 - xy - y^2 = -3 \\ 2x - 3y = 5 \end{cases}$$

$$16. \begin{cases} 9x^2 + 49y^2 = 85 \\ 3x - 7y = 13 \end{cases}$$

$$17. \begin{cases} y - 3x = 1 \\ xy + x^2 = 5 \end{cases}$$

$$18. \begin{cases} 2x - y = 4 \\ xy = 6 \end{cases}$$

$$19. \begin{cases} x + 2y = 12 \\ xy + y^2 = 32 \end{cases}$$

$$20. \begin{cases} 2x + 4y = 20 \\ x^2 + y^2 = 25 \end{cases}$$

$$21. \begin{cases} x + y = 7 \\ x^2 - xy + y^2 = 19 \end{cases}$$

$$22. \begin{cases} x^2 - xy = 160 \\ 1.5x - .5y = 6 \end{cases}$$

$$23. \begin{cases} x^2 + y^2 = 26 \\ 2x - 3y = 7 \end{cases}$$

$$24. \begin{cases} x + y = 11 \\ 4x^2 - 3y^2 = -8 \end{cases}$$

$$25. \begin{cases} 2x + 3y = 14 \\ x^2 - y^2 = 12 \end{cases}$$

Problems

1. The sum of two numbers is 8 and their product is 15. Find the numbers.

2. The sum of two numbers is 9 and the difference between their squares is 9. What are the numbers?

3. The product of two numbers is 48 and their quotient is 3. What are the numbers?

4. The perimeter of a rectangular field is 90 rods and its area is 450 square rods. Find its dimensions.

5. A diagonal path across a rectangular field is 26 rods long, and the perimeter of the field is 68 rods. Find the dimensions of the field.

6. The sum of two numbers is 16. The sum of their squares exceeds 13 times the larger number by 4. What are the numbers?

7. Find two numbers whose sum is 3 times their difference and the difference of whose squares is 675.

When both equations are symmetric

An equation that is not affected by interchanging the unknown numbers involved is called a **symmetric equation**.

$x^2 + xy + y^2 = 7$ and $3x^2 + 3y^2 = 4$ are symmetric equations.

Equations of this class may sometimes be solved advantageously first for $x + y$ and $x - y$ and then for x and y .

EXERCISES

1. Solve the equations:

$$(1) \quad \begin{cases} x^2 + y^2 = 13 \\ x + y = 5 \end{cases}$$

$$(2) \quad \begin{cases} x^2 + y^2 = 13 \\ x + y = 5 \end{cases}$$

SOLUTION

$$(3) \quad x^2 + 2xy + y^2 = 25$$

(2) squared

$$(4) \quad 2xy = 12$$

(3) - (1)

$$(5) \quad x^2 - 2xy + y^2 = 1$$

(1) - (4)

$$(6) \quad x - y = \pm 1$$

Square root of (5)

$$(7) \quad x = 3 \text{ or } 2$$

From (2) and (6)

$$(8) \quad y = 2 \text{ or } 3$$

From (2) and (6)

2. Solve the equations $\begin{cases} x + y = 11 \\ xy = 30 \end{cases}$

SUGGESTION. From the square of the first equation subtract 4 times the second equation and then proceed as in exercise 1.

Solve the following simultaneous equations:

$$3. \quad \begin{cases} x + y = 7 \\ xy = 6 \end{cases}$$

$$9. \quad \begin{cases} x^2 + y^2 = 37 \\ x + y = 7 \end{cases}$$

$$4. \quad \begin{cases} xy = 36 \\ x + y = 15 \end{cases}$$

$$10. \quad \begin{cases} x^2 + y^2 = 13 \\ x + y = 5 \end{cases}$$

$$5. \quad \begin{cases} x^2 + y^2 = 17 \\ xy = 4 \end{cases}$$

$$11. \quad \begin{cases} x + y = 7 \\ x^2 + y^2 = 25 \end{cases}$$

$$6. \quad \begin{cases} x + y = 4 \\ x^2 + xy + y^2 = 13 \end{cases}$$

$$12. \quad \begin{cases} x + y = 3 \\ x^2 + xy + y^2 = 7 \end{cases}$$

$$7. \quad \begin{cases} x^2 + xy + y^2 = 49 \\ x + y = 8 \end{cases}$$

$$13. \quad \begin{cases} x^2 + y^2 = 26 \\ x^2 - xy + y^2 = 21 \end{cases}$$

$$8. \quad \begin{cases} x^2 + xy + y^2 = 57 \\ x^2 + y^2 = 50 \end{cases}$$

$$14. \quad \begin{cases} x^2 + y^2 = 8 \\ x^2 - xy + y^2 = 4 \end{cases}$$

Special devices

Many systems of equations may be solved readily by *special devices*. Though it is impossible to lay down any

fixed line of procedure, the object often aimed at is to find values for *any two* of the expressions $x + y$, $x - y$, and xy from which the values of x and y may be obtained.

EXERCISES

1. Solve the equations :

$$\begin{aligned} (1) \quad & \begin{cases} x^2 + xy = 20 \\ xy + y^2 = 5 \end{cases} \end{aligned}$$

SOLUTION

$$\begin{array}{ll} (3) \quad x^2 + 2xy + y^2 = 25 & (1) + (2) \\ (4) \quad \quad \quad x + y = +5 \text{ or } -5 & \text{Square root of (3)} \\ (5) \quad \quad \quad x^2 - y^2 = 15 & (1) - (2) \\ (6) \quad \quad \quad x - y = +3 \text{ or } -3 & (5) \div (4) \\ (7) \quad \quad \quad \quad \quad x = 4 \text{ or } -4 & (4) + (6) \\ (8) \quad \quad \quad \quad \quad y = 1 \text{ or } -1 & (4) - (6) \end{array}$$

NOTE. The first value of $x - y$ corresponds only to the first value of $x + y$, and the second value only to the second value.

Consequently there are only two pairs of values of x and y .

$$2. \text{ Solve the equations } \begin{cases} x^2 + y^2 + x + y = 14 \\ xy = 3 \end{cases}$$

SUGGESTION. Adding twice the second equation to the first, we have

$$x^2 + 2xy + y^2 + x + y = 20, \text{ or } (x + y)^2 + (x + y) = 20$$

This equation may be solved for $x + y$ and the results combined with $xy = 3$.

When both equations are **symmetric except as to sign**, that is, when one equation is symmetric and when the other would be so if some of its signs were changed or when both are of the latter type, the method of solution is the same as on page 314.

$$3. \text{ Solve the equations } \begin{cases} x^2 + y^2 = 53 \\ x - y = 5 \end{cases}$$

SUGGESTION. Subtract the square of the second equation from the first equation, obtaining $2xy = 28$. Add this equation to the first equation and solve the resulting equation for $x + y$.

4. Solve the equations
$$\begin{cases} \frac{1}{x^2} + \frac{1}{y^2} = 74 \\ \frac{1}{x} - \frac{1}{y} = 2 \end{cases}$$

SUGGESTION. Proceed as in exercise 3, solving for $\frac{1}{x} + \frac{1}{y}$, then for $\frac{1}{x}$ and $\frac{1}{y}$, and finally for x and y .

The reduction of equations of higher degree to quadratic equations is often effected by **dividing one of the given equations by the other, member by member.**

5. Solve the equations
$$\begin{cases} x^3 - y^3 = 54 \\ x - y = 6 \end{cases}$$

SUGGESTION. Divide the first equation by the second equation, obtaining $x^2 + xy + y^2 = 9$, and solve the system made up of this result and the second equation.

An equation *all* of whose terms are of the same degree with respect to the unknown numbers is called a **homogeneous equation.**

$x^2 - xy = y^2$ and $3x^2 + y^2 = 0$ are homogeneous equations.

An equation like $x^2 - xy + y^2 = 21$ is said to be **homogeneous in the unknown terms.**

6. Solve the equations:

$$\begin{aligned} (1) \quad & \begin{cases} x^2 + xy = 12 \\ xy + 2y^2 = 5 \end{cases} \\ (2) \quad & \end{aligned}$$

SOLUTION

$$(3) \quad 5x^2 + 5xy = 60 \qquad 5 \times (1)$$

$$(4) \quad 12xy + 24y^2 = 60 \qquad 12 \times (2)$$

$$(5) \quad 5x^2 - 7xy - 24y^2 = 0 \qquad (3) - (4)$$

(6) $(5x + 8y)(x - 3y) = 0$ from which we obtain the two equations $5x + 8y = 0$ and $x - 3y = 0$. Solve the systems made up of these two equations and (1).

Using the methods illustrated in exercises 1 to 6, solve:

- | | |
|---|--|
| 7. $\begin{cases} x^2 + 2y^2 = 17 \\ xy - y^2 = 2 \end{cases}$ | 15. $\begin{cases} y^2 + z^2 = 41 \\ y - z = 1 \end{cases}$ |
| 8. $\begin{cases} x^2 - xy + y^2 = 21 \\ x^2 + 2y^2 = 27 \end{cases}$ | 16. $\begin{cases} x^3 + y^3 = 126 \\ x + y = 6 \end{cases}$ |
| 9. $\begin{cases} x^2 - y^2 = 24 \\ xy = 35 \end{cases}$ | 17. $\begin{cases} \frac{1}{x^2} + \frac{1}{y^2} = 13 \\ \frac{1}{x} - \frac{1}{y} = 1 \end{cases}$ |
| 10. $\begin{cases} x^3 - y^3 = .056 \\ x - y = .2 \end{cases}$ | 18. $\begin{cases} r^2 - s^2 = 54 \\ r - s = 6 \end{cases}$ |
| 11. $\begin{cases} x^3 + y^3 = 35 \\ x + y = 5 \end{cases}$ | 19. $\begin{cases} c^4 + c^2d^2 + d^4 = 3 \\ c^2 - cd + d^2 = 3 \end{cases}$ |
| 12. $\begin{cases} x^3 - y^3 = 130 \\ x - y = 8 \end{cases}$ | 20. $\begin{cases} \frac{1}{x^3} - \frac{1}{y^3} = \frac{7}{8} \\ \frac{1}{x} - \frac{1}{y} = \frac{1}{2} \end{cases}$ |
| 13. $\begin{cases} x^2 + xy = 30 \\ xy + y^2 = 6 \end{cases}$ | |
| 14. $\begin{cases} p^2 + q^2 + p + q = 36 \\ pq = -15 \end{cases}$ | |

SUMMARY OF SIMULTANEOUS QUADRATIC EQUATIONS

It has been shown that when a system of quadratic equations consists of one linear equation, or equation of the first degree, and one equation of the second degree, a solution may be obtained. Several special forms of equations have been solved by quadratic methods; but, in general, simultaneous quadratic equations cannot be solved by such simple methods.

The general case in which the two equations are of the form $ax^2 + bxy + cy^2 + dx + ey + f = 0$ is more difficult and leads to equations of the fourth degree. The solution of equations as simple as $x^2 + y = 7$ and $x + y^2 = 2$ involves equations of the fourth degree. The solutions of such equations is studied in higher algebra.

The number of solutions cannot exceed the product of the degrees of the two equations. Frequently the maximum number of solutions is not found, as in systems like $x + y = 3$ and $x^3 + y^3 = 117$; $x^2 - xy = 5$ and $2x^2 + xy = 7$. In such cases the work should be looked over carefully to see that there are no mistakes. The solutions should be grouped in pairs, preferably in the form $(x, y) = (a, b)$, and each pair should be checked by substitution in the original equations.

MISCELLANEOUS EXERCISES

Solve the following simultaneous equations:

1. $\begin{cases} x - y = 4 \\ xy = -4 \end{cases}$
2. $\begin{cases} x^2 + y^2 = 52 \\ 3x = 2y \end{cases}$
3. $\begin{cases} 1 + x = y \\ x^2 + y^2 = 61 \end{cases}$
4. $\begin{cases} 2x - y = 2 \\ 2x^2 + y^2 = \frac{3}{2} \end{cases}$
5. $\begin{cases} x^2 + 3xy = y^2 + 23 \\ x + 3y = 9 \end{cases}$
6. $\begin{cases} x + 2y = 8 \\ x^2 + y^2 = 13 \end{cases}$
7. $\begin{cases} x + y = 3 \\ x^3 + y^3 = 117 \end{cases}$
8. $\begin{cases} x^3 + y^3 = 341 \\ x + y = 11 \end{cases}$
9. $\begin{cases} x^2 + 3xy - y^2 = 43 \\ x + 2y = 10 \end{cases}$
10. $\begin{cases} x^2 - xy + y^2 = 7 \\ 2x - 3y = 0 \end{cases}$
11. $\begin{cases} 4x + 5y = 6 \\ xy = -2 \end{cases}$
12. $\begin{cases} 6y^2 - xy = 2x^2 \\ 9y - 12 = -4x \end{cases}$
13. $\begin{cases} 3x^2 - 2y^2 = 19 \\ 2x^2 - 3y^2 = 6 \end{cases}$
14. $\begin{cases} x^2 + xy = 77 \\ xy - y^2 = 12 \end{cases}$
15. $\begin{cases} x^2 - xy = 48 \\ xy - y^2 = 12 \end{cases}$
16. $\begin{cases} a + ab + 28 = 0 \\ b + ab + 40 = 0 \end{cases}$
17. $\begin{cases} xy = -12 \\ x^2 + y^2 + x + y = 26 \end{cases}$
18. $\begin{cases} x^2 + xy - y^2 = -31 \\ x^2 - xy + y^2 = 49 \end{cases}$
19. $\begin{cases} m^4 - n^4 = 65 \\ m^2 - n^2 = 5 \end{cases}$
20. $\begin{cases} p^2 - q^2 = p + q \\ p + 2q = 7 \end{cases}$
21. $\begin{cases} m + n + 2\sqrt{m + n} = 24 \\ m - n + 3\sqrt{m - n} = 10 \end{cases}$

Solve:

$$22. \begin{cases} r^2 + rs + s^2 = 37 \\ r^2 + s^2 = 25 \end{cases}$$

$$30. \begin{cases} x^2 + y^2 = 369 \\ x^2 - y^2 = 9 \end{cases}$$

$$23. \begin{cases} t^2 + u^2 + t + u = 26 \\ tu = -12 \end{cases}$$

$$31. \begin{cases} x^2 + xy + y^2 = 84 \\ x^2 - xy + y^2 = 6 \end{cases}$$

$$24. \begin{cases} x^3 - y^3 = 127 \\ x^2y - xy^2 = 42 \end{cases}$$

$$32. \begin{cases} x + y = 25 \\ \sqrt{x} + \sqrt{y} = 7 \end{cases}$$

$$25. \begin{cases} x^2 + xy = -6 \\ xy + y^2 = 15 \end{cases}$$

$$33. \begin{cases} \frac{1}{m} + \frac{1}{n} = \frac{1}{2} \\ \frac{1}{mn} - \frac{1}{18} = 0 \end{cases}$$

$$26. \begin{cases} x^2 + 3y^2 = 28 \\ 3x^2 + 3y^2 = 60 \end{cases}$$

$$34. \begin{cases} x^2 - xy = a^2 + b^2 \\ xy - y^2 = 2ab \end{cases}$$

$$27. \begin{cases} x^2 + y^2 = 40 \\ xy = 12 \end{cases}$$

$$35. \begin{cases} x - 2y = 2(a + b) \\ xy + 2y^2 = 2b(b - a) \end{cases}$$

$$28. \begin{cases} p(p + q) = p \\ q(p - q) = -1 \end{cases}$$

$$36. \begin{cases} x^2 - xy - 5 = 0 \\ 2x^2 + xy - 7 = 0 \end{cases}$$

$$29. \begin{cases} x^2 + xy + y^2 = 151 \\ x^2 + y^2 = 106 \end{cases}$$

$$37. \begin{cases} r^2 + s^2 + 6\sqrt{r^2 + s^2} = 55 \\ r^2 - s^2 = 7 \end{cases}$$

$$38. \text{ Solve for } a \text{ and } t \begin{cases} s = \frac{1}{2} at^2 \\ v = at \end{cases}$$

$$39. \text{ Solve for } v \text{ and } t \begin{cases} s = 6t + \frac{1}{2} at^2 \\ v = at \end{cases}$$

Problems

1. The sum of two numbers is 16 and their product is 48. What are the numbers?

2. The difference between two numbers is 4 and their product is 77. Find the numbers.

3. The product of two numbers is 108 and their quotient is $1\frac{1}{3}$. Find the numbers.

(4) The sum of two numbers is 8 and the sum of their squares is 40. Find the numbers.

5. The sum of two numbers is 82 and the sum of their square roots is 10. What are the numbers?

6. The perimeter of a rectangle is 20 inches and its area is 24 square inches. Find its dimensions.

7. The product of two numbers is s^2 and the difference between them is 8 times the smaller number. What are the numbers?

8. The perimeter of a floor is 44 feet and its area is 120 square feet. Find its length and its width.

9. Two numbers have the ratio of 1 to 3. If the difference between their squares is 1800, what are the numbers?

10. If a side of one square is added to a side of another square, the sum is 12. If the difference between the areas of the two squares is 3, what is the side of each?

11. The area of a rectangular field is 3 acres and its length is 4 rods more than its width. Find its dimensions.

12. An Indian blanket has an area of 35 square feet. If its width were 1 foot less and its length 1 foot more, the former dimension would be $\frac{1}{2}$ of the latter. Find its dimensions.

13. The product of two numbers is 18 less than 10 times the larger number and 8 less than 10 times the smaller number. Find the numbers.

14. If a two-digit number is multiplied by its units' digit, the result is 24. If the sum of the digits is added to the number, the result is 15. What is the number?

15. The perimeter of a right triangle is 12 feet and its hypotenuse is 1 foot longer than its base. Find its base.

16. If a two-digit number is multiplied by the sum of its digits, the result is 198. If it is divided by the sum of its digits, the result is $5\frac{1}{2}$. Find the number.

17. The area of a right triangular lot is $1\frac{1}{2}$ acres. The sum of the two shorter sides exceeds the other side by 12 rods. Find the dimensions of the lot.

18. The base of a triangle was 7 inches longer than its altitude and its area was $\frac{1}{2}$ of a square foot. Find the dimensions of the triangle.

19. The difference between two numbers is $2a$ and their product is b . Find the numbers.

20. A certain door mat has an area of 882 square inches. If its length had been 6 inches less and its width $5\frac{1}{2}$ inches more, the mat would have been square. Find the dimensions of the door mat.

21. I paid 75¢ for ribbon. If it had cost 10¢ less a yard, I could have bought 2 yards more for the same money. How many yards did I buy and what was the price per yard?

22. A man spent \$6 for canvas. If it had cost \$.04 less a yard, he would have received 5 yards more. How many yards did he buy and at what price per yard?

23. The central court of the New York State Capitol has an area of 12,604 square feet. What are the dimensions of the court, if the width is 2 feet more than twice the difference between the length and the width?

24. The radius of one circle is $\frac{2}{3}$ that of another circle. If the sum of the areas of the circles is 117π square feet, how long is the radius of each circle?

25. A grocer sold carrots for \$4.40. If the number of bunches had been 4 less and the price per bunch \$.01 more, he would still have received \$4.40. Find the price per bunch.

26. One machine sticks 720,000 pins into the papers a day. If the machine ran 2 hours longer daily and stuck into the papers 18,000 pins less hourly, the result would be the same. How long does the machine run per day?

27. A merchant bought a piece of cloth for \$147. He cut 12 yards that were damaged from the piece and then sold the remainder for \$120.25 at a gain of \$.25 a yard. How many yards did he buy? What was the cost per yard?

28. A ship was loaded with 2000 tons of coal. If 50 tons more had been put on per hour, it would have taken 1 hour 20 minutes less time to load the whole amount. How long did it take to load the coal?

29. The perimeter of a certain square is the same as that of a rectangle, but the area of the square is 9 more than the area of the rectangle. The diagonal of the rectangle is to the diagonal of the square as 5 is to 4. Find the dimensions of the figures.

SUGGESTION. Let the side of the square equal $\frac{l+w}{2}$.

30. A boy has a large blotter, 4 inches longer than it is wide, and 480 square inches in area. He wishes to cut away enough to leave a square 256 square inches in area. How many inches must he cut from the length and from the width?

31. The total area of a rug whose length is 3 feet more than its width is 108 square feet. The area of the rug exclusive of the border is 54 square feet. Find the width of the border.

32. After a mowing machine had made the circuit of a 7-acre rectangular hay field 11 times, cutting a swath 6 feet wide each time, 4 acres of grass were still standing. Find the dimensions of the field in rods.

33. A bank deposit at the end of 1 year amounts to \$3990. At 1% less interest, with \$200 more principal, the amount would be \$4160. Find the principal and the rate.

34. My income from an investment is \$60. If the principal were \$500 less and the rate of interest 1% more, my income would be the same. Find the principal and the rate.

35. A sum of money on interest for 1 year at a certain per cent amounted to \$11,130. If the rate had been 1% less and the principal \$100 more, the amount would have been the same. Find the principal and the rate.

36. The sum of two numbers is 8 and the sum of their cubes is 152. Find the numbers.

CHAPTER XV

REVIEW THROUGH QUADRATIC EQUATIONS

DEFINITIONS AND PRINCIPLES

1. Distinguish between known and unknown numbers.
2. (a) When $\times$ and $\div$ or both occur in connection with $+$ and $-$ or both in an expression, what is the sequence of operations?
(b) Illustrate by finding the value of $7 - 3 \times 2 + 6 \div 2$.
3. Mention and illustrate three ways of indicating multiplication; two ways of indicating division.
4. What is a fraction? When is a fraction in its lowest terms? What are similar fractions?
5. By what principle may cancellation be used in reducing fractions to lowest terms?
6. Define power; root; like terms; systems of equations; surd. What other name is used for systems of equations?
7. Express the following without parentheses:
 $(a^2x^m)^n$ $- [- (a^2)^2]^2$ $(a^3)^4$ $(a^4)^3$ $(- a^2)^5$
8. What is the sign of any power of a positive number? of any even power of a negative number? of any odd power of a negative number?
9. Define similar surds; mixed surd; quadratic surd.
10. Represent graphically: $\sqrt{10}$, $2\sqrt{5}$, $\sqrt{10\frac{1}{4}}$.
11. Define coördinate axes; abscissa; ordinate; axiom; elimination.
12. How is the degree of an equation determined? What is the degree of $x + b = c$? of $x^2 + 3x = 7$? What is the degree of $5x + xy = 11$?

13. What name is given to an equation of the first degree? of the second degree?

14. What is an incomplete quadratic equation? a complete quadratic equation? Illustrate each.

15. (a) What is the root of an equation? What is the principle relating to the roots of an incomplete quadratic equation?

(b) Illustrate by solving the equation $7x^2 - 5 = 23$.

16. (a) Give a method of completing the square in the solution of complete quadratic equations.

(b) Solve the following equations by this method:

$$3x^2 + 5x = 22 \qquad 5x^2 = 7x + 8$$

17. (a) Outline the method of solving quadratic equations by factoring.

(b) Illustrate by solving $2x^2 - 5x = 12$. Show that 4 checks as a root, but 5 does not check.

18. State the quadratic formula. Show how this formula is obtained.

19. Using the quadratic formula solve the equations:

$$2x^2 - 3x = 9 \qquad 3x^2 - 9 = 5x$$

20. Explain how, in the solution of problems, negative roots of quadratic equations, while mathematically correct, are often inadmissible.

21. Why is it important to check the values of the unknown number found in the solution of a radical equation?

22. What is a monomial? a binomial? a trinomial? a polynomial? Illustrate each.

23. Define and illustrate coefficient; exponent; significant figure; index of a root.

24. How many roots has a linear equation? a quadratic equation?

25. Define ratio; proportion; mean proportion.

26. How may an extraneous root of an equation be introduced in the solution? How may a root be lost?

27. Give and illustrate two principles relating to proportions. Show how these principles are derived.

28. In the following proportions indicate the means; the extremes; a mean proportional: $a:b = c:d$; $\frac{x}{y} = \frac{y}{z}$.

29. Find a mean proportional between 8 a and 2 a .

30. Form two proportions from the equality

$$18 \times 2 = 3 \times 12.$$

FORMULAS

Read the following formulas as complete sentences. Evaluate them. When there is a decimal fraction in the result, give the answer correct to the nearest tenth. Use $\pi = 3\frac{1}{7}$.

1. Formula: $A = bh$ (Page 70)

(a) Find A when $b = 12.8$, $h = 12.2$.

~~(b)~~ Find A when $b = 4.25$, $h = 6.12$.

(c) Find b in terms of A and h .

(d) Find b when $A = 42.23$, $h = 5.1$.

2. Formula: $A = s^2$ (Page 70)

(a) Find A when $s = 15.5$.

~~(b)~~ Find A when $s = 28.6$.

3. Formula: $A = \frac{1}{2}bh$ (Page 71)

(a) Find A when $b = 10$, $h = 8.5$.

(b) Find A when $b = 14.2$, $h = 11.7$.

4. Formula: $A = \frac{1}{2}h(b + b')$ (Page 71)

(a) Find A when $h = 12$, $b = 6.3$, $b' = 4.8$.

(b) Find A when $h = 12.2$, $b = 11.8$, $b' = 14.4$.

~~(c)~~ Find h in terms of A , b , and b' .

(d) Find h when $A = 60.55$, $b = 12.3$, $b' = .5$.

5. Formula: $A = \frac{1}{2}ap$ (Page 71)

(a) Find A when $a = 6.45$, $p = 68.5$.

(b) Find A when $a = 8.25$, $p = 70.6$.

6. Formula: $c^2 = a^2 + b^2$ (Page 71)

(a) Find c when $a = 6.2$, $b = 8.3$.

(b) Find c when $a = 6.5$, $b = 9.5$.

(c) Find a in terms of b and c .

(d) Find a when $b = 35$, $c = 39$.

7. Formulas: (1) $c = \pi d$ (2) $c = 2\pi r$ (Page 72)

(a) Find c : (1) when $d = 24.5$; (2) when $r = 7.8$.

(b) Find c : (1) when $d = 7.2$; (2) when $r = 5.6$, $\pi = 3.1$.

8. Formulas: (1) $A = \frac{1}{4}\pi d^2$ (2) $A = \pi r^2$ (3) $A = \frac{1}{2}cr$
(Page 58)

(a) Find A : (1) when $d = 7.1$; (2) when $r = 7.35$;
(3) when $c = 22$, $r = 3.5$.

(b) Find A : (1) when $d = 6.25$; (2) when $r = 4.28$;
(3) when $c = 8.64\pi$, $r = 4.32$.

9. Formula: $V = lwh$ (Page 58)

(a) Find V when $l = 6.32$, $w = 8.05$, $h = 4.05$.

(b) Find V when $l = 4.36$, $w = 6.24$, $h = 8.66$.

(c) Solve for l in terms of V , w , and h .

(d) Find l when $V = 31$, $w = 2.5$, $h = 3.1$.

10. Formula: $V = e^3$ (Page 57)

(a) Find V when $e = 8.75$.

(b) Find V when $e = 2.22$.

11. Formula: $S = 6e^2$ (Page 72)

(a) Find S when $e = 5.56$.

(b) Find S when $e = 4.25$.

12. Formula: $V = \frac{1}{3}Bh$ (Page 72)

(a) Find V when $B = 36.8$, $h = 10.2$.

(b) Find V when $B = 14.2$, $h = 4.24$.

(c) Solve for B in terms of V and h .

(d) Find B when $V = 36.4$, $h = 5.2$.

13. Formula: $S = \frac{1}{2} lp$ (Page 73)

- (a) Find S when $l = 12.7$, $p = 28.4$.
- (b) Find S when $l = 8.6$, $p = 42.6$.
- (c) Solve for p in terms of S and l .
- (d) Find p when $S = 153.72$, $l = 4.2$.

14. Formula: $V = \pi r^2 h$ (Page 73)

- (a) Find V when $r = 7.63$, $h = 6.15$.
- (b) Find V when $r = 4.28$, $h = 5.27$.
- (c) Solve for r .
- (d) Find r when $V = 192.5$, $h = 5$.

15. Formula: $S = 2 \pi r h$ (Page 73)

- (a) Find S when $r = 14.77$, $h = 8.34$.
- (b) Find S when $r = 6.29$, $h = 7.53$.

16. Formula: $V = \frac{1}{3} \pi r^2 h$ (Page 73)

- (a) Find V when $r = 3.57$, $h = 11.83$.
- (b) Find V when $r = 6.68$, $h = 7.24$.
- (c) Solve for h .
- (d) Find h when $V = 110.88$, $r = 4.2$.

17. Formula: $S = \pi r l$ (Page 73)

- (a) Find S when $r = 21.7$, $l = 12.3$.
- (b) Find S when $r = 4.8$, $l = 5.3$.

18. Formula: $S = 4 \pi r^2$ (Page 74)

- (a) Find S when $r = 7.35$.
- (b) Find S when $r = 5.50$.
- (c) Solve for r .
- (d) Find r when $S = 200$.

19. Formula: $V = \frac{4}{3} \pi r^3$ (Page 74)

- ~~(a)~~ Find V when $r = 4.25$.
- (b) Find V when $r = 6.10$.

20. Formula: $s = vt$ (Page 66)
- Find s when $v = 6.3$, $t = 9.7$.
 - Find s when $v = 7.6$, $t = 6.5$.
21. Formula: $s = \frac{1}{2}at^2$ (Page 74)
- Find s when $a = 9.83$, $t = 8.57$.
 - Find s when $a = 6.27$, $t = 8.53$.
 - Solve for t .
 - Find t when $s = 52.7$, $a = 8.6$.
22. Formula: $h = \frac{a}{2}\sqrt{3}$ (Page 263)
- Find h when $a = 12.2$.
 - Find a when $h = 5.3$.
23. Formula: $M = mv$ (Page 74)
- Find M when $m = 10.25$, $v = 1000$.
 - Find M when $m = 62.57$, $v = 900$.
24. Formula: $f = \frac{mv}{t}$ (Page 75)
- Find f when $m = 20.2$, $v = 30.9$, $t = 10.8$.
 - Find f when $m = 7.0$, $v = 10$, $t = 3.5$.
 - Solve for m .
 - Find m when $f = 30.5$, $v = 18.3$, $t = 15.7$.
25. Formula: $w = mg$ (Page 75)
- Find w when $m = 100$, $g = 980$.
 - Find w when $m = 6.25$, $g = 980$.
26. Formula: $C = \frac{5}{9}(F - 32^\circ)$ (Page 75)
- Find C when $F = -8.5^\circ$; when $F = 125.2^\circ$.
 - Solve for F .
 - Find F when $C = 120.8^\circ$; when $C = -4^\circ$.
27. Formula: $a = p(1 + rt)$ (Page 56)
- Solve for t .
 - Find t when $a = \$295.30$, $p = \$250.25$, $r = 4\frac{1}{2}\%$.

GRAPHS

Represent graphically each of the following, choosing a convenient scale and indicating the reason for your choice.

1. The daily receipts in a store for one week were :

Day	MON.	TUES.	WED.	THURS.	FRI.	SAT.
Receipts	\$155.00	\$142.50	\$138.75	\$146.00	\$164.25	\$184.50

2. Of the 80 students in the senior class of a high school, 25 students expected to go to college, 10 students expected to return to high school for continued study, 30 students planned to go to work, while the rest hoped to get work and continue their studies at night school. Draw an appropriate graph to represent these conditions.

3. The temperature report on a winter day was :

		8 P.M.	MAX.			8 P.M.	MAX.
Albany		40	42	New York		42	46
Boston		43	47	Philadelphia		44	48
Buffalo		40	42	Pittsburgh		30	34
Chicago		41	42	Portland, Maine		40	46
Cincinnati		20	26	Portland, Oregon		48	52
Denver		-2	5	San Francisco		56	56
Helena		-8	-1	St. Louis		10	14
Kansas City		-3	2	Seattle		42	46

Draw two graphs on the same scale, one to represent the temperatures at 8 P.M. and the other to represent the maximum temperatures.

4. Draw a graph to represent the growth in population of a city for 25 years, using the following table :

Year	1893	1900	1905	1910	1915	1920
Population	65,200	69,700	71,500	73,900	79,400	85,800

From the graph determine what was the probable population in 1903; in 1914; in 1918. By extending the graph, estimate the approximate population in 1922; in 1924.

5. Keep a record of the temperature at the beginning of each hour from 8 A.M. to 6 P.M. one day and draw a graph to represent this record.

6. The following is a sample home budget :

Rent	20%
Food	25%
Clothing	15%
Fuel and light	15%
Insurance and medical services	5%
Savings and incidentals	20%

7. In a business the per cent on the capital invested during the first eight years was as follows :

YEAR	PER CENT RETURN	YEAR	PER CENT RETURN
1	3.7	5	5.4
2	4.5	6	6.8
3	5.8	7	7.6
4	6.9	8	5.8

8. Represent graphically the number of telephone calls per day in a telephone exchange, reported as follows :

Sunday	2250
Monday	2900
Tuesday	2875
Wednesday	2965
Thursday	3026
Friday	2978
Saturday	3015

9. Draw on the same scale graphs of the monthly income and expenses of a business as shown in the table below :

	INCOME	EXPENSES
January	\$25,050	\$20,750
February	28,000	22,250
March	30,500	24,500
April	26,750	25,000
May	25,250	20,500
June	20,750	22,250
July	24,000	21,000
August	27,750	23,500
September	30,250	26,250
October	26,750	20,500
November	30,750	25,750
December	27,500	22,000

10. Represent on the same scale the monthly production and consumption of petroleum in the United States one year as shown by the following table :

MONTH	PRODUCTION [In Millions of Barrels]	CONSUMPTION [In Millions of Barrels]
January	43,000	45,000
February	41,000	40,000
March	46,500	47,000
April	44,500	40,000
May	46,500	48,500
June	45,500	49,000
July	46,500	50,000
August	46,500	50,000
September	45,000	49,000
October	47,500	52,000
November	38,000	45,000
December	42,000	47,000

11. During the first six months of a year the postal receipts in the United States were approximately as follows :

MONTH	RECEIPTS
January	\$21,000,000
February	20,300,000
March	22,200,000
April	22,000,000
May	22,300,000
June	22,100,000

12. Represent graphically the quotations of a certain stock as reported on the first day of each month for a year :

Jan. 1	96 $\frac{1}{4}$	July 1	97 $\frac{3}{4}$
Feb. 1	95 $\frac{3}{4}$	Aug. 1	95 $\frac{1}{4}$
March 1	96 $\frac{1}{2}$	Sept. 1	93 $\frac{1}{2}$
April 1	97 $\frac{1}{2}$	Oct. 1	91
May 1	98 $\frac{1}{4}$	Nov. 1	92 $\frac{1}{2}$
June 1	98 $\frac{1}{4}$	Dec. 1	93 $\frac{1}{4}$

13. A formula for the selling price (s) when the gain is reckoned as 25% of the selling price is $s = c + .25 s$ (c = cost). Draw a graph of this formula. From the graph determine the cost when the selling price is \$3.20. What will be the selling price when the cost is \$1.60?

14. If a profit of 50% is reckoned on the cost, the formula for the sales price is $s = 1.50c$. From the graph of this formula determine the sales price if the cost is \$1.25; \$1.90. What will be the cost when the sales price is \$2.50?

15. Separate formulas may be used to designate the cost and the selling price of a number (n) of articles. Two such formulas are $c = 2.25n + 250$ and $s = 4.25n$. Graph these formulas on the same scale, letting each horizontal space represent 25 articles and each vertical space \$50.00. From the graph determine the gain or the loss if 75 articles are sold; 200 articles.

How many articles must be sold before any profit is made? What is a possible explanation of the number 250 in the cost formula?

FUNDAMENTAL OPERATIONS

1. Add $2a + b$, $3b - c$, $d - 2a$, $c + d$, and $b - d$.
2. Add $\frac{1}{4}x + \frac{1}{2}y - z$, $\frac{1}{2}x - y + \frac{1}{4}z$, and $x + \frac{1}{4}y + \frac{1}{2}z$.
3. $2c - 7d + 6n + 11m - 3c - 5n + 8d - 3m + 10c + 7n - 2d - 8c + 4d - 3n - 8m - 6n + m - 3d + 2m = ?$
4. $5am - 3a^2m^2 + 4 - 4am + a^2m^2 - 2 + 5 + a^2m^2 - 6 + 3am + 2a^2m^2 - 3am - 3 + 8am + 2 - a^2m^2 = ?$
5. $6\sqrt{x} - 5\sqrt{xy} + 3\sqrt{y} - 4\sqrt{x} + 6\sqrt{xy} - \sqrt{x} - \sqrt{y} + 3\sqrt{y} - 2\sqrt{xy} + \sqrt{x} + 2\sqrt{xy} - 3\sqrt{y} + 6\sqrt{x} + 4\sqrt{xy} + \sqrt{y} = ?$

What expression added to the first will give the second?

- | | |
|--------------------------|------------------------------|
| ⑥ $5x + y$, $9x - 8y$ | ⑨ $10x + 3y$, $18x - 7y$ |
| 7. $3c - 4d$, $7c + 6d$ | 10. $-12a - 5b$, $7a + 2b$ |
| 8. $4p - 4q$, $-5p + q$ | 11. $16c + 9d$, $-10c - 5d$ |

What expression subtracted from the first will give the second? Check your answers.

- | | |
|-------------------------------|---------------------------------|
| 12. $r + s - v$, $r - s + v$ | 14. $x - 2y + z$, $x + 2y + z$ |
| ⑬ $x + y - z$, $3x - y + z$ | 15. $b - c - 3d$, $b + c + 2d$ |

By how much does the first expression exceed the second?

16. $p - q + 5r$, $3p + q - r$

17. $2a - b - c$, $4a + b - c$

18. $x^2 - xy + y^2$, $x^2 + xy + y^2$

19. $a^2 + 2ab + b^2$, $a^2 - 2ab + b^2$

20. Take $4x^m + 2x^my^n + 5y^n$ from $7x^m + 2x^my^n + 9y^n$.

21. To what must $r^2 - 4s^2$ be added to produce $3s^2 - r^2$?

22. From $5x - 2y$ subtract the sum of $2x - y$ and $x - 2y$.

23. Take $6m^s + 11m^sn^t + 5n^t$ from $10m^s + 11m^sn^t + 8n^t$.

24. From the sum of $1 + x$ and $1 - x^2$ subtract $1 - x + x^2 - x^3$.

25. What number added to $a^2 - ab - ac + b^2 - c^2$ gives 0?

26. From the sum of $3x^2 - 2x + 1$ and $2x - 5$ subtract the sum of $x - x^2 + 1$ and $2x^2 - 4x + 3$.

Simplify :

27. $2(\frac{1}{2}a + b)^2 - (4a - b)^2$

28. $(3ab - 4cd)^2 - (5ab + 2cd)^2$

29. $5(7x + 3y)^2 - 3(2x - 5y)^2$

30. $(x^3 - 8)^2 + 3(4 - x^3)^2$

31. $(x^a - y^b)^2 - (2x^a + 3y^b)^2$

32. $(y^2 - 7)(y^2 + 7) + (y^2 + 3)(5 - y^2)$

33. $3(x^2 - 6)(x^2 + 9) - (9 + x^2)(x^2 - 9)$

34. $(a - 3b)(a + 6b) - 2(a - 4b)^2$

35. $(3a + 5b)(2a - 3b) + (a - 7b)(2a - 5b)$

36. $(.2x - 1.3y)^2 - (.3x + 2.5y)^2$

Multiply, and check by division :

37. $(x^{2n} + 2x^ny^m + y^{2m})(x^n - y^m)$

38. $(5x - 5x^2 + 10)(12 - 30x + 2x^2)$

39. $(4x - 3x^2 + 2x^3)(3x - 10x^2 + 10)$

40. $(2a^2 - 3b^2 - ab)(3a^2 - 4ab - 5b^2)$

Multiply, and check by division :

41. $(a^{2n} - 2 a^n b^n + b^{2n})(a^{2n} + 2 a^n b^n + b^{2n})$

42. $(x^{a+1}y^{c-1} + x^{c-1}y^{a+1} + 1)(x^{c-1}y^{a+1} - x^{a+1}y^{c-1} + 1)$

43. $(\frac{1}{2}z^{2a+1} - \frac{1}{2}z^{2a} + \frac{1}{2}z^{2a-1})(2z^{2a-1} + 2z^{2a-2} + 2z^{2a-3})$

Divide, and check by multiplication :

44. $35 - 31x + 6x^2$ by $2x - 7$

45. $a^{2n} + 5a^nb + 4b^2$ by $a^n + b$

46. $4x^3 - .7x^2 - 2.75x + .6$ by $2.5x - 2$

47. $2a^4 - 5a^3 - 28a^2$ by $2a^2 + 5a - 1$

48. $3x^3 + x^2 + 10x - 7$ by $x^2 + x - 1$

49. $x^{a+3} + x^{a+1} + x^{a-1}$ by $x^2 - x + 1$

50. $(a^6 + b^6)$ by $(a + b)$, to five terms of the quotient

FACTORING

Factor :

- | | |
|------------------------------------|---------------------------------|
| 1. $m^2 + 2mn$ | 18. $4c^2d^2 + 9abcd + 5a^2b^2$ |
| 2. $16a^2 - 8ab + b^2$ | 19. $2m^2 - 3mn - 2n^2$ |
| 3. $9 - 12xy + 4x^2y^2$ | 20. $6c^2 - 7cd - 20d^2$ |
| 4. $4a^{2m} + 16a^mb^m + 16b^{2m}$ | 21. $5xy - x^2 - 6y^2$ |
| 5. $ax^4 - 4ay^2$ | 22. $9x^2 - 3x^3y$ |
| 6. $27a^2b^2 - 3$ | 23. $4a^2b + 12ab^2$ |
| 7. $8 - 98c^2d^2$ | 24. $ab(x - 2y) - 4(x - 2y)$ |
| 8. $24b^2 - 6a^2d^2$ | 25. $am^2 + 6amy^2 + 9ay^4$ |
| 9. $x^2y - 3xy + 2y$ | 26. $75b^2 - 27c^2$ |
| 10. $5a^2 + 45a + 70$ | 27. $49a^{2n} - 36$ |
| 11. $xy^2 - 5xy + 6x$ | 28. $2ar^2s^2 - 32ay^2$ |
| 12. $12ax^2 - 33ax - 9a$ | 29. $12b^2 - 3a^2d^2$ |
| 13. $42 + 35r - 42r^2$ | 30. $16 - 196c^2d^2$ |
| 14. $24ay^2 + 22ay + 4a$ | 31. $5x^2 + 15axy - 20a^2y^2$ |
| 15. $6x^{2n} + 38x^n - 28$ | 32. $7z^2 - 21abz - 126a^2b^2$ |
| 16. $a^2b^2 - 3abc - 4c^2$ | 33. $2b^2 + 12abc - 32a^2c^2$ |
| 17. $10d^2 + 21dx + 9x^2$ | 34. $-5x^2 + 5x + 60$ |

- | | |
|--------------------------------|--|
| 35. $3y^2 - 3ay - 6a^2$ | 42. $.04a^2 - .25b^2$ |
| 36. $14 + 21r - 14r^2$ | 43. $1.08b^2 - 1.47c^2$ |
| 37. $4a^3 - 6a^2b - 4ab^2$ | 44. $.01a^2 + .06ab + .09b^2$ |
| 38. $18a^2 - 21ac - 60c^2$ | 45. $r(t + 3) - s(t + 3)$ |
| 39. $21ab + 42a^2c + 21a^2b^2$ | 46. $acx^2 - bcx + adx - bd$ |
| 40. $12x^4 - 16x^2$ | 47. $x^{3m} - x^{2m} + x^{2m+1} - x^{m+1}$ |
| 41. $180a^4 - 405b^4$ | 48. $6x^2 - 3ay - 9xy + 2ax$ |

FRACTIONS

Unite the terms. Check exercises 1 to 5 with $x=3$, $a=2$, $b=3$.

1. $\frac{5}{x} + \frac{3}{2x} + \frac{27}{8x^2}$
2. $\frac{2x+5}{3x} - \frac{13}{8x^2} - \frac{1}{2x}$
3. $\frac{a-x}{x} + \frac{a+x}{a} + \frac{a^2-x^2}{2ax}$
4. $\frac{5(x-3)}{x^2-x-2} - \frac{2(x+2)}{x^2+4x+3} - \frac{x-1}{6-x-x^2}$
5. $\frac{a^2+2ab+b^2}{a^2+b^2} - \frac{2ab}{b^2-a^2} - 1$
6. $\frac{x-5}{2x^2+x-21} - \frac{x-3}{2x^2-3x-35}$
7. $\frac{x+1}{x-1} + \frac{x-1}{x+1} - \frac{x+2}{x-2} - \frac{x-2}{x+2}$

NOTE. It is sometimes best to combine the sum of two fractions with the sum of the others.

8. $\frac{x}{x-1} - \frac{x}{x+2} - \frac{x}{x+1} + \frac{x}{x-2}$
9. $\frac{a+5}{a^2+a-20} + \frac{a-5}{a^2-9a+20} - \frac{a+4}{a^2-25}$
10. $\frac{5x+4}{x-2} + \frac{3x-2}{3-x} - \frac{x^2-2x-17}{x^2-5x+6}$

Reduce the following mixed expressions to fractions :

- | | | |
|----------------------------|---------------------------|-----------------------------------|
| 11. $3y - \frac{a}{4}$ | 14. $3r - \frac{r+s}{2}$ | 17. $4x - \frac{2xy}{x-y}$ |
| 12. $ab + \frac{1}{x}$ | 15. $\frac{2}{x+1} + 3$ | 18. $a - b + \frac{a^2}{a+b}$ |
| 13. $\frac{a^2}{z^2} - 4n$ | 16. $\frac{a+b}{a-b} - 1$ | 19. $x + y - \frac{x^2+y^2}{x+y}$ |

Simplify :

- | | |
|--|---|
| 20. $\frac{3a^3c^2}{4xyz} \times \frac{12xz}{27a^4c^2}$ | 24. $\frac{x^2+2x+1}{y} \cdot \frac{4y^2}{x^2-1}$ |
| 21. $\frac{a^2-b^2}{a^2+ab} \times \frac{4a}{a-b}$ | 25. $\frac{x^4-y^4}{8x^3+8xy^2} \cdot \frac{4}{x^2-y^2}$ |
| 22. $\frac{3a^2}{5a-10} \cdot \frac{3a-6}{4a^3}$ | 26. $\frac{2a^5b^4}{3c^3} \cdot \frac{5c^2}{4b^3} \cdot \frac{18bc}{25a^3}$ |
| 23. $\frac{xy-x^2y^2}{axy} \div \frac{a-axy}{a^2x+a^2y}$ | 27. $\frac{(r-2s)^2}{s-r} \div \frac{r^2-rs-2s^2}{r^2-s^2}$ |
28. $\frac{c+d}{5x} \cdot \frac{10x^2}{c^2-cd-2d^2} \cdot \frac{ac-2ad}{2x}$
29. $\frac{a-b}{(a+b)^2} \cdot \frac{a^2-b^2}{3} \cdot \frac{4}{a^2-2ab+b^2}$
30. $\frac{x^2+x-2}{4x-12} \cdot \frac{3}{x^2-x-6} \cdot \frac{x-3}{x-1}$
31. $\frac{x^2-2xy-3y^2}{x^2+2xy+y^2} \div \frac{(x-3y)^2}{x+y}$
32. $\frac{2a^2+ab-3b^2}{2a^2+5ab+3b^2} \div \frac{a-b}{a+b}$
33. $\left(\frac{10}{x-x^3}\right) \div \left(\frac{1}{1-x} - \frac{1}{1+x}\right)$
34. $\left(a - \frac{1}{a}\right)^2 \div \left(1 - \frac{1}{a}\right)^2$
35. $\left(x + \frac{xy}{x-y}\right) \times \left(x - \frac{xy}{x+y}\right) \div \frac{x^2+y^2}{x^2-y^2}$

Simplify:

$$36. \left(\frac{x}{1+x} + \frac{1-x}{x} \right) \div \left(\frac{x}{1+x} - \frac{1-x}{x} \right)$$

$$37. \left[\frac{5}{2(x-3)} - \frac{3}{2(x-1)} \right] \times \left[\frac{5}{x+2} - \frac{4}{x+1} \right]$$

$$38. \left(a + \frac{1}{a+1} \right) \div \left(1 + \frac{a}{1+a} \right)$$

$$39. \frac{\frac{x^2+y^2}{x^2-y^2}}{\frac{x-y}{x+y}}$$

$$41. \frac{\frac{3}{4} + \frac{2}{3} - 1}{\frac{5}{6} + \frac{1}{2}}$$

$$40. \frac{\frac{a+x}{a-x} - \frac{a-x}{a+x}}{\frac{4ax}{a^2-x^2}}$$

$$42. \frac{\frac{1}{1-x} + \frac{1}{1+a}}{\frac{1}{1-x} - \frac{1}{1+a}}$$

FRACTIONAL EQUATIONS

Solve, and check:

$$1. \frac{3}{x-2} - \frac{6x}{2-x} = 9$$

$$2. \frac{6}{2r+1} - \frac{2+4r}{2r-1} = \frac{8r^2}{1-4r^2}$$

$$3. \frac{6m-3}{2m+7} = \frac{3m-2}{m+5}$$

$$9. \frac{x}{3} - \frac{x-3}{3} = 12 - \frac{x+4}{2} - x$$

$$4. \frac{3r-1}{r-5} = \frac{5r+4}{r+8} - 2$$

$$10. \frac{5}{x-7} - \frac{7}{x+3} = \frac{5x+1}{x^2-4x-21}$$

$$5. \frac{t^2+5}{t+3} - (t-7) = 3$$

$$11. \frac{m-2}{m+3} = \frac{3}{4}$$

$$6. \frac{\frac{1}{4}-p}{\frac{1}{4}+p} + \frac{1}{4} = \frac{p}{\frac{1}{4}+p} - \frac{1}{4}$$

$$12. \frac{3P}{P-3} = \frac{3P+3}{P-4}$$

$$7. \frac{6m+8}{2m+1} = 1 + \frac{2m+38}{m+12}$$

$$(13) \frac{10m+20}{3m-5} = \frac{25}{2}$$

$$8. 10 + \frac{10y}{y+3} = \frac{20y}{y+7}$$

$$14. \frac{8}{x-2} - \frac{6x}{2-x} = 9$$

ALGEBRAIC REPRESENTATION

- ① Twenty-six persons pay d dollars each as dues to a society; the expenses of the society are s dollars. How many dollars are left in the treasury?
2. Express in feet the sum of y yards, 6 feet, and i inches.
3. Two automobiles travel away from each other, one at the rate of a miles an hour and the other at the rate of b miles an hour. How far apart will they be at the end of h hours?
4. A boy bought d dozen oranges at s cents an orange and had c cents left. How many cents did he have at first?
5. If the length of a room is l yards, the width w yards, and the height h yards, write in terms of these letters an expression to represent the total surface of (1) the four walls; (2) the ceiling.
6. From a board b feet in length, a piece c feet long is cut from one end and a piece d inches long is cut from the other end. How many inches in length remain?
- ⑦ If c represents the cost of an article and r the gain per cent, express the selling price in terms of c and r .
8. What is the dividend which, divided by x , gives a quotient of y and a remainder of z ?
- ⑨ If a apples are sold for a dime, how many can be bought for c cents?
10. A salesman received d dollars a week for expenses, and a commission of r per cent on n articles which he sold during the week at c dollars each. Write a formula to represent his income, I , each week.
11. The three digits of a number are a , b , and c . Represent the number. Represent the number with its digits reversed.
12. A man earns e dollars a month and spends s dollars a month. How many dollars will he save in 3 years?
- ⑬ John is y years old now. Write an equation showing that 4 times his age 3 years ago equals twice his age 5 years hence.

14. The width of a rectangle is x yards and its length is y yards. Express its area and its perimeter.

15. What will be the amount at simple interest at the end of 5 years of p dollars invested at r per cent?

16. If $2n + 1$ is an odd number, express the next two higher even numbers; the next two higher odd numbers.

17. If the list price of an article is represented by L and the discount from the list price by d per cent, how should you represent the selling price in terms of L and d ? Representing the selling price by S , make a formula for the selling price.

18. If a man's rate of rowing in still water is s miles an hour and the current flows at the rate of c miles an hour, express the man's rate (1) when rowing with the current; (2) when rowing against the current.

LINEAR EQUATIONS

1. What number must be subtracted from the numerator and the denominator of the fraction $\frac{1}{3}\frac{3}{2}$ to make it equal $\frac{2}{5}$?

2. What number must be added to the numerator and the denominator of the fraction $\frac{1}{2}\frac{7}{5}$ to make it equal $\frac{3}{4}$?

3. A man can lay a cement walk in 5 days; his assistant, in 8 days. They work together for 2 days; then the man is called away. How long will it take the assistant to finish the work?

4. One contractor could lay a pavement on a street in 20 days; another, in 10 days. The first contractor worked for 8 days, then stopped. How long should it take the second contractor to finish the work?

5. By using a tractor of one make, a field can be plowed in 4 hours; by using a tractor of a different make, the work can be done in $4\frac{1}{2}$ hours. If both tractors are used at the same time, how long will it take to plow the field?

⑥ An oil tank can be filled by one of two pipes in 18 minutes and by the other in 12 minutes; it can be emptied by a third in 9 minutes. In what time will it be filled with all three pipes running, starting with an empty tank?

⑦ A certain automobile radiator holds $4\frac{1}{2}$ gallons. How much alcohol must be mixed with the water so that when the radiator is full the mixture shall be 20% alcohol? If, after a month's time, 3 quarts of the mixture leak out, how much alcohol must be added so that when the radiator is full the mixture shall be 25% alcohol?

⑧ In a mixed tea 40% is green tea and the rest is black tea. How much green tea must be added to 10 pounds so that the resulting mixture shall be 50% green tea?

9. A man found that he had \$2.45 in change, consisting of nickels, dimes, and 5 quarters. If there were twice as many nickels as dimes, how many were there of each?

10. A man who can walk 4 miles an hour takes a walk into the country on a holiday. He is away 6 hours. How far does he go if he returns by a trolley car which averages 12 miles an hour?

SIMULTANEOUS LINEAR EQUATIONS

Solve:

$$1. \begin{cases} 3x + 2y - 18 = 0 \\ x + 4 = y \end{cases}$$

$$4. \begin{cases} 5x - 3y = -.8 \\ 7x + 8.5 = 6y \end{cases}$$

$$2. \begin{cases} \frac{x-3}{x-1} = \frac{y-8}{y-5} \\ \frac{2x-3}{2(y-1)} = \frac{5x-6}{5y-7} \end{cases}$$

$$\begin{cases} \frac{2x-5}{4y+3} = \frac{x}{2y+7} \\ \frac{3y+2}{x+6} = \frac{9y-10}{3x-4} \end{cases}$$

$$3. \begin{cases} \frac{3x}{2} + \frac{2y}{3} = 3 \\ \frac{9x}{2} - 2y = 3 \end{cases}$$

$$6. \begin{cases} \frac{x+y}{3} - \frac{x-y}{4} = 11 \\ \frac{x+y}{2} - \frac{x-y}{3} = 8 \end{cases}$$

7. A number is expressed by two digits whose sum is 12. If the digits are interchanged, the resulting number is 36 smaller than the original number. Find the number.

8. A number has 3 digits. The sum of the first two exceeds the third by 4; the sum of all 3 digits is 10; and if the order of the digits is reversed, the resulting number exceeds the original number by 99. Find the original number.

9. A, B, and C can build a summer cottage in 20 days; A and B, in 30 days; and B and C, in 40 days. How long would it take each to build the cottage alone?

10. A, B, and C have together \$1285. A's share is \$25 more than $83\frac{1}{3}\%$ of B's, and C's share is $26\frac{2}{3}\%$ of B's. Find the share of each.

11. In building a new road the county paid $33\frac{1}{3}\%$ as much as the state, and the town paid 70% as much as the county. If the road through the town cost \$35,250, find the amount paid by each.

12. John can row 12 miles down a stream and 12 miles back in $3\frac{1}{2}$ hours. He finds that he can row as far in 30 minutes going downstream as he can row in 40 minutes going upstream. Find his rate of rowing and the rate of the current.

RATIO AND PROPORTION

1. Two numbers are in the ratio of 3 to 2. If each is increased by 4, the sums are in the ratio of 4 to 3. What are the numbers?

2. Divide 16 into two parts such that their product is to the sum of their squares as 3 is to 10.

3. Divide 25 into two parts such that the greater increased by 1 is to the less decreased by 1 as 2 is to 1.

4. The sum of two numbers is 4, and the square of their sum is to the sum of their squares as 8 is to 5. What are the numbers?

5. Find a number that added to each of the numbers 1, 2, 4, and 7 will give four numbers in proportion.

6. Find the angles of a triangle if their ratio is 2:3:4 (see page 25, ex. 14).

7. A study hall held 48 pupils. The ratio of boys to girls was 9:7. Several boys withdrew for a recitation, leaving 3 girls for every 2 boys. Find the number of boys that withdrew.

8. What number must be added to each of the numbers 11, 17, 2, and 5 so that the sums shall be in proportion when taken in the order given?

9. Four numbers are in proportion. The difference between the first and the third is $2\frac{2}{3}$; the sum of the second and the third is $6\frac{1}{3}$; the third is to the fourth as 4 is to 5. Find the numbers.

GEOMETRIC FORMULAS

1. Two lots are in the shape of similar polygons. The length of a side of the first is 16.6 rods, and the length of the corresponding side of the second is 66.4 rods. Compare (a) their areas, (b) the amount of fence necessary to inclose each.

2. Two rectangular building lots have the same width. The depth of the first is 100 feet and the depth of the second is 125 feet. How do their areas compare?

3. The diameter of a circular disk is 12.5 inches; the diameter of a second circular disk is 62.5 inches. Compare their areas and their circumferences.

4. The perimeters of two similar polygons are 96 inches and 144 inches. If a side of the first is 21 inches, what is the corresponding side of the second?

5. The diameters of two circles are 24 inches and 36 inches, respectively. What is the ratio of their areas? of their circumferences?

6. The area of one circle is $2\frac{1}{5}$ times the area of a second circle. If the diameter of the first is 12 inches, what is the diameter of the second?

7. If 3 feet is the altitude of a prism whose volume is 54 cubic feet, find the altitude of a similar prism whose volume is 128 cubic feet.

8. Two bins, similar in shape, hold, respectively, 1000 bushels and 1728 bushels of grain. If the first bin is 5 feet long, find the length of the second bin.

9. A sphere of a certain material 5 inches in diameter weighs 250 centigrams. What is the weight of a sphere of the same material whose diameter is 6 inches?

NOTE. The *weights* of similar solids may be compared in the same way as their *volumes*.

10. A cone whose altitude is 7 inches weighs 686 centigrams. Find the weight of a similar cone whose altitude is 8 inches.

11. Solve $c^2 = a^2 + b^2$ for a . What is the value of a , to the nearest tenth, if $c = 32$ and $b = 24$?

12. What is the length of the altitude of an equilateral triangle if each side is 21.5 inches long? Find the result correct to one decimal place.

13. Find, to the nearest inch, the length of each side of an equilateral triangle if the altitude is 16.45 inches long.

14. What is the area, to the nearest square inch, of each of the triangles in exercises 12 and 13?

15. Find, to the nearest tenth, the value of $\frac{1}{2}\sqrt{2(a^2+b^2)-c^2}$ when $a = 7.2$, $b = 4.8$, and $c = 6.5$.

16. If $T = \frac{2}{b+c}\sqrt{bcs(s-a)}$, find the value of T in the simplest radical form and to the nearest tenth when $a = 17$, $b = 10$, $c = 15$, and $s = \frac{a+b+c}{2}$.

EXPONENTS AND RADICALS

1. Simplify the following:

$$\frac{1}{2}\sqrt{72 a^2 b} + 4 a \sqrt{\frac{9 b}{8}} - \sqrt{2 b(6 a - 1)^2}$$

2. Find the product of $(a^2 - b^2)^{\frac{1}{2}}$ and $(a^2 + 4 ab + 3 b^2)^{\frac{1}{2}}$. Reduce the answer to the simplest form. Check with $a = 5$ and $b = 3$.

3. Find the product of $(\sqrt{6} - \sqrt{10})$ and $(\sqrt{3} - \sqrt{5})$. Express the result in the simplest form.

4. Multiply $a^{\frac{3}{4}} + a^{\frac{1}{2}} - 3 a^{\frac{1}{4}} + 1 - a^{-\frac{1}{4}} + a^{-\frac{1}{2}}$ by $a^{\frac{1}{4}} - 2 + a^{-\frac{1}{4}}$. Express the result with positive exponents. Check by letting $a = 16$.

5. Without solving the equations, determine whether $x = 2\sqrt{5}$ and $y = 3\sqrt{5}$ satisfy the equations $x + y = 5\sqrt{5}$ and $3x^2 - 4xy + y^2 = 65$.

Simplify the following:

$$6. \left(6 a \left(\frac{2}{3}\right)^{\frac{1}{2}} - \frac{b}{3}(54)^{\frac{1}{2}} + (24 b^2)^{\frac{1}{2}} - (6 a^2)^{-\frac{1}{2}}\right)$$

$$7. \sqrt{16 a^3 - 48 a^2 b} + \sqrt{9 a b^2 - 27 b^3}$$

$$8. (2\sqrt{3} - 3\sqrt{5})(\sqrt{3} + 2\sqrt{5})$$

$$9. (\sqrt{a+x} - \sqrt{a-x})^2 \quad 13. (\sqrt{x+3} - 5)^2$$

$$10. \sqrt{\frac{2}{5}} + \sqrt{\frac{1}{10}} - \sqrt{\frac{1}{40}} \quad 14. \sqrt{\frac{a^2}{2}} + 4 \sqrt{\frac{2 a^2}{9}} + 3 a \sqrt{\frac{1}{8}}$$

$$11. \sqrt{162} - 4\sqrt{\frac{1}{2}} + \frac{1}{5}\sqrt{50} \quad 15. \frac{8 - \sqrt{3}}{2 - \sqrt{3}} + \frac{2\sqrt{3} - \sqrt{2}}{\sqrt{3} - \sqrt{2}}$$

$$12. \left[\frac{2+\sqrt{5}}{\sqrt{5}-1}\right]^2 - \left[\frac{\sqrt{5}-3}{2+\sqrt{5}}\right]^2 \quad 16. \frac{\sqrt{a-x} + \sqrt{a+x}}{\sqrt{a-x} - \sqrt{a+x}}$$

SQUARE ROOT

Find the square roots of the following:

1. $a^4 + 12a^3 - 216a + 324$

2. $13x^2 - 12x - 6x^3 + x^4 + 4$

3. $x^4 - x^3 + \frac{x^2}{4} + 4x - 2 + \frac{4}{x^2}$

4. $\frac{a^2}{4} + b^2 + \frac{c^2}{4} + ab - \frac{ac}{2} - bc$

5. $y^2 - by + \frac{a^2}{9} + \frac{b^2}{4} - \frac{ab}{3} + \frac{2ay}{3}$

6. $\frac{a^2}{25} + \frac{9b^2}{25} + c^2 - \frac{6}{5}bc + \frac{2}{5}ac - \frac{6ab}{25}$

QUADRATIC EQUATIONS

Solve, and check:

1. $y^2 + 3.9y + 1.4 = 0$

3. $x^2 + .25x = .015$

2. $.2x^2 - x = 2.8$

4. $.5x^2 - .2x - 1.6 = 0$

5. $2x^2 - .1x - .08 = .003$

6. $(x + 3)(x - 2) = (3x + 1)(x - 3)$

Find the value of x , correct to the nearest tenth:

7. $.3x^2 - .5x = 2$

10. $2x^2 - 4.3x = 27.3$

8. $5x - 3 = x^2$

11. $3x^2 + .75x = .145$

9. $3x^2 - .4x = 0$

12. $y^2 + 4y = -.4$

13. $(x + 1)(2x - 4) = (3x + 2)(2x + 3)$

Solve:

14. $\frac{x - 2}{x - 3} + \frac{x + 3}{x + 2} = 6$

17. $2 = \frac{(r + 2)^2}{r^2 - 4} - \frac{r + 5}{r - 3}$

15. $\frac{6x + 3}{2x + 3} = \frac{3x + 3}{3x + 2}$

18. $\frac{1 + x}{x - 3} - \frac{x - 1}{x - 2} = \frac{4}{5}$

16. $\frac{y + 2}{3y - 2} = \frac{15}{3y + 2}$

19. $\frac{9x}{2x^2 + x} + \frac{3}{x - 3} = 4$

20. Find two consecutive integers the sum of whose squares is 145.

21. If one side of a square is increased by 2 and another side by 3, the area of the resulting rectangle will be 31 square units larger than that of the square. Find the side of the square.

22. Find two numbers which are in the ratio 3:4 and the sum of whose squares is 100.

23. Divide 20 into two parts whose product is 69.75.

24. Two boys planted 400 hills of corn. The number of hills in each row was 9 more than the number of rows. Find the number of rows.

25. One leg of a right triangle is 4 feet longer than twice the other leg and 2 feet less than the hypotenuse. Find the length of each side.

26. A train makes a run of 117 miles in $3\frac{1}{2}$ hours. After proceeding the first 60 miles, it increases the rate 8 miles an hour. Find the rate at first.

27. A man was offered \$72 for his potato crop, but decided not to sell. Later in the season he sold all but 4 barrels of his potatoes at \$.20 a barrel more than the first price, and received \$72. How large was his potato crop?

28. The length of a rectangular lot is 3 times its width and the number of square feet in its area is $7\frac{1}{2}$ times the number of feet around the field. Find its length and width.

29. There are 980 trees in an orchard, set out in rows. The number of trees in each row is 7 less than the number of rows. Find the number of rows.

30. A man drove his automobile a distance of 90 miles and back in 6 hours and 45 minutes. The rate returning was 6 miles an hour faster than the rate going. Find the rate in each direction.

31. A man by increasing the rate of driving his automobile 6 miles an hour gains 1 hour and 20 minutes in going a distance of 126 miles. Find his first rate.

32. Two men can dig a trench in 20 days. It would take the first alone 9 days longer than it would take the second. How long would it take each alone to dig the trench?

33. If an equilateral triangle has its sides increased 7 inches, 4 inches, and 1 inch, respectively, a right triangle is formed. Find the sides of the right triangle.

34. The distance between the two opposite corners of a rectangular field is 26 rods and the distance around the field is 68 rods. Find the dimensions of the field.

35. Divide a line 25 inches long into two parts such that the longer part shall be a mean proportional between the whole line and the shorter part.

SIMULTANEOUS QUADRATIC EQUATIONS

Solve:

$$1. \begin{cases} x^3 + y^3 = 28b^3 \\ x + y = 4b \end{cases}$$

$$2. \begin{cases} 2y - 1 = -x \\ x^2 - 6x = 11 - 4y^2 \end{cases}$$

$$3. \begin{cases} 2x + y = 13 \\ 3x - xy + 2y = 2 \end{cases}$$

$$4. \begin{cases} x^3 - y^3 = 26 \\ x - y = 2 \end{cases}$$

$$5. \begin{cases} 2x - 3y = 13 \\ 3x^2 + 33 = 5y^2 \end{cases}$$

$$6. \begin{cases} 1.5x - .5y = 6 \\ x^2 - xy = 160 \end{cases}$$

$$7. \begin{cases} x^2 + xy = 5 \\ xy + y^2 = 3 \end{cases}$$

$$8. \begin{cases} x - y = 1 \\ \frac{x}{y} - \frac{y}{x} = \frac{5}{6} \end{cases}$$

$$9. \begin{cases} \frac{x}{2} + \frac{y}{5} = 5 \\ \frac{2}{x} + \frac{5}{y} = \frac{5}{6} \end{cases}$$

$$10. \begin{cases} \frac{x}{y} - \frac{y}{x} = \frac{16}{15} \\ x - y = 2 \end{cases}$$

$$11. \begin{cases} x + y = 5 \\ \frac{1}{x} + \frac{1}{y} = \frac{5}{6} \end{cases}$$

$$12. \begin{cases} \frac{1}{x} + \frac{2}{y} = 8 \\ xy = \frac{1}{6} \end{cases}$$

13. Separate 12 into two parts such that their product shall be to the sum of their squares as 2 is to 5.

14. The difference between two numbers is 37 and the difference between their cube roots is 1. Find the numbers.

15. The diagonal of a rectangle is 25 inches and its area is 168 square inches. Find its length and its breadth.

16. Two boys run in opposite directions around a rectangular field, the area of which is 4840 square yards. They start at one corner and meet 39 feet from the opposite corner. One boy runs 5 feet while the other runs 6 feet. Find the dimensions of the field.

17. Find two numbers whose sum multiplied by the greater is 126 and whose difference multiplied by the less is 20.

18. The value of a certain fraction is $\frac{1}{3}$. The product of its numerator and denominator is 12. Find the fraction.

EXAMINATION QUESTIONS

1. Divide $12x^4 - x^3 - 27x^2 - 3x + 10$ by $4x^2 + x - 2$ and check your work, letting $x = 2$.

2. Find the prime factors of the following:

(a) $3mr^2 - 363m^3$

(d) $9x^{4a} + 30x^{2a} + 25$

(b) $3y^2 - 7y - 6$

(e) $15a^2 - 7ab - 2b^2$

(c) $M^2 - 8MN + 16N^2 - 4R^2$

(f) $x^{2a} - 2x^ay^a + y^{2a}$

3. Simplify the following and express the result as a fraction in its lowest terms:

$$\left(\frac{5}{x^2 - 4} + \frac{2}{2 - x}\right) \div \left(2 + \frac{3}{x - 2}\right)$$

4. Solve, and check:

$$4 - \frac{5x - 15}{4} + \frac{2(x + 2)}{3} = \frac{5(x - 1)}{6}$$

5. Solve for x and y :

$$\begin{cases} \frac{x}{2a} - \frac{y}{5b} = \frac{2}{5} \\ \frac{x}{a} + \frac{y}{b} = 5 \end{cases}$$

6. Simplify each term and unite the results into a single term:

$$(a) \sqrt{\frac{25a^2}{8}} + \frac{1}{4}\sqrt{98a^2} - 3\sqrt{2(a^2 - 4a + 4)}$$

$$(b) \left(\frac{8}{m-x} - \frac{4}{x^2 - m^2} \right) \div \left(\frac{1}{m-x} - \frac{1}{m+x} \right)$$

$$(c) 4\sqrt{\frac{x^2}{2}} - \frac{1}{12}\sqrt{288x^2} - \sqrt{2(x-1)^2}$$

7. Solve, and check:

$$(a) \frac{1}{2}(y+1) - \frac{y}{3}(2y-1) = \frac{2}{3}$$

$$(b) \frac{x}{4} - \frac{x}{x+4} = \frac{x-1}{6}$$

$$(c) \frac{2}{3}(x-1) - \frac{1}{4}(x+1) - (x-4) = \frac{x+1}{12} - \frac{1}{3}$$

8. Solve the following formula for n :

$$C = \frac{en}{R + nr}$$

9. Solve the following formula for l :

$$s = \frac{n}{2}(a + l)$$

10. Find to the nearest tenth the value of s in the formula in exercise 9 if $n = 9$, $a = 4.37$, and $l = 13$.

11. A rectangular field is l rods long. What must be its width (w) if its perimeter is p rods?

12. Two automobiles m miles apart travel toward each other, one at the rate of a miles an hour, the other at the rate of b miles an hour. If they have not yet met, how far apart are they at the end of h hours?

13. Find, to the nearest tenth, the roots of $\frac{x^2}{4} - \frac{3x}{2} = \frac{5}{2}$.

14. A merchant purchased a number of overcoats for \$720, paying the same amount for each. He sold them for \$40 each, making a profit equal to the cost of 8 of the coats. How many coats did he buy?

15. Solve for x and y , correctly group your answers, and check:

$$(a) \begin{cases} 3x + 2y = 5 \\ xy + 3 = 6x \end{cases} \quad (b) \begin{cases} 2x^2 - y^2 = 14 \\ x - y = 1 \end{cases}$$

16. Evaluate the expression $\frac{3a\sqrt{a^2 - 2ab}}{4b}$ when $a = 8.5$ and $b = 3.2$, expressing the result to the nearest tenth.

17. Two numbers are in the ratio 3:5. If 12 is added to each number, their ratio becomes 2:3. What are the numbers?

18. If m dollars represents my money and I lend $\frac{1}{5}$ of it for 1 year at 6%, what interest shall I receive?

19. If n is a certain number, write the equation which shows that $\frac{1}{4}$ of the number, increased by 1, equals the number diminished by 2.

20. If 3 is added to the numerator and 5 is subtracted from the denominator of a certain fraction, the value of the fraction is $\frac{8}{7}$. If 2 is subtracted from the numerator and 3 is added to the denominator, the value becomes $\frac{1}{5}$. Find the fraction.

21. Extract the square root of the following polynomial and check the result:

$$4x^4 + 17x^2 - 12x - 12x^3 + 4$$

22. Two automobiles starting from Albany travel east and west respectively, the first one traveling $\frac{3}{4}$ as fast as the second. In 6 hours they are 378 miles apart. Find the rate of each.

CHAPTER XVI

IMAGINARY NUMBERS. PROPERTIES OF QUADRATIC EQUATIONS

INTRODUCTION TO IMAGINARY NUMBERS

Our number system now comprises *natural numbers*, 1, 2, 3, ...; *fractions*, arising from the indicated division of one natural number by another; *negative numbers* (the opposite of *positive numbers*), arising from the subtraction of a number from a less number; *irrational numbers*, arising from the attempt to extract a root of a number that is not a perfect power; and, finally, **imaginary numbers**, arising from the attempt to extract an even root of a negative number.

Before the introduction of imaginary numbers, the only numbers known were those *whose squares are positive*; now called **real numbers** to distinguish them from **imaginary numbers**, *whose squares are negative*.

Since the square of an imaginary number is negative, imaginary numbers present an apparent exception *regarding signs*. Apparently $\sqrt{-1} \times \sqrt{-1}$ would equal $\sqrt{(-1)(-1)}$, which equals $\sqrt{+1}$, or ± 1 ; but by the definition of a root, the square of the square root of a number is the number itself.

$$\text{Hence } \sqrt{-1} \times \sqrt{-1} = (\sqrt{-1})^2 = -1, \text{ not } +1. \quad (A)$$

For brevity $\sqrt{-1}$ is often written i .

It will be assumed that imaginary numbers obey the same laws as real numbers, the signs being determined by (A), which we call the **fundamental property of imaginaries**.

Every imaginary number can be expressed as the product of a real number and the imaginary unit $\sqrt{-1}$; as, $\sqrt{-4} = 2\sqrt{-1}$, or simply $2i$; $\sqrt{-a} = i\sqrt{a}$.

A number composed of the sum of a real number and an imaginary number is called a **complex number**; as, $5 + 2\sqrt{-1}$.

Only a brief introduction to imaginary numbers will be presented here. For our present needs it is sufficient to be able to recognize imaginary numbers, to know their meaning, and to express, in terms of i , square roots of negative numbers.

Powers of $\sqrt{-1}$

$$(\sqrt{-1})^1 = + \sqrt{-1}$$

$$(\sqrt{-1})^2 = (\sqrt{-1})(\sqrt{-1}) = -1$$

$$(\sqrt{-1})^3 = (\sqrt{-1})^2 \sqrt{-1} = (-1)\sqrt{-1} = -\sqrt{-1}$$

$$(\sqrt{-1})^4 = (\sqrt{-1})^2 (\sqrt{-1})^2 = (-1)(-1) = +1$$

$$(\sqrt{-1})^5 = (\sqrt{-1})^4 \sqrt{-1} = (+1)\sqrt{-1} = +\sqrt{-1}$$

and so on. Hence if $n = 0$ or a positive integer,

$$\left. \begin{aligned} (\sqrt{-1})^{4n+1} &= + \sqrt{-1}; & (\sqrt{-1})^{4n+2} &= -1 \\ (\sqrt{-1})^{4n+3} &= - \sqrt{-1}; & (\sqrt{-1})^{4n+4} &= +1 \end{aligned} \right\} \quad (B)$$

Hence *any even power of $\sqrt{-1}$ is real and any odd power is imaginary.*

EXERCISES

Find the value of:

- | | | |
|-----------------------|-----------------------|-------------------------------|
| 1. $(\sqrt{-1})^6$ | 5. $(\sqrt{-1})^{18}$ | 9. $(\sqrt{-2})(\sqrt{-2})$ |
| 2. $(\sqrt{-1})^7$ | 6. $(\sqrt{-1})^{15}$ | 10. $(\sqrt{-3})(\sqrt{-27})$ |
| 3. $(\sqrt{-1})^{10}$ | 7. $(-i)^3$ | 11. $(\sqrt{-6})(\sqrt{-24})$ |
| 4. $(\sqrt{-1})^{21}$ | 8. $(-i)^8$ | 12. $(2\sqrt{-1})(\sqrt{-4})$ |

13. Add $\sqrt{-a^4}$ and $\sqrt{-16a^4}$.

SOLUTION

$$\sqrt{-a^4} + \sqrt{-16a^4} = a^2i + 4a^2i = 5a^2i$$

Perform the indicated operations:

14. $\sqrt{-4} + \sqrt{-49}$

20. $(3\sqrt{-20})(\sqrt{-80})$

15. $\sqrt{-9} + \sqrt{-64}$

21. $(3\sqrt{-5})(2\sqrt{-15})$

16. $2\sqrt{-4} + 3\sqrt{-1}$

22. $(4\sqrt{-27})(\sqrt{-12})$

17. $\sqrt{-12} + 4\sqrt{-3}$

23. $(2\sqrt{-8})(5\sqrt{-3})$

18. $5\sqrt{-18} - \sqrt{-72}$

24. $(\sqrt{-18}) \div (\sqrt{-3})$

19. $\sqrt{-16} - 3\sqrt{-9}$

25. $(\sqrt{-27}) \div (\sqrt{3})$

PROPERTIES OF QUADRATIC EQUATIONS

Nature of the roots

In the following discussion the student should keep in mind the distinctions between *rational* and *irrational*, *real* and *imaginary* numbers.

For example, 2 and $\sqrt{4}$ are *rational* and also *real*; $\sqrt{2}$ and $\sqrt{5}$ are *irrational*, but *real*; $\sqrt{-2}$ and $\sqrt{-5}$ are *irrational* and also *imaginary*.

Every quadratic equation may be reduced to the form

$$ax^2 + bx + c = 0$$

in which a is positive and b and c are positive or negative.

Denote the roots by r_1 and r_2 . Then (page 291)

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

An examination of the above values of r_1 and r_2 will show that the nature of the roots, as real or imaginary, rational or irrational, may be determined by observing whether $\sqrt{b^2 - 4ac}$ is real or imaginary, rational or irrational.

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PRINCIPLES. In any quadratic equation, $ax^2 + bx + c = 0$, when a , b , and c represent real and rational numbers:

1. If $b^2 - 4ac$ is **positive**, the roots are **real and unequal**.
2. If $b^2 - 4ac$ equals **zero**, the roots are **real and equal**.
3. If $b^2 - 4ac$ is **negative**, the roots are **imaginary and unequal**.
4. If $b^2 - 4ac$ is a **square** or equals **zero**, the roots are **rational**; otherwise they are **irrational**.

The expression $b^2 - 4ac$ is called the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$.

If a is positive and b and c are positive or negative, the **signs of the roots** of $ax^2 + bx + c = 0$, that is, the signs of

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

may be determined from the signs of b and c .

Thus, if c is positive, $-b$ is numerically greater than $\pm\sqrt{b^2 - 4ac}$, whence both roots have the sign of $-b$; if c is negative, $-b$ is numerically less than $\pm\sqrt{b^2 - 4ac}$, whence r_1 is positive and r_2 is negative. The root having the sign opposite to that of b is the greater numerically.

PRINCIPLE 5. If c is positive, both roots have the sign opposite to that of b ; if c is negative, the roots have opposite signs, and the numerically greater root has the sign opposite to that of b .

NOTE. If $b = 0$, the roots have opposite signs.

EXERCISES

1. What is the nature of the roots of $x^2 - 7x - 8 = 0$?

SOLUTION

Since $b^2 - 4ac = 49 + 32 = 81 = 9^2$, a positive number and a square, by Prin. 1 and 4 above, the roots are real and unequal and rational.

Since c is negative, by Prin. 5 above, the roots have opposite signs and, b being negative, the positive root is the greater numerically.

2. What is the nature of the roots of $3x^2 + 5x + 3 = 0$?

SOLUTION

Since $b^2 - 4ac = 25 - 36 = -11$, a negative number, by Prin. 3 above, both roots are imaginary.

Find, without solving, the nature of the roots of:

3. $x^2 - 5x - 75 = 0$

8. $x^2 + x + 2 = 0$

4. $x^2 + 5x + 6 = 0$

9. $x^2 - 3x + 5 = 0$

5. $x^2 + 7x - 30 = 0$

10. $x^2 + 3x - 5 = 0$

6. $4x^2 - 4x + 1 = 0$

11. $4x^2 + 16x + 7 = 0$

7. $4x^2 + 6x - 4 = 0$

12. $9x^2 + 12x + 4 = 0$

13. For what values of m will the equation $2x^2 + 3mx + 2 = 0$ have equal roots? imaginary roots?

SOLUTION

The roots will be equal if the discriminant equals zero (Prin. 2, page 354); that is, if $(3m)^2 - 4 \cdot 2 \cdot 2 = 0$, $m = \frac{4}{3}$ or $-\frac{4}{3}$.

The roots will be imaginary if the discriminant is negative (Prin. 3, page 354); that is, if $(3m)^2 - 4 \cdot 2 \cdot 2$ is negative, which will be true when m is numerically less than $\frac{4}{3}$.

14. For what values of m will the equation $9x^2 - 5mx + 25 = 0$ have equal roots? real roots? imaginary roots?

15. For what values of a will the roots of the equation $4x^2 - 2(a - 3)x + 1 = 0$ be real and equal? real and unequal? imaginary?

16. Find the values of m for which the roots of the equation $4x^2 + mx + x + 1 = 0$ are equal.

17. For what values of n are the roots of the equation $3x^2 + 1 = n(4x - 2x^2 - 1)$ real and equal?

18. For what value of a are the roots of the equation $ax^2 - (a - 1)x + 1 = 0$ numerically equal but opposite in sign? Find the roots for this value of a .

19. For what values of d has the equation $x^2 + (2 - d)x = 3d^2 - 27$ a zero root? Find both roots for each of these values of d .

20. For what values of m will the roots of the equation $(m + \frac{5}{2})x^2 - 2(m + 1)x + 2 = 0$ be equal?

21. Solve for x and y :
$$\begin{cases} 3x^2 - 4y^2 = 8 \\ 5(x - k) - 4y = 0 \end{cases}$$

For what values of k are the roots real? imaginary? equal?

Relation of roots and coefficients

In the general quadratic equation $ax^2 + bx + c = 0$, if we divide both members by a and then substitute p for $\frac{b}{a}$ and q for $\frac{c}{a}$, we have $x^2 + px + q = 0$, whose roots by actual solution are found to be

$$r_1 = \frac{-p + \sqrt{p^2 - 4q}}{2} \quad \text{and} \quad r_2 = \frac{-p - \sqrt{p^2 - 4q}}{2}$$

Add the roots: $r_1 + r_2 = \frac{-2p}{2} = -p$

Multiply the roots: $r_1 r_2 = \frac{p^2 - (p^2 - 4q)}{4} = q$

Hence we have the following:

PRINCIPLE. *The sum of the roots of a quadratic equation having the form $x^2 + px + q = 0$ is equal to the coefficient of x with its sign changed, and their product is equal to the absolute term.*

Formation of quadratic equations

Substituting $-(r_1 + r_2)$ for p , and $r_1 r_2$ for q in the equation $x^2 + px + q = 0$, we have

$$x^2 - (r_1 + r_2)x + r_1 r_2 = 0$$

Expand: $x^2 - r_1 x - r_2 x + r_1 r_2 = 0$

Factor: $(x - r_1)(x - r_2) = 0$

Hence to form a quadratic equation whose roots are given,

- (1) *we may make the proper substitutes for p and q , as above, or*
- (2) *we may subtract each root from x and place the product of the remainders equal to zero.*

EXERCISES

1. Form the equation whose roots are -5 and 2 .

SOLUTION

(1) Since the sum of the roots with their signs changed is $+5 - 2$, or 3 , and the product of the roots is -10 , the equation is $x^2 + 3x - 10 = 0$.

(2) $(x + 5)(x - 2) = 0$. Hence the equation is $x^2 + 3x - 10 = 0$.

Form the equation whose roots are :

- | | | |
|---------------------------------|-----------------------------------|------------------------------------|
| 2. 6, 4 | 8. $a, -3a$ | 14. $3 \pm \sqrt{-2}$ |
| 3. 5, -3 | 9. $a + 2, a - 2$ | 15. $2 \pm \sqrt{5}$ |
| 4. $3, -\frac{1}{3}$ | 10. $b + 1, b - 1$ | 16. $2 \pm \sqrt{-3}$ |
| 5. $\frac{2}{3}, \frac{5}{3}$ | 11. $a + b, a - b$ | 17. $-\frac{1}{2}(3 \pm \sqrt{6})$ |
| 6. $-2, -\frac{1}{2}$ | 12. $3.7a, 5.3a$ | 18. $\frac{1}{2}(-1 \pm \sqrt{2})$ |
| 7. $-\frac{1}{2}, -\frac{3}{2}$ | 13. $\frac{1}{2}(a \pm \sqrt{b})$ | 19. $a(2 \pm 2\sqrt{-5})$ |

20. What is the sum of the roots of $2m^2x^2 - (5m - 1)x = 6$? For what values of m is the sum equal to 2?

21. When one of the roots of $ax^2 + bx + c = 0$ is twice the other, what is the relation between a, b , and c ?

SOLUTION

Write $ax^2 + bx + c = 0$ in the following form :

$$(1) \quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Represent the roots by r and $2r$:

$$(2) \quad r + 2r = 3r = -\frac{b}{a}$$

$$(3) \quad r \cdot 2r = 2r^2 = \frac{c}{a}$$

Substitute the value of r obtained from (2) in (3) and reduce :

$$b^2 = \frac{9}{2}ac$$

22. Obtain an equation expressing the condition that one root of $4x^2 - 3ax + b = 3$ is twice the other.

23. Find the condition that one root of $ax^2 + bx + c = 0$ shall be greater than the other by 3.

24. When one root of the general quadratic equation $ax^2 + bx + c = 0$ is the reciprocal of the other, what is the relation between a and c ?

25. If the roots of $ax^2 + bx + c = 0$ are r_1 and r_2 , write an equation whose roots are $-r_1$ and $-r_2$.

26. If one root of the equation $x^2 + 5x + c = 0$ exceeds the other by 1, find the value of c .

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27. Without solving the equation, obtain the sum of the squares of the roots of $2x^2 - 12x + 3 = 0$.

SOLUTION

(1)	Sum of roots $= r_1 + r_2 = 6$	
(2)	Product of roots $= r_1 r_2 = \frac{3}{2}$	
(3)	$r_1^2 + 2r_1 r_2 + r_2^2 = 36$	(1) squared
(4)	$2r_1 r_2 = 3$	$2 \times (2)$
	$r_1^2 + r_2^2 = 33$	(3) - (4)

Find, without solving the equation :

28. The sum of the squares of the roots of $x^2 - 5x - 6 = 0$.

29. The sum of the cubes of the roots of $2x^2 - 3x + 1 = 0$.

30. The difference between the roots of $12x^2 + x - 1 = 0$.

31. The square root of the sum of the squares of the roots of $x^2 - 7x + 12 = 0$.

32. The sum of the reciprocals of the roots of $ax^2 + bx + c = 0$.

SUGGESTION.
$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{r_1 + r_2}{r_1 r_2}$$

33. The difference between the reciprocals of the roots of $8x^2 - 10x + 3 = 0$.

The number of roots of a quadratic equation

It has been seen (page 356) that any quadratic equation may be reduced to the form $x^2 + px + q = 0$, which has two roots, as r_1 and r_2 . To prove that the equation has two roots and not more than two roots, it is necessary to assume that every equation has at least one root. This can be proved in more advanced algebra.

If the equation $x^2 + px + q = 0$ has one root, designate it by r_1 . When both members are divided by $x - r_1$, an equation of the first degree is obtained. If this equation has a root, call the root r_2 .

Then (1) $x^2 + px + q = (x - r_1)(x - r_2) = 0$

If the equation has a third root, suppose it is r_3 .

Substituting r_3 for x in (1), we have $(r_3 - r_1)(r_3 - r_2) = 0$ which is impossible if r_3 differs from both r_1 and r_2 .

PRINCIPLE. A quadratic equation has two roots and only two roots.

CHAPTER XVII

VARIATION AND DEPENDENCE

If an automobile is pulling a trailer, the automobile cannot change its rate without causing a corresponding change in the rate of the trailer. If the automobile changes its direction, it will make a difference in the behavior of the trailer. What the trailer does depends on the automobile to which it is attached.

Many things are *related* so definitely that one thing cannot be changed without *changing* something that *depends* on it. In driving from home to a football game, we cannot increase the speed of our car without decreasing the time it takes us to arrive at our destination. So definitely are factors related that a change produced in one factor can be measured by the change in another factor. The kind and amount of change produced in the second can be determined by the change in the first.

Algebra furnishes tools by which common relationships can be expressed and measured.

ORAL EXERCISES

Supply the missing words :

1. The number of pounds of steak that can be bought for a dollar depends on the — per pound.
2. The interest which Mr. Adams pays Mr. Jones depends on the —, the —, and the —.
3. If a man works for \$.50 an hour, his daily wages depend on —.

4. The time it takes to drive from school to the ball field depends on the — of driving.
5. The area of a circle depends on its —.
6. The perimeter of a rectangle depends on its — and its —.
7. If $y = 2x + 3$, the value of y depends on —.

Relationship shown by formulas

In the preceding work with formulas, many illustrations of dependence have been given. The formula $F = \frac{9}{5}C + 32^\circ$ expresses a relationship. From the formula we can tell that when the reading on a centigrade scale is 10° , the corresponding Fahrenheit reading is 50° ; or that when the centigrade reading is 20° , the Fahrenheit reading is 68° . C and F are so related that the value of C cannot be changed without changing the value of F .

ORAL EXERCISES

1. Draw three circles with different radii, say $2''$, $3''$, and $5''$. The formula for the area of a circle is $A = \pi r^2$. By looking at the formula, you see that the value of A depends on the value of r . As r is increased, does the value of A become greater or less? Compute the area of each circle drawn.
2. The formula $i = .04\frac{1}{4}p$ tells how much interest a Liberty bond will yield annually at $4\frac{1}{4}\%$. What effect on the value of i will an increase in p have?
3. The formula $d = 20t$ shows that the distance a driver can drive at a constant rate of 20 miles an hour depends on the number of hours he drives. If t is decreased, what will be the effect on d ?
4. Use the formula $V = e^3$ to compute the volume of a cubical box whose side, e , is 3; 5; 10. What change is produced in V when e is increased?

The notion of a variable. The meaning of a constant

In computing the areas of circles of different sizes by the formula $A = \pi r^2$, two kinds of algebraic numbers are used.

On the one hand, the number r may have any size. In other words, the formula can be used for large or small circles. The algebraic number r can change or *vary*. Such a number is called a **variable**. A also is a variable, for it *depends on*, or changes with, the variable r .

On the other hand, the number π for any circle, large or small, has one *fixed* value. It is called a **constant**.

The formula $p = 4s$ expresses the fact that the perimeter of a square equals 4 times the length of a side. Whatever the size of the square the number 4 remains the same and is a *constant*, while the numbers p and s change and are *variables*.

ORAL EXERCISES

Tell which quantities in these formulas and equations are *variables* and which are *constants*.

- | | | |
|----------------------------------|-------------------------|----------------|
| 1. $H = s^3$ | 4. $c = \pi d$ | 7. $y = x + 3$ |
| 2. $F = \frac{9}{5}C + 32^\circ$ | 5. $r = \frac{d}{2}$ | 8. $d = 20t$ |
| 3. $p = 2a + 2b$ | 6. $i = \frac{6p}{100}$ | 9. $i = prt$ |

Functions

A variable expression involving one or more variables is called a **function** of those variables.

We have already seen several examples of functions:

Thus: (1) The area of a rectangle is a function of the base and the altitude: $A = bh$

(2) The circumference of a circle is a function of r :

$$c = 2\pi r$$

(3) $A = \pi r^2$ is a function of r .

(4) $3x - 2$ is a function of x . This may be expressed by the equation $f(x) = 3x - 2$.

We see in these and other functions the idea of the *dependence* of one *variable* on another.

In $A = bh$, A depends for its value on the values of b and h .

In $c = 2\pi r$, c depends for its value on the value of r .

In $A = \pi r^2$, A depends for its value on the value of r .

1. In the formula $A = bh$, what change is produced in the value of A when the value of h or of b is doubled? when the value of h or of b is tripled?

2. In the formula $c = 2\pi r$, what change is produced in the value of c when the value of r is doubled? when the value of r is tripled? when the value of r is divided by 2?

3. In the formula $A = \pi r^2$, what change is produced in the value of A when the value of r is doubled? when the value of r is tripled? when the value of r is divided by 2?

Practical importance of the function idea

Every formula or equation represents one quantity as a function of one or more quantities. Graphs and statistical tables are essentially functional in character. In our study of mathematics we should develop the power of observing the relationship between variable quantities.

Mechanics, farmers, merchants, housewives, as well as scientists and engineers, have to work constantly with quantitative values that are dependent on other values. The tariff, rates of postage and express, insurance rates, and income taxes, all are fundamentally functional in character.

ILLUSTRATION

If a man travels in an automobile at an average rate of 25 miles an hour, the distance covered may be found by the formula d (number of miles) $= 25 t$ (number of hours). The following table gives the distance covered at the end of each hour:

Number of hours (t) . . .	1	2	3	4	5	6	7	8
Number of miles (d) . . .	25	50	75	?	?	?	?	?

EXERCISES

1. Find the distance covered in $2\frac{1}{2}$ hours by a man who travels in an automobile at the rate of 25 miles an hour; the distance covered in 9.2 hours; in 12.4 hours.

2. In the table on page 362 what effect does an increase in the time have upon the distance? What effect does an increase in the distance have upon the time?

3. What is the value of the ratio $\frac{d}{t}$? Is this the same or different when we compare each distance with its corresponding time?

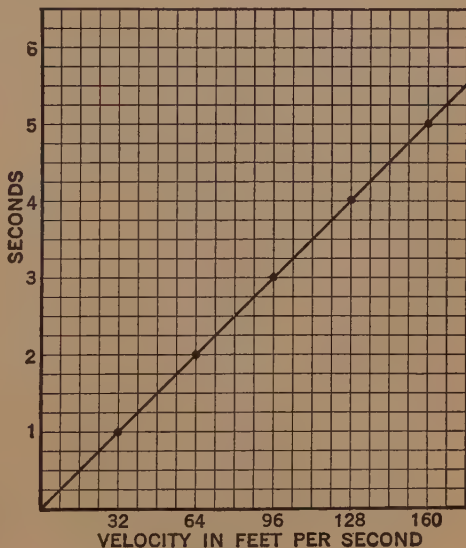
4. Draw a graph of the formula $d = 25 t$.

Graphs showing direct variation

In studying the motion of falling bodies the physicist employs the two formulas $v = gt$ and $s = \frac{1}{2} gt^2$. The force of gravity represented by g is a *constant* for any locality and is approximately 32. Hence the formulas become $v = 32 t$ and $s = 16 t^2$.

The formula $v = 32 t$ states

that a freely falling body acquires a velocity (v) of 32 feet for each second (t) of the fall.



GRAPH OF $v = 32 t$

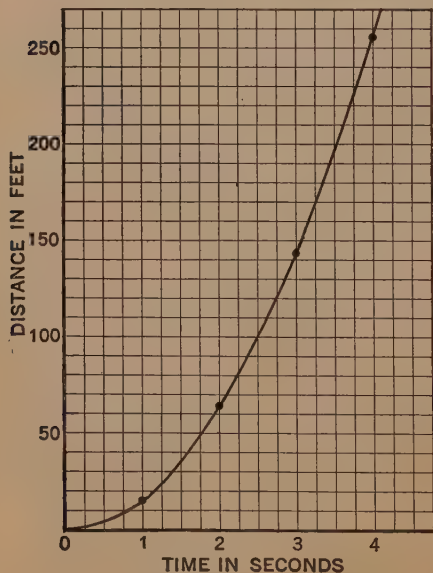
The table below also shows this relation :

When t is	0	1	2	3	4	5	6
Then v is	0	32	64	?	?	?	?

The graph on page 363 shows the variation plainly. Construct such a graph.

EXERCISES

1. What is the value of v in the formula $v = 32t$ when $t = 1\frac{1}{4}$? $2\frac{1}{2}$? $3\frac{1}{4}$? 5? $5\frac{1}{4}$? $5\frac{1}{2}$?
2. What is the value of t in the formula $v = 32t$ when $v = 40$? 60? 80? 120? 136? 168?
3. What does the graph of $v = 32t$ show with respect to the rate of increase of v as compared with t ?



GRAPH OF $s = 16t^2$

The graph of $s = 16t^2$ at the left represents the distance a body will fall in t seconds. Make a copy of this graph.

4. What is the value of s when $t = 1\frac{1}{2}$? 2? $2\frac{1}{2}$? 4?
5. What is the value of t when $s = 50$? 75? 100? 150?
6. How does the rate of increase of the time compare with that of the distance?
7. Draw a graph to show the rate of

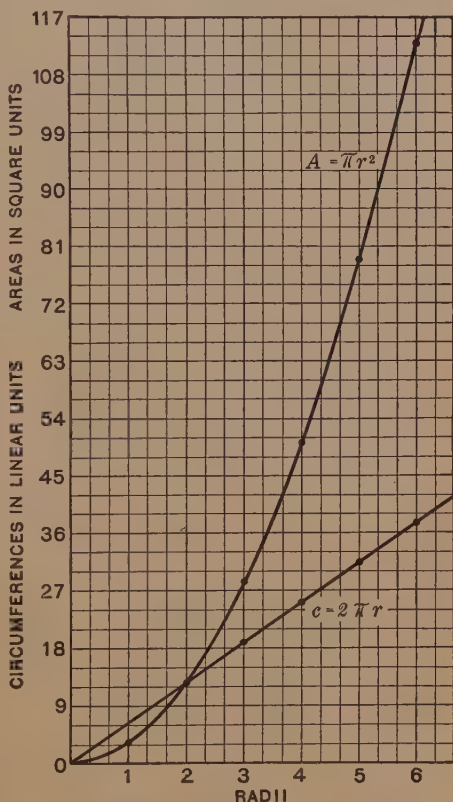
increase of the area of a rectangle as compared with the base when the altitude is 5. ($A = 5b$.)

8. If we double the length of the base, what is the change in the area? If the area is tripled, what change is made in the base?

Sometimes we can compare different rates of change advantageously by two graphs on the same scale. The graphs at the right enable one to compare the rate of increase of the circumference of a circle with its area as the radius changes. The vertical scale is given in linear units for $c = 2\pi r$ and in square units for $A = \pi r^2$.

9. Construct these two graphs.

10. For what values of r does the numerical value of the circumference exceed that of the area? What is the value of r when the numerical value of the circumference equals that of the area?



GRAPHS OF THE TWO FORMULAS

$$c = 2\pi r$$

$$A = \pi r^2$$

11. From the graph determine the approximate area when $r = 2$; when $r = 4$; when $r = 6$. What will be the values of the circumferences for the same values of r ?

12. If the value of r is multiplied by 4, how much is A increased? How much is c increased? If r is multiplied by 5, how much is A increased? How much is c increased?

The circumference varies directly as the radius of the circle, and the area varies directly as the square of the radius.

13. On the same scale construct the graphs of the formulas for the area and the perimeter of a square; that is, of $A = s^2$ and $p = 4s$.

14. Using the graphs in exercise 13, state the parts not given in the following table:

	s (SIDE)	p (PERIMETER)	A (AREA)
1	3	?	?
2	4	?	?
3	$2\frac{1}{2}$	?	?
4	?	20	?
5	?	?	36
6	?	14	?

Variation and proportion

A mechanic charges 80 cents an hour for repairing an automobile. The relation between the cost and the time is expressed in the following table:

The value of c is	80	120	160	200	240
When t is	1	1.5	2	2.5	3

Any value of c divided by the corresponding value of t equals any other value of c divided by the corresponding value of t . If we call c_1 the "first value of c " and t_1 the corresponding value of t , we can write the proportion

$\frac{c_1}{t_1} = \frac{c_2}{t_2}$. Testing the proportion with the second and third sets of numbers of our table, we get $\frac{120}{1.5} = \frac{160}{2}$.

Hence we see that when we have *direct variation*, the numbers, if properly chosen, are proportional.

The equation $\frac{c_1}{t_1} = \frac{c_2}{t_2}$ can be solved if at least three of the terms are known, and not otherwise. What is the value of c_2 if $c_1 = 80$, $t_1 = 1$, and $t_2 = 3.75$? As a practical problem, we could state this as follows:

“If a mechanic always charges at the rate of \$.80 for working 1 hour, how much should he charge for working $3\frac{3}{4}$ hours?”

EXERCISES

1. In many localities fire wood is sold by the cord, which is a pile 4 feet wide, 4 feet high, and 8 feet long, containing 128 cubic feet. If the width and the height of the pile remain constant, the number of cords of wood varies directly as the length of the pile. This is expressed by the proportion $\frac{c_1}{c_2} = \frac{l_1}{l_2}$. If a pile of wood 12 feet long contains $1\frac{1}{2}$ cords, how many cords will there be in a pile 10 feet long? 16 feet long? 30 feet long? How long would the pile be to contain $5\frac{1}{2}$ cords? 3.75 cords?

2. If l_1 and l_2 are the lengths of two pendulums and t_1 and t_2 the number of seconds they take for oscillation, then

$$\frac{t_1^2}{t_2^2} = \frac{l_1}{l_2}$$

A pendulum which makes one oscillation a second is approximately 39.1 inches long. How many seconds will it take a pendulum 156.4 inches long to oscillate?

3. The circumferences of two circles vary directly as their diameters. If a circumference of a circle 10 inches in diameter is 31.4 inches, what is the circumference of a circle whose diameter is 8 inches?

4. The area of a circle varies directly as the square of the diameter. The area of a circle whose diameter is 20 inches is 314 square inches. What is the area of a circle whose diameter is 5 inches?

Inverse variation

We have seen how one quantity varies directly as another. Now we shall see that there are instances when one quantity increases as another quantity correspondingly decreases in value.

For example, in an automobile trip of 120 miles, if the rate of the automobile is 20 miles an hour, the time for the journey is 6 hours ; but if the rate is 30 miles an hour, the time is only 4 hours, while if the rate is 40 miles an hour, the time is only 3 hours.

The following table will show that an increase of the rate per hour will result in a decrease in the time. The distance covered is 120 miles.

Rate in miles per hour . . .	10	15	20	25	30	35	40
Total time in hours	12	8	6	$4\frac{4}{5}$	4	$3\frac{3}{7}$	3

We see that the rates *vary inversely* as the time.

For example, 10 is not in the same ratio to 15 as 12 is to 8, but is in the same ratio as 8 is to 12. In other words, $\frac{10}{15}$ does not equal $\frac{12}{8}$, but equals $\frac{12}{8}$ inverted ; that is, $\frac{10}{15} = \frac{8}{12}$. Again, $\frac{20}{30} = \frac{4}{6}$. Here also we see that the ratio of the rates per hour varies inversely as the total time of the journey, if the automobile goes at different rates.

We may generalize and say that, if r_1 and r_2 represent the rates per hour and t_1 and t_2 the corresponding times,

$$\frac{r_1}{r_2} = \frac{t_2}{t_1}$$

EXERCISES

1. The time it takes a boy to drive his motor cycle from home to the ball field varies inversely as the rate. Express this as a proportion. *Ans.* $\frac{t_1}{t_2} = \frac{r_2}{r_1}$

2. How long will it take the boy to go to the ball field when driving at the rate of 30 miles an hour if it takes him 24 minutes to go at the rate of 25 miles an hour?

3. The rate of travel for a fixed distance *varies inversely* as the time taken. The average rate of a 20-hour train between New York and Chicago is 49 miles an hour. What is the average rate of a 24-hour train between these cities?

4. It has been learned by experiment that when gas is put under pressure the volume varies inversely as the pressure. State this fact, which is known as Boyle's law, as a proportion.

5. If the volume of a gas is 100 cubic centimeters when the pressure is 76 centimeters, what will be the volume when the pressure is 38 centimeters?

6. The time it will take to finish laying a sidewalk varies inversely as the number of men engaged. State this as a proportion.

7. If 3 men can lay a sidewalk in $2\frac{1}{2}$ days, how long will it take 4 men?

8. The illumination from a source of light is inversely proportional to the square of the distance from the source. If a book is 8 feet from a lamp, how near the lamp must it be moved in order that the light on it may be doubled?

9. Make up and solve original problems which are applications of the principles given in exercises 1, 3, 4, and 6.

Graphic representation of inverse variation

Using the example on page 368, in which in an automobile journey of 120 miles we noted the changes in time

due to changes in rate, we shall see that inverse variation may be represented graphically.

EXERCISES

1. Using the graph below, answer the following questions :

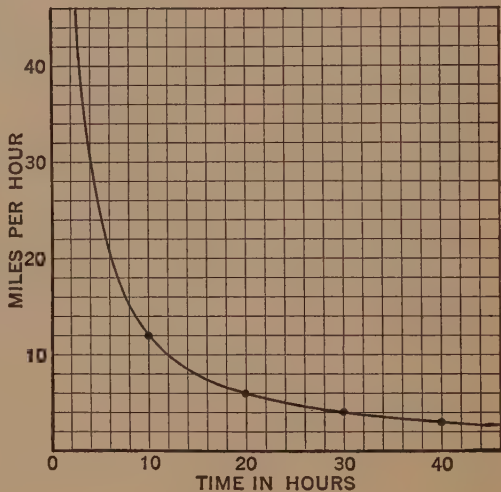
(a) If the rate is 30 miles an hour, what is the time?

(b) If the rate is 12 miles an hour, what is the time?

(c) If the time is 30 hours, what is the rate?

(d) If $\frac{10}{30} = \frac{4}{t_1}$, what does t_1 equal?

(e) If $\frac{r_1}{40} = \frac{3}{6}$, what does r_1 equal?



GRAPH OF $120 = rt$

2. Draw a graph showing the inverse variation between the length and the width of a rectangle whose area is 48 square inches and whose length varies as follows: 2 inches, 4 inches, 6 inches, 8 inches, 10 inches, 12 inches, 14 inches, 16 inches, 20 inches, 24 inches, 32 inches.

3. Assume that you desire to save \$60, putting aside a certain amount each month. If you vary the amount saved each month, how will this affect the time necessary to save the whole amount? Represent this by means of a graph.

4. Think of another instance of inverse variation. Represent this (1) by means of a table showing the variable quantities; (2) by means of a graph.

CHAPTER XVIII

ADDITIONAL CASES IN DIVISION AND FACTORING

SYNTHETIC DIVISION

Equations in one unknown number may sometimes be solved by factoring (page 288) and by setting each factor equal to zero. This method may be used to solve equations of a higher degree than the second degree. In order to factor an expression, it is frequently necessary to divide it by a binomial. This can be done in the usual way; but when the divisor is of the form $x \pm a$ the work may be greatly abbreviated, as shown below.

ILLUSTRATIVE EXAMPLES

1. Divide $x^3 + 4x^2 + 7x - 6$ by $x - 2$.

SOLUTIONS

$$\begin{array}{r|l}
 (a) & x^3 + 4x^2 + 7x - 6 \\
 x^3 - 2x^2 & x - 2 \\
 \hline
 6x^2 + 7x & x^2 + 6x + 19 \\
 6x^2 - 12x & \\
 \hline
 19x - 6 & \\
 19x - 38 & \\
 \hline
 32, \text{ remainder} &
 \end{array}$$

In (b) below the work is shortened by omitting the literal factors and using only the detached coefficients.

$$\begin{array}{r|l}
 (b) & 1 + 4 + 7 - 6 \\
 1 - 2 & 1 - 2 \\
 \hline
 6 + 7 & 1 + 6 + 19 \\
 6 - 12 & \\
 \hline
 19 - 6 & \\
 19 - 38 & \\
 \hline
 32, \text{ remainder} & \\
 371 &
 \end{array}$$

3. Divide $x^4 - 3x^2 - 4$ by $x - 2$.

SOLUTION

In this case the first and third powers of x are lacking, hence it is necessary to indicate their places by zeros.

$$\begin{array}{r} 1 + 0 - 3 + 0 - 4 \quad | \quad 2 \\ + 2 + 4 + 2 + 4 \\ \hline 1 + 2 + 1 + 2 \end{array}$$

The quotient is $x^3 + 2x^2 + x + 2$.

EXERCISES

Divide by synthetic division :

1. $x^3 + 4x^2 + 5x + 2$ by $x + 1$
2. $1 + 2x + 3x^2 + 4x^3$ by $1 + x$
3. $5v^3 - 12v + 3v^2 + 4$ by $v + 2$
4. $y^4 + 3y^2 - 4y^3 + 8y - 24$ by $y - 3$
5. $4x^4 - 3x^3 - 1$ by $x - 1$
6. $m^5 - 19m - 6$ by $m + 2$
7. $x^5 + 32$ by $x + 2$
8. $y^4 - 81$ by $y - 3$
9. $A^3 + 125$ by $A + 5$
10. $x^4 - 3x^2 - x + 3$ by $x - 1$
11. $2x^5 - 3x^3 + 2x - 40$ by $x - 2$
12. $x^3 - 49x - 115$ by $x + 5$

REVIEW OF FACTORING

The most common types of factoring are given in Chapter VII. You will, however, sometimes meet with expressions that are factored by other methods. Some of these are given in this chapter.

Review Chapter VII so as to be familiar with the types given there; then work the exercises on page 374. Always look for the prime factors.

REVIEW EXERCISES

Find the prime factors of :

- | | |
|---|-------------------------------|
| 1. $9x^2 - 36y^2$ | 10. $a^2 - b^2 - 9c^2 + 6bc$ |
| 2. $16x^4 - 72x^2y^2 + 81y^4$ | 11. $12x^2 + 12x - 9$ |
| 3. $75a^2 - 15ab + .75b^2$ | 12. $20x^{2a} - 30x^a - 50$ |
| 4. $5a^2 - 35ab - 90b^2$ | 13. $y^4 + y^3 + y^2 + y$ |
| 5. $7a^2 + 35abc - 252b^2c^2$ | 14. $9x^{4n} + 43x^{2n} - 10$ |
| 6. $10a^2 + 11ab - 6b^2$ | 15. $4x^{2a} - 30x^a + 50$ |
| 7. $80x^4 - 405y^4$ | 16. $51x^3 + 75x^2 - 54x$ |
| 8. $14a^2 - 95ab - 75b^2$ | 17. $10a^3 - 13a^2b - 3ab^2$ |
| 9. $4a^2 - 12ab + 9b^2 - 9c^2$ | 18. $27a^4 - 3a^2b^2 - 4b^4$ |
| 19. $25a^2 - 30ab^2 + 9b^4$ | |
| 20. $9a^{2m} + 24a^mb^{n+1} + 16b^{2n+2}$ | |
| 21. $(2a + b)^2 + 5(2a + b) - 24$ | |
| 22. $bx - acx + abcy - b^2y$ | |
| 23. $36ab - 18b^2 - 18ac + 9bc$ | |
| 24. $75x^3 - 90x^2y^2 + 27xy^4$ | |

FACTORING BINOMIAL EXPRESSIONS

1. Prove the following by synthetic division :

$$(a^2 - b^2) \div (a - b) = a + b$$

$$(a^3 - b^3) \div (a - b) = a^2 + ab + b^2$$

$$(a^4 - b^4) \div (a - b) = a^3 + a^2b + ab^2 + b^3$$

The difference of the same powers of two numbers is divisible by the difference of the numbers.

“Divisible” means “exactly divisible”; that is, divisible without a remainder.

2. Prove the following by synthetic division :

$$(a^2 - b^2) \div (a + b) = a - b$$

$$(a^3 - b^3) \div (a + b) = a^2 - ab + b^2, \text{ rem., } -2b^3$$

$$(a^4 - b^4) \div (a + b) = a^3 - a^2b + ab^2 - b^3$$

The difference of the same powers of two numbers is divisible by the sum of the numbers only when the powers are even.

3. Prove the following by synthetic division :

$$(a^2 + b^2) \div (a - b) = a + b, \text{ rem., } 2b^2$$

$$(a^3 + b^3) \div (a - b) = a^2 + ab + b^2, \text{ rem., } 2b^3$$

The sum of the same powers of two numbers is not divisible by the difference of the numbers.

4. Prove the following by synthetic division :

$$(a^2 + b^2) \div (a + b) = a - b, \text{ rem., } 2b^2$$

$$(a^3 + b^3) \div (a + b) = a^2 - ab + b^2$$

The sum of the same powers of two numbers is divisible by the sum of the numbers only when the powers are odd.

Hence the preceding conclusions may be summarized as follows, n being a positive integer :

PRINCIPLES. 1. $a^n - b^n$ is always divisible by $a - b$.

2. $a^n - b^n$ is divisible by $a + b$ only when n is even.

3. $a^n + b^n$ is never divisible by $a - b$.

4. $a^n + b^n$ is divisible by $a + b$ only when n is odd.

The following law of signs may be inferred readily :

When $a - b$ is the divisor, the signs in the quotient are plus.

When $a + b$ is the divisor, the signs in the quotient are alternately plus and minus.

The following law of exponents also may be inferred :

In the quotient the exponent of a decreases and that of b increases by 1 in each successive term.

! To factor the sum or the difference of two cubes

By applying the principles above :

$$\frac{a^3 + b^3}{a + b} = a^2 - ab + b^2 \text{ and } \frac{a^3 - b^3}{a - b} = a^2 + ab + b^2$$

$$\text{Hence } a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$\text{and } a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

By use of these forms any expression that can be written as the sum or the difference of two cubes can be factored.

EXERCISES

1. Factor
- $x^6 + y^6$
- .

SOLUTION

$$x^6 + y^6 = (x^2)^3 + (y^2)^3 = (x^2 + y^2)(x^4 - x^2y^2 + y^4)$$

2. Factor
- $a^9 - 125 b^3$
- .

SOLUTION

$$a^9 - 125 b^3 = (a^3)^3 - (5 b)^3 = (a^3 - 5 b)(a^6 + 5 a^3b + 25 b^2)$$

Factor, and check each result :

- | | | |
|------------------|-----------------------|---------------------------|
| 3. $x^3 + y^3$ | 8. $64 r^9 + 216 s^9$ | 13. $27 x^3y^6 - 216$ |
| 4. $x^3 - y^3$ | 9. $x - x^4$ | 14. $343 n^3 + 1000$ |
| 5. $m^3 - 1$ | 10. $v^7 + 27 v$ | 15. $r^{3x} - 729 s^{3z}$ |
| 6. $1 + 125 m^3$ | 11. $a^3b^3 - c^3d^3$ | 16. $1 + (a + b)^3$ |
| 7. $x^3 - y^6$ | 12. $r^6 + 64 s^3$ | 17. $(x - y)^3 - 8$ |

To factor the sum or the difference of the same odd powers of two numbers

By applying the principles of page 375, any expression that can be written as the sum or the difference of the same odd powers of two numbers can be resolved into two factors. Thus,

$$a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$$

$$a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$$

EXERCISES

1. Factor
- $m^5 + 32 x^5$
- .

SOLUTION

$$\begin{aligned} m^5 + 32 x^5 &= m^5 + (2 x)^5 \\ &= (m + 2 x)(m^4 - 2 m^3x + 4 m^2x^2 - 8 mx^3 + 16 x^4) \end{aligned}$$

Factor :

- | | | |
|-----------------------|-------------------|---------------------------|
| 2. $m^5 + n^5$ | 6. $x^6 - y^3$ | 10. $a^6b - ab^6$ |
| 3. $m^5 - n^5$ | 7. $x^5 + y^{10}$ | 11. $x^{5m} - y^{5n}$ |
| 4. $32 x^5 + 243 y^5$ | 8. $a^5 + 32$ | 12. $1 - a^5b^{10}c^{15}$ |
| 5. $x^6 - x$ | 9. $64 - 2 a^5$ | 13. $x^{10} + 243 a^5$ |

To factor the difference of the same even powers of two numbers

EXERCISES

1. Factor $a^6 - b^6$.

SOLUTION

$$\begin{aligned} a^6 - b^6 &= (a^3)^2 - (b^3)^2 = (a^3 + b^3)(a^3 - b^3) \\ &= (a + b)(a^2 - ab + b^2)(a - b)(a^2 + ab + b^2) \end{aligned}$$

Considering the even powers as squares, the process may be regarded as factoring the difference of two squares.

Separate into prime factors:

- | | | |
|----------------|-----------------|----------------|
| 2. $x^6 - y^6$ | 5. $x^4 - 16$ | 8. $729 - b^6$ |
| 3. $64x^6 - 1$ | 6. $64x^4 - 81$ | 9. $64 - y^6$ |
| 4. $a^8 - b^8$ | 7. $a^4 - 625$ | 10. $1 - x^8$ |

All the preceding methods of finding binomial factors are really *special* methods. The following is a *general* method of finding binomial factors, when they exist.

FACTORING BY THE FACTOR THEOREM

Zero multiplied by any number is equal to 0.

Conversely, if a product is equal to zero, at least one of the factors must be 0 or a number equal to 0.

If $5x = 0$, since 5 is not equal to 0, x must equal 0.

If $5(x - 3) = 0$, since 5 is not equal to 0, x must have such a value as to make $x - 3$ equal to 0; that is, x must equal 3.

If $5(x - 3)$, or $5x - 15$, or any other polynomial in x reduces to 0 when $x = 3$, $x - 3$ is a factor of the polynomial.

Sometimes a polynomial in x reduces to 0 for more than one value of x . For example, $x^2 - 5x + 6$ equals 0 when $x = 3$ and also when $x = 2$; or when $x - 3 = 0$ and $x - 2 = 0$. In this case both $x - 3$ and $x - 2$ are factors of the polynomial.

$$\therefore x^2 - 5x + 6 = (x - 3)(x - 2)$$

Factor Theorem

If a polynomial in x , having positive integral exponents, reduces to zero when r is substituted for x , the polynomial is exactly divisible by $x - r$.

The letter r represents any number that we may substitute for x .

PROOF. Let D represent any rational integral function of x , and let D reduce to zero when r is substituted for x .

It is to be proved that D is exactly divisible by $x - r$.

Suppose that the dividend D is divided by $x - r$ until the remainder does not contain x . Denote the remainder by R and the quotient by Q .

Then $D = Q(x - r) + R$ (1)

But since D reduces to zero when $x = r$, that is, when $x - r = 0$, (1) becomes $0 = 0 + R$; whence $R = 0$

That is, the remainder is zero, and the division is exact.

EXERCISES

1. Factor $x^3 - x^2 - 4x + 4$.

FIRST SOLUTION

When $x = 1$, $x^3 - x^2 - 4x + 4 = 1 - 1 - 4 + 4 = 0$

Therefore $x - 1$ is a factor of the given polynomial.

Dividing $x^3 - x^2 - 4x + 4$ by $x - 1$ gives the quotient $x^2 - 4$.

$$x^2 - 4 = (x + 2)(x - 2)$$

$$\therefore x^3 - x^2 - 4x + 4 = (x - 1)(x + 2)(x - 2)$$

SUGGESTIONS. 1. Only factors of the absolute term of the polynomial need be substituted for x in seeking factors of the polynomial of the form $x - r$, for if $x - r$ is one factor, the absolute term of the polynomial is the product of r and the absolute term of the other factor.

2. In substituting the factors of the absolute term, try them in order, beginning with the numerically smallest.

3. When 1 is substituted for x , the value of the polynomial is equal to the sum of its coefficients; then $x - 1$ is a factor when the sum of the coefficients is equal to 0.

SECOND SOLUTION

Sometimes an expression like this can be factored quickly by making use of the method of grouping terms on page 127.

$$\begin{aligned} x^3 - x^2 - 4x + 4 &= (x^3 - x^2) - (4x - 4) = x^2(x - 1) - 4(x - 1) \\ &= (x - 1)(x^2 - 4) \\ &= (x - 1)(x + 2)(x - 2) \end{aligned}$$

2. Factor $17x^3 - 14x^2 - 37x - 6$.

SOLUTION

Since the sum of the coefficients is not equal to 0, $x - 1$ is not a factor.

When $x = -1$, $17x^3 - 14x^2 - 37x - 6 = -17 - 14 + 37 - 6 = 0$.

Therefore $x - (-1)$, or $x + 1$, is a factor of the given polynomial.

Dividing $17x^3 - 14x^2 - 37x - 6$ by $x + 1$ gives the quotient $17x^2 - 31x - 6$, which in turn may be tested for factors by the factor theorem.

Substituting factors of -6 for x , we find that:

When $x = 2$, $17x^2 - 31x - 6 = 68 - 62 - 6 = 0$.

Therefore $x - 2$ is a factor of $17x^2 - 31x - 6$.

Dividing by $x - 2$ gives the other factor, $17x + 3$.

$$\therefore 17x^3 - 14x^2 - 37x - 6 = (x + 1)(x - 2)(17x + 3)$$

 3. Factor $2x^3 + x^2y - 5xy^2 + 2y^3$.

SUGGESTION. When $x = y$,

$$2x^3 + x^2y - 5xy^2 + 2y^3 = 2y^3 + y^3 - 5y^3 + 2y^3 = 0$$

Therefore $x - y$ is a factor of $2x^3 + x^2y - 5xy^2 + 2y^3$.

NOTE. The method of synthetic division will in general shorten the work.

Find the prime factors:

4. $x^3 - 4x^2 - 7x + 10$

16. $x^3 - 19x + 30$

5. $x^3 - 6x^2 - 9x + 54$

17. $x^3 - 39x - 70$

6. $x^3 - 7x^2 + 7x + 15$

18. $x^3 - 67x - 126$

7. $x^3 - 12x^2 + 41x - 30$

19. $x^3 - 21xy^2 + 20y^3$

8. $2x^3 - 3x^2 - 17x - 12$

20. $x^3 - 13xy^2 + 12y^3$

9. $x^3 - 13x^2 + 13x - 169$

21. $a^3 + 4a^2 - 11a - 30$

10. $x^3 - 16x^2 + 71x - 56$

22. $a^3 + 9a^2 + 26a + 24$

11. $2x^3 - 9x^2 - 2x + 24$

23. $2n^3 - 7n^2 - 7n + 30$

12. $n^3 + 12n^2 + 41n + 42$

24. $x^4 - 9x^2y^2 + 12xy^3 - 4y^4$

13. $x^3 + 2x^2y - xy^2 - 2y^3$

25. $x^5 - 4x^4 + 19x^2 - 28x + 12$

14. $x^3 + 4x^2y + 5xy^2 + 2y^3$

26. $x^4 + x^3 - 53x + 51$

15. $x^3 - 7x + 6$

27. $x^4 - 12x^2 + 8x + 3$

SPECIAL APPLICATIONS AND DEVICES

The principle by which the difference of two squares is factored has several special applications.

EXERCISES

1. Factor $a^4 + a^2b^2 + b^4$.

SOLUTION

Since $a^4 + a^2b^2 + b^4$ lacks $+ a^2b^2$ of being a square, and since the value of the polynomial will not be changed by adding a^2b^2 and also subtracting a^2b^2 , the polynomial may be written

$$a^4 + 2a^2b^2 + b^4 - a^2b^2,$$

which is the difference of two squares.

$$\begin{aligned}\therefore a^4 + a^2b^2 + b^4 &= a^4 + 2a^2b^2 + b^4 - a^2b^2 \\ &= (a^2 + b^2)^2 - a^2b^2 \\ &= (a^2 + ab + b^2)(a^2 - ab + b^2)\end{aligned}$$

2. Factor $a^4 - 21a^2b^2 + 36b^4$.

SUGGESTION. $a^4 - 21a^2b^2 + 36b^4 = a^4 - 12a^2b^2 + 36b^4 - 9a^2b^2$
 $= (a^2 - 6b^2)^2 - (3ab)^2$

3. Factor $a^4 + 4$.

SUGGESTION. $a^4 + 4 = a^4 + 4a^2 + 4 - 4a^2 = (a^2 + 2)^2 - 4a^2$

Factor the following:

4. $x^4 + x^2y^2 + y^4$

5. $a^8 + a^4b^4 + b^8$

6. $9x^4 + 20x^2y^2 + 16y^4$

7. $4a^4 + 11a^2b^2 + 9b^4$

8. $16a^4 - 17a^2x^2 + x^4$

9. $25x^4 - 29x^2y^2 + 4y^4$

10. $49a^4 - 15a^2b^2 + 121b^4$

11. $3a^4 + 48b^4 - 36a^2b^2$

12. $16x^4 + 4x^2y^2 + y^4$

13. $4a^4 - 13a^2 + 9$

14. $25a^4 - 14a^2b^4 + b^8$

15. $9a^4 + 26a^2b^2 + 25b^4$

16. $b^4 + 64$

17. $a^4 + 4b^4$

18. $m^8 + 4$

19. $a^4 + 324$

20. $a^8 - 16$

21. $m^5 + 4mn^4$

22. $x^4 + 64y^4$

23. $4a^4 + 81$

24. $x^5y^2 + 4xy^2$

25. $9m^8 - 19m^4n^2 + 25n^4$

26. $ab^8 + 16ab^4 + 256a$

27. $25x^4 - 171x^2y^2 + 81y^4$

28. $4c^4 - 37c^2d^2 + 9d^4$

29. $25x^4 - 56x^2y^2 + 16y^4$

Some polynomials may be written in the form $x^2 + px + q$, x^2 and x being replaced by polynomials.

EXERCISES

- Factor $9x^2 + 4y^2 + 12z^2 + 21xz + 14yz + 12xy$.

SOLUTION

$$\begin{aligned} & 9x^2 + 4y^2 + 12z^2 + 21xz + 14yz + 12xy \\ &= (9x^2 + 12xy + 4y^2) + (21xz + 14yz) + 12z^2 \\ &= (3x + 2y)^2 + 7z(3x + 2y) + 4z \cdot 3z \\ &= (3x + 2y + 4z)(3x + 2y + 3z) \end{aligned}$$

Factor the following:

- $a^2 + 2ab + b^2 + 8ac + 8bc + 15c^2$
- $x^2 - 6xy + 9y^2 + 6xz - 18yz + 5z^2$
- $m^2 + n^2 - 2mn + 7mp - 7np - 30p^2$
- $16n^2 + 55 - 64n - 16m + m^2 + 8mn$
- $9m^4 + k^2 - 30 + 39m^2 + 13k + 6m^2k$
- $25c^2 + y^2 + 10x^2 + 10ay - 35ax - 7xy$

REVIEW OF FACTORING

SUMMARY OF ALL THE CASES

- Monomial factors of expressions whose terms have a common factor; as,

$$nx + ny + nz$$

- Binomials that are the difference of two squares; as,

$$a^2 - b^2$$

- Trinomials that are squares; as,

$$a^2 \pm 2ab + b^2$$

- Trinomials of the form

$$ax^2 + bx + c$$

NOTE. $x^2 + bx + c$ is a special case of this type.

- Polynomials whose terms may be grouped to show a common polynomial factor; as,

$$ax + bx + ay + by$$

VI. Binomials that are the sum or the difference of the same odd powers of two numbers ; as,

$$a^n \pm b^n \text{ (when } n \text{ is odd)}$$

NOTE. $a^3 \pm b^3$ is a special case of this type.

VII. Binomials that are the difference of the same even powers of two numbers ; as,

$$a^n - b^n \text{ (when } n \text{ is even)}$$

NOTE. Such binomials are factored like type II.

VIII. Polynomials that have binomial factors.

IX. Polynomials that can be made into the difference of two squares ; as,

$$a^4 + a^2b^2 + b^4$$

NOTE. Polynomials may sometimes be grouped so as to come under one of the forms II to V.

General directions for factoring polynomials

- (1) *Remove the monomial factors if there are any.*
- (2) *Try to bring the polynomial under some one of the cases II to IX.*
- (3) *When other methods fail, try the factor theorem.*
- (4) *Resolve into prime factors.*
- (5) *Group all the prime factors together as a product.*

NOTE. Each factor should be taken out of the given expression as soon as found in order to simplify the discovery of the remaining factors.

EXERCISES

Factor :

- | | | |
|----------------------|------------------|------------------------|
| 1. $x^2 - 3x$ | 8. $x^6 - 1$ | 15. $a^5 + 27a^2$ |
| 2. $x^2 + x$ | 9. $a - a^7$ | 16. $125 - 8x^6$ |
| 3. $4a^2 + 4a$ | 10. $x^{10} - 1$ | 17. $4m^3 + .004$ |
| 4. $8x^3 - 2x^2$ | 11. $b^7 + b$ | 18. $25x^4 - 64x^2y^2$ |
| 5. $x^3 + 8y^6$ | 12. $p^4 + 4$ | 19. $8a^3 - 64$ |
| 6. $y^{3n} - x^{3b}$ | 13. $1 + x^{12}$ | 20. $9d^3 - 81d$ |
| 7. $1 - x^8$ | 14. $a^9 - b^3$ | 21. $a^3 - a^2y$ |

22. $4x^4 - 4x$ 26. $b^{2a} - 196$ 30. $a^{13} - ab^{12}$
 23. $125 - 45x^4$ 27. $9a^4 - 169$ 31. $7n^7 + 7n$
 24. $9a^4 - (b - c)^2$ 28. $64 - 2y^5$ 32. $7y^4 - 175$
 25. $5x^2 + 5x - 30$ 29. $6b^4 + 24$ 33. $8 - 27a^3x^3$
 34. $5ap - p^2 - 6a^2$ 60. $(a^2 + a)^3 - 8$
 35. $3x^{2n} + 19x^n - 14$ 61. $n^4 + n^2a^2b^4 \neq a^4b^8$
 36. $6x^2 + 2xy - 48y^2$ 62. $5x^{10} + 10x^5 - 15$
 37. $a^2x^2 - 3abx - 4b^2$ 63. $16x^4 + 4x^2y^2 + y^4$
 38. $4x^2y^2 + 6xy - 40$ 64. $x^9 + 4x$
 39. $16q^2 + 8pq - 3p^2$ 65. $10a^4 - 33a^3 - 7a^2$
 40. $9x^2 + 21dx + 10d^2$ 66. $3x^6 + 96x$
 41. $18b^4 + 3ab^2 - 10a^2$ 67. $(a + b)^6 - 1$
 42. $10a^2 - 25ab^2 - 60b^4$ 68. $(a + x)^4 - x^4$
 43. $5a^2b^2 + 9abcd + 4c^2d^2$ 69. $1 + (x + 1)^3$
 44. $3x^2 + 15x + 18$ 70. $a^3 - 2a^2 + 1$
 45. $25x^2 - 4y^2 + 30x + 9$ 71. $b^3 - 4b^2 + 8$
 46. $x^4 - 4x + 3$ 72. $a^3 - (a + b)^3$
 47. $14x^2 + 19x - 3$ 73. $x^7 - 2x^6 + x$
 48. $4x^3 - 3x - 1$ 74. $a^4b^2 + a^2b - 12$
 49. $3x^{2n+4} - 27$ 75. $1000x^3 - 27y^3$
 50. $16a^2 - 9b^2 - 4c^2 + 12bc$ 76. $x^5 - x^2 - x^4 + x^3$
 51. $x^2 - x^3 + 42x$ 77. $17x^2 + 25x - 18$
 52. $1 + 19x - 66x^2$ 78. $x^3 - xy - x^2y + y^2$
 53. $x^3 - 3x^2 - x + 3$ 79. $(x + y)^3 + (x - y)^3$
 54. $x^5 - 9x^3 + 8x^2 - 72$ 80. $(a - 2)^3 + (a - 1)^3$
 55. $ab - bx + ac - cx$ 81. $4x^3 + x^2 - 8x - 2$
 56. $9x^4 + 11x^2y^2 + 4y^4$ 82. $x^2 + 5x + ax + 5a$
 57. $a^4 + a^2 + 1$ 83. $x^4 - 119x^2y^2 + y^4$
 58. $a^4 + 14a^2 + 81$ 84. $(a + b)^4 - (b - c)^4$
 59. $128a^2 - 250a^5$ 85. $m^3 + m^2 - mn - mn^2$

Factor:

86. $(a^2 + b^2 - c^2)^2 - 4a^2b^2$ 98. $ab - bx^n + x^ny^m - ay^m$
 87. $(x^2 - y^2)^2 - (x^2 - xy)^2$ 99. $a^4 - b^4 - (a+b)(a-b)$
 88. $3ab(a+b) + a^3 + b^3$ 100. $x^3 + 15x^2 + 75x + 125$
 89. $x^{2n-2} + b^2y^2 + 2x^{n-1}by$ 101. $x^2 - 3ax + 4bx - 12ab$
 90. $x^3 - 13x^2 + 55x - 75$ 102. $x^4 - 6x^2y^2 + y^4$
 91. $x^3 - 7x + 6$ 103. $49a^2 + 4 - 36b^2 - 28a$
 92. $4x^3 - 13x + 6$ 104. $r^8 - r^5 + 32r^3 - 32$
 93. $6x^3 + 7x^2 - x - 2$ 105. $7x^2 - 20xy - 3y^2$
 94. $6x^3 + 7x^2 - 5x$ 106. $3y^2 - 3y + \frac{3}{4}$
 95. $15x^4 + 31x^3 + 10x^2$ 107. $9a^2 - b^2 - c^2 + 2bc$
 96. $8x^3 - 12x^2 + 6x - 1$ 108. $2x^4 - 5x^3 + 5x - 2$
 97. $2x^3 - 3x^2 - 3x + 2$ 109. $10x^4 - 27x^3 - 10x^2 + 3x$
 110. $ax^3 - 9ax^2 + 26ax - 24a$
 111. $12ax - 8bx - 9ay + 6by$
 112. $25x^2 - 9y^2 - 24yz - 16z^2$
 113. $abx^3 + 3abx^2 - abx - 3ab$
 114. $3bmx + 2bm - 3anx - 2an$
 115. $20ax^3 - 28ax^2 + 5a^2x - 7a^2$
 116. $x^2 - z^2 + y^2 - a^2 - 2xy + 2az$
 117. $2b^2m - 3ab^2 + 2bmx - 3abx$
 118. Factor $32 - x^5$ by the factor theorem.
 119. Factor $16 + 5x - 11x^2$ by the factor theorem.
 120. If n is odd, factor $x^n - a^n$ by the factor theorem.
 121. If n is odd, factor $x^n + r^n$ by the factor theorem.
 122. Factor $x^3 - 6bx^2 + 12b^2x - 8b^3$ by the factor theorem.

123. Discover by the factor theorem for what values of n , between 1 and 20, $x^n + a^n$ has no binomial factors.

CHAPTER XIX

THE BINOMIAL THEOREM

The binomial theorem is one by means of which any indicated power of a binomial may be obtained.

By actual multiplication, we have :

EXPANSION OF $(a + b)^n$	BINOMIAL COEFFICIENTS
$(a+b)^0 = 1$	1
$(a+b)^1 = a + b$	1 1
$(a+b)^2 = a^2 + 2ab + b^2$	1 2 1
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$	1 3 3 1
$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$	1 4 6 4 1
$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$	1 5 10 10 5 1

The triangular array of coefficients is known as **Pascal's triangle**. Each number is the sum of the one directly above and the one to the left of the number above.

From the expansions given above, it will be observed that for any *positive integral* value of n in the expansion of $(a + b)^n$:

1. The first term is a^n , the last term is b^n , and the number of terms is $n + 1$.
2. The exponent of a decreases 1 in each succeeding term; b appears in the second term and its exponent increases 1 in each succeeding term.
3. The coefficient of any term may be found by multiplying the coefficient of the preceding term by the exponent of a in that term, and dividing this product by the number of the term.
4. All terms are positive if both a and b are positive; they are alternately positive and negative if b is negative.
5. The sum of the coefficients of the expansion of $(a + b)^n$ equals 2^n ; the sum of the coefficients of $(a - b)^n$ equals 0.

Note that the coefficients of the latter half of the expansion are the same as those of the first half, written in reverse order.

EXERCISES

1. Write by inspection the expansion of $(a - b)^4$.

SOLUTION

$$\begin{aligned}(a - b)^4 &= a^4 + 4(a)^3(-b) + 6(a)^2(-b)^2 + 4(a)(-b)^3 + (-b)^4 \\ &= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4\end{aligned}$$

Check the results by giving numerical values to the letters.

Expand, and check the results :

- | | | |
|----------------|----------------|--------------------|
| 2. $(a - c)^3$ | 5. $(x - y)^6$ | 8. $(r - s)^8$ |
| 3. $(a + b)^3$ | 6. $(x + y)^8$ | 9. $(c + d)^7$ |
| 4. $(c + d)^5$ | 7. $(m - n)^7$ | 10. $(3a - b^2)^5$ |

SOLUTION TO EXAMPLE 10

$$\begin{aligned}(3a - b^2)^5 &= (3a)^5 - 5(3a)^4(b^2) + 10(3a)^3(b^2)^2 - 10(3a)^2(b^2)^3 + 5(3a)(b^2)^4 - (b^2)^5 \\ &= 243a^5 - 405a^4b^2 + 270a^3b^4 - 90a^2b^6 + 15ab^8 - b^{10}\end{aligned}$$

Expand :

- | | | |
|-------------------------------------|--------------------------------------|---------------------------------------|
| 11. $(x + 2)^3$ | 17. $(x^2 - 10)^4$ | 23. $(2ax - b)^5$ |
| 12. $(a - 3)^4$ | 18. $(3a + b^2)^4$ | 24. $(1 - x)^7$ |
| 13. $(m - pn)^4$ | 19. $(1 - 3x^2)^4$ | 25. $(1 - 2x)^6$ |
| 14. $(x + 2y)^4$ | 20. $(x^2 + 5y)^3$ | 26. $(a^2x + 4)^5$ |
| 15. $(2x - y)^3$ | 21. $(1 + a^2b^2)^4$ | 27. $(x - \frac{1}{2})^6$ |
| 16. $(2x - 5)^3$ | 22. $(5x^2 - ab)^3$ | 28. $(\frac{1}{2}x - \frac{1}{3}y)^4$ |
| 29. $(\frac{1}{3} + x)^3$ | 33. $(3a^2 + \frac{b}{6})^3$ | 37. $(\frac{1}{a} - a)^5$ |
| 30. $(2a + \frac{1}{2})^5$ | 34. $(\frac{3}{4} - 2x)^4$ | 38. $(x + \frac{1}{x})^6$ |
| 31. $(\frac{x}{y} - \frac{y}{x})^4$ | 35. $(1 + \frac{3x}{2})^5$ | 39. $(\frac{1}{2x} - 2x)^5$ |
| 32. $(\frac{x}{y} - \frac{y}{x})^6$ | 36. $(\frac{3}{5} + \frac{5x}{3})^4$ | 40. $(2a^2 - \frac{b^3}{2})^3$ |

41. Expand $(a + b - c)^3$.

SUGGESTION. $(a + b - c)^3 = [(a + b) - c]^3$, a binomial form. Expand this and then expand each binomial factor in the terms of the resulting expansion.

42. Expand $(r - s - t + v)^3$.

SUGGESTION. $(r - s - t + v)^3 = [(r - s) - (t - v)]^3$, a binomial form.

Expand:

$$43. (a - b + c)^3 \quad 45. (b - c - d)^3 \quad 47. (a + b + c + d)^3$$

$$44. (x + y + 2)^3 \quad 46. (x + 3y - 2z)^3 \quad 48. (a + b - x - y)^3$$

Compound interest

The amount of \$1.00 at compound interest for 1 year at 4% is 1.04; for 2 years, 1.04^2 ; for 3 years, 1.04^3 ; etc. In general, the amount of \$1.00, when interest at r per cent is compounded periodically, is $(1 + r)^n$ for n periods. The compound interest is found by subtracting the principal from the amount. The amount of p dollars at r per cent compounded for n periods is given by the formula $a = p(1 + r)^n$ and the interest is given by the formula $I = p[(1 + r)^n - 1]$.

EXERCISES

Find the amount of:

1. 1.06^3

SOLUTION

$$\begin{aligned} 1.06^3 &= (1 + .06)^3 = 1^3 + 3(1)^2(.06) + 3(1)(.06)^2 + (.06)^3 \\ &= 1 + .18 + .0108 + .000216 = 1.191016 \end{aligned}$$

2. 1.04^2

5. 1.05^3

8. 1.05^4

3. $1.03\frac{1}{2}^2$

6. 1.02^6

9. 1.07^4

4. $1.04\frac{1}{2}^2$

7. 1.02^8

10. 1.06^4

11. Find the amount of \$200 at 5% for 5 years if interest is compounded annually.

12. A man deposited \$500 in a savings bank that paid 4%, compounded semiannually. How much would his account amount to at the end of $2\frac{1}{2}$ years?

SUGGESTION. $500(1.02)^5 = ?$

13. If the interest in exercise 12 had been compounded quarterly, how much interest would the man have received?

SUGGESTION. Since the result is to be accurate to two decimals only, find the first five terms of the expansion of 1.01^{10} .

14. How much will the interest be on \$850 left in a savings bank for 3 years if interest at 4% is compounded semi-annually?

Finding the r th term of the expansion of $(a + b)^n$

The product of the successive integers from 1 to n , or from n to 1, inclusive, is called **factorial n** , written $n!$ or $\underline{n}$.

$$5! = 1 \times 2 \times 3 \times 4 \times 5, \text{ or } 5 \times 4 \times 3 \times 2 \times 1$$

$$n! = 1 \cdot 2 \cdot 3 \cdots (n-2)(n-1)n, \text{ or } n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$$

From the observations on page 385, it is evident that the following powers of $(a + b)$ may be written:

$$(a + b)^2 = a^2 + 2ab + \frac{2 \cdot 1}{1 \cdot 2} b^2$$

$$(a + b)^3 = a^3 + 3a^2b + \frac{3 \cdot 2}{1 \cdot 2} ab^2 + \frac{3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3} b^3$$

$$(a + b)^4 = a^4 + 4a^3b + \frac{4 \cdot 3}{1 \cdot 2} a^2b^2 + \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} ab^3 + \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4} b^4$$

If the law of development revealed in the above is applied to the expansion of the n th power of $(a + b)$, the result is

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!} a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!} a^{n-3}b^3 \dots$$

This is known as the **binomial formula**.

From the binomial formula it is evident that in *any term*:

1. The exponent of b is 1 less than the number of the term.
2. The exponent of a is n minus the exponent of b .
3. The number of factors in the numerator and in the denominator of the coefficient is 1 less than the number of the term.

Hence the **formula for the r th term** of the expansion is:

$$\frac{n(n-1)(n-2) \cdots (n-r+2)}{(r-1)!} a^{n-r+1} b^{r-1}$$

Any term of the expansion of a power of a binomial may be obtained by substitution in this formula for the r th term.

In the expansion of a power of the difference of two numbers $(a - b)^n$, since the exponent of b in the r th term is $r - 1$, the sign of the general term is $-$ if r is *even*, and $+$ if r is *odd*.

EXERCISES

1. Find the 12th term of $(a - b)^{14}$.

SOLUTION

$$\begin{aligned} \text{12th term} &= \frac{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11} a^3 (-b)^{11} \\ &= - \frac{14 \cdot 13 \cdot 12}{1 \cdot 2 \cdot 3} a^3 b^{11} = - 364 a^3 b^{11} \end{aligned}$$

Or, since there are 15 terms, the coefficient of the 12th term, or the 4th term from the end, is equal to that of the 4th term from the beginning.

$$\therefore \text{12th term} = - \frac{14 \cdot 13 \cdot 12}{1 \cdot 2 \cdot 3} a^3 b^{11} = - 364 a^3 b^{11}$$

Without actually expanding, find the following :

- (2.) 4th term of $(a + 2)^{10}$
- (5.) 20th term of $(1 + x)^{24}$
3. 8th term of $(x - y)^{11}$
6. 18th term of $(1 - 2x)^{20}$
- (4.) 5th term of $(x - 2y)^{12}$
7. 13th term of $(a^2 - a^{-2})^{15}$
8. Find the middle term of $(a + 3b)^6$.
- (9.) Find the 6th term of $\left(x + \frac{1}{x}\right)^{10}$.
10. Find the middle term of $\left(\frac{x}{y} - \frac{y}{x}\right)^8$.
11. Find the two middle terms of $\left(\frac{a}{b} - \frac{b}{a}\right)^9$.
12. Find the 6th term of $(2m^2 - x^{\frac{1}{2}})^8$.
13. Find the 5th term of $(x^{-\frac{1}{2}} + \sqrt{x})^6$.
14. Find the $r + 1$ st term of $(2a + b)^n$.
15. Find the 5th term of $\left(3a - \frac{b}{5}\right)^{n+2}$.

CHAPTER XX

LOGARITHMS

Logarithms were invented more than 300 years ago in order to shorten arithmetic work. With their aid we can make many arithmetic calculations that would otherwise be very complicated.

A **logarithm** of a number is the exponent of the power to which a fixed number, called the **base**, must be raised to produce the given number. Logarithms, then, are exponents and follow the laws of exponents.

It follows from the definition of *logarithm* that, since $8 = 2^3$, the logarithm of 8 to the base 2 is 3. This is written $\log_2 8 = 3$. Similarly, since $16^{\frac{3}{4}} = 8$, $\log_{16} 8 = .75$. If $a^x = m$, then x is the logarithm of m to the base a , or $\log_a m = x$.

Logarithms may be computed with any arithmetic number except 1 as a base. The base of the *common*, or *Briggs*, system of logarithms is 10. When no base is expressed, the base 10 is understood. The logarithm of 100 to the base 10 is written $\log 100 = 2$.

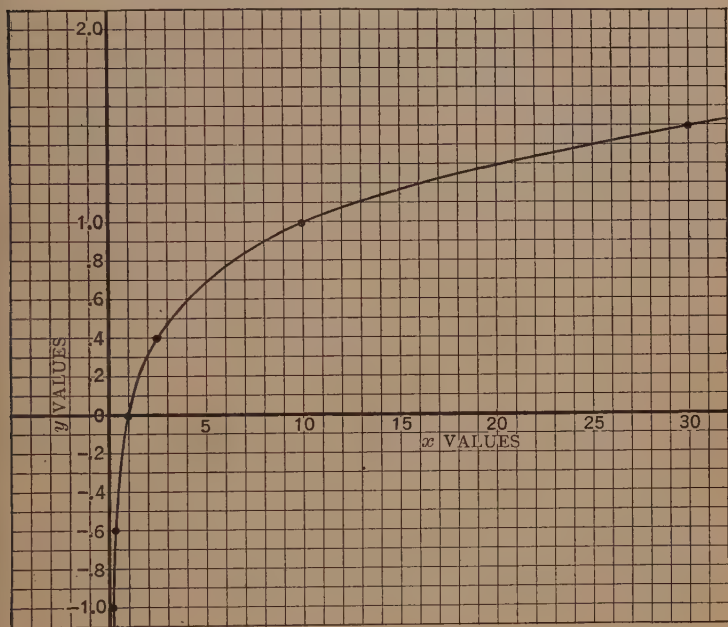
The logarithms of numbers which are powers of 10 may be easily seen from the following table :

Since $10^{-3} = \frac{1}{1000} = .001$,	$\log .001 = -3$.
Since $10^{-2} = \frac{1}{100} = .01$,	$\log .01 = -2$.
Since $10^{-1} = \frac{1}{10} = .1$,	$\log .1 = -1$.
Since $10^0 = 1$,	$\log 1 = 0$.
Since $10^1 = 10$,	$\log 10 = 1$.
Since $10^2 = 100$,	$\log 100 = 2$.
Since $10^3 = 1000$,	$\log 1000 = 3$.

The logarithm of any number between 1 and 10 lies between 0 and 1; the logarithm of any number between 10 and 100 lies between 1 and 2. For example, the logarithm of

4 is approximately 0.6021; the logarithm of 50 is approximately 1.6990. Most logarithms are endless decimals.

The relation between the logarithm of a number and the number can be readily seen by inspecting the graph of $x = 10^y$ for the values of x between .1 and 30.



GRAPH OF $x = 10^y$

EXERCISES

1. For what values of x do the corresponding logarithms change most rapidly?
2. How does the rate of change of y compare with that of x for values of x between 1 and 10?
3. For what values of x are the values of y negative?
4. What is the approximate value of y when $x = 8$?
15? 28?

5. What is the number whose logarithm is .2? .4? 1.2? 1.4?

6. Show that in the graph $\log 10$ is approximately equal to $\log 5 + \log 2$; that $\log 5$ is approximately equal to $\log 25 - \log 5$; that $\log 27$ is approximately equal to $3 \log 3$.

Logarithms consist of an integral part and a decimal part. The integral part is called the **characteristic**; the decimal part, the **mantissa**. If $70 = 10^{1.8451}$, $\log 70 = 1.8451$. The characteristic is 1 and the mantissa is .8451. The following illustrate characteristics and mantissas:

It is known that 4.58 is approximately equal to $10^{0.6609}$. From this it follows that:

$4.58 = 10^{0.6609}$	$\log 4.58 = 0.6609$
$45.8 = 10 \times 4.58 = 10^{1+0.6609}$	$\log 45.8 = 1.6609$
$458 = 100 \times 4.58 = 10^{2+0.6609}$	$\log 458 = 2.6609$
$4580 = 1000 \times 4.58 = 10^{3+0.6609}$	$\log 4580 = 3.6609$

Again,

$.458 = 10^{-1} \times 4.58 = 10^{-1+0.6609}$	$\log .458 = -1 + .6609$
$.0458 = 10^{-2} \times 4.58 = 10^{-2+0.6609}$	$\log .0458 = -2 + .6609$
$.00458 = 10^{-3} \times 4.58 = 10^{-3+0.6609}$	$\log .00458 = -3 + .6609$

From the preceding examples it is evident that:

PRINCIPLES. 1. *The characteristic of the logarithm of a number greater than 1 is either zero or a positive integer 1 less than the number of digits in the integral part of the number.*

2. *The characteristic of the logarithm of a number less than 1, that is, of a decimal, is negative and numerically 1 greater than the number of ciphers immediately following the decimal point.*

3. *The mantissa of the logarithm is always a positive decimal.*

From the table above, $\log .458 = -1 + .6609$. This is sometimes written $\log .458 = \bar{1}.6609$. To avoid writing negative characteristics, however, and to simplify the work, it is usual to add 10 or some multiple of 10 to the negative characteristic, and to indicate that the number added is to be subtracted from the whole logarithm.

Thus, $\bar{1}.6609$ is written $9.6609 - 10$; $\bar{2}.3010$ is written $8.3010 - 10$ or sometimes $18.3010 - 20$ or $28.3010 - 30$, etc.

It is evident, also, from the examples, that in the logarithms of numbers expressed by the same figures in the same order, the decimal parts, or mantissas, are the same, and the logarithms differ only in their characteristics. Hence tables of logarithms contain only the *mantissas*.

The table of logarithms on pages 396 and 397 gives the mantissas, to the nearest fourth place, of the common logarithms of all numbers from 1 to 1000. Results obtained by the use of a four-place table are not accurate beyond four significant figures. When greater accuracy is required, tables must be used which give the mantissas correct to a greater number of decimal places.

To find the logarithm of a number

EXERCISES

1. Find the logarithm of 765.

SOLUTION

In the table the letter **N** designates a vertical column of numbers from **10** to **99** inclusive, and also a horizontal row of figures **0, 1, 2, 3, 4, 5, 6, 7, 8, 9**. The first two figures of 765 appear as the number **76** in the vertical column marked **N** on page 397, and the third figure **5** appears in the horizontal row marked **N**. In the same horizontal row as **76** are found the mantissas of the logarithms of the numbers 760, 761, 762, 763, 764, 765, etc. The mantissa of the logarithm of 765 is found in this row under **5**, the third figure of 765. It is 8837 and means .8837.

By Prin. 1, the characteristic of the logarithm of 765 is 2.

Hence the logarithm of 765 is 2.8837.

2. Find the logarithm of 4.

SOLUTION

Since $4 = 4.00$, and since (page 392) the mantissa of the logarithm of 4.00 is the same as that of 400, which is .6021, the mantissa of the logarithm of 4 is .6021.

By Prin. 1, the characteristic of the logarithm of 4 is 0.

Therefore the logarithm of 4 is 0.6021.

Verify the following from the table :

3. $\log 10 = 1.0000$	9. $\log .02 = 8.3010 - 10$
4. $\log 100 = 2.0000$	10. $\log 542 = 2.7340$
5. $\log 110 = 2.0414$	11. $\log 345 = 2.5378$
6. $\log 2 = 0.3010$	12. $\log 5.07 = 0.7050$
7. $\log 20 = 1.3010$	13. $\log 78.5 = 1.8949$
8. $\log 200 = 2.3010$	14. $\log .981 = 9.9917 - 10$

Interpolation

The table contains the mantissas of logarithms of numbers that can be expressed by three figures, and of such numbers multiplied by some power of 10, as 625, 6250, 62,500.

The mantissas of logarithms of such numbers as 6253 are found by the principle known as **interpolation**. This principle assumes that the logarithm of 6253 bears the same relation to the logarithms of 6250 and 6260 that the number 6253 bears to the numbers 6250 and 6260. As a matter of fact, the logarithms are not proportional to the numbers ; but for practical purposes interpolation is sufficiently accurate for finding mantissas not given in the table.

ILLUSTRATIVE EXAMPLE

Find the logarithm of 6253.

SOLUTION

To find the logarithm of 6253, we compute the characteristic, 3 (Prin. 1), and use the table to obtain the mantissa. Since the table does not give the mantissa of the logarithm of 6253, we look up the mantissas of the logarithms of the numbers immediately before and after it, that is, of 6250 and 6260, and find them to be .7959 and .7966, respectively. Observe that as the number increases from 6250 to 6260 the mantissa increases from .7959 to .7966, or an increase of 10 in the number produces an increase of .0007 in the mantissa. It may be assumed that an increase of 3 in the number will produce an increase of $\frac{3}{10}$ of .0007, or approximately .0002, in the mantissa. Therefore the mantissa of the logarithm of 6253 is $.7959 + .0002$, or .7961. The logarithm of 6253 has thus been found to be 3.7961.

The work may be arranged as follows :

$$\begin{array}{r} \log 6260 = 3.7966 \\ \log 6253] = \quad ? \\ \log 6250 = 3.7959 \\ \quad .3 \text{ of } \quad .0007 = .0002 \\ \log 6253 = 3.7959 + .0002 = 3.7961 \end{array}$$

NOTE. The difference between two successive mantissas in the table is called the *tabular difference*.

EXERCISES

Find the logarithm of :

1. 1054	6. 3.060	11. 54.67	16. 63.24
2. 1272	7. 441.1	12. .0988	17. 721.6
3. .0165	8. .7854	13. .9846	18. .4376
4. 1906	9. .0909	14. 21.32	19. .02164
5. 21.09	10. .1012	15. 8.432	20. 4.362

To find a number whose logarithm is given

The number that corresponds to a given logarithm is called its **antilogarithm**.

Thus, since the logarithm of 62 is 1.7924, the antilogarithm of 1.7924 is 62.

EXERCISES

1. Find the number whose logarithm is 0.9472.

SOLUTION

The two mantissas adjacent to the given mantissa are .9469 and .9474, corresponding to the numbers 8.85 and 8.86 (Prin. 1). The given mantissa is 3 ten-thousandths greater than the mantissa of the logarithm of 8.85, and the mantissa of the logarithm of 8.86 is 5 ten-thousandths greater than that of the logarithm of 8.85.

Since the numbers 8.85 and 8.86 differ by 1 one-hundredth, and the mantissas of their logarithms differ by 5 ten-thousandths, it may be assumed as sufficiently accurate that each increase of 1 ten-thousandth in the mantissa is produced by an increase of $\frac{1}{5}$ of 1 one-hundredth in the number. Hence an increase of 3 ten-thousandths in the mantissa is produced by an increase of $\frac{3}{5}$ of 1 one-hundredth, or .006, in the number.

Hence the number whose logarithm is 0.9472 is 8.856.

TABLE OF COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N	0	1	2	3	4	5	6	7	8	9

TABLE OF COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

2. Find the antilogarithm of $9.4180 - 10$.

SOLUTION

Mantissa next larger: .4183 (Mantissa of 2620)

Given mantissa: .4180

Mantissa next less: .4166 (Mantissa of 2610)

$$\frac{14}{17} \times 10 = 8$$

Adding 8 to 2610, we have 2618.

Since the characteristic is $9 - 10$, or -1 , the number is a decimal with no ciphers immediately following the decimal point (Prin. 2).

Hence the antilogarithm of $9.4180 - 10$ is .2618.

Find the antilogarithm of:

3. 0.3010	11. 0.8294	19. $\bar{1}.2122$
4. 1.6021	12. 1.9039	20. $\bar{2}.6431$
5. 2.9031	13. $9.3685 - 10$	21. $\bar{1}.5858$
6. 1.6669	14. $8.9932 - 10$	22. $\bar{3}.6644$
7. 2.7971	15. $8.9535 - 10$	23. $\bar{0}.6894$
8. 3.9545	16. $7.7168 - 10$	24. $\bar{1}.7639$
9. 0.8794	17. $6.7016 - 10$	25. $\bar{2}.3939$
10. 2.9371	18. $\bar{2}.8464$	26. $\bar{2}.5634$

27. If $\log_{16} n = .75$, what is the value of n ?

28. If $\log_3 B = 2.3333\frac{1}{3}$, find B .

29. If $A = 10^x$ and $B = 10^y$, find to the base 10 $\log A$; $\log B$; AB ; $\log AB$; $\frac{A}{B}$; $\log \frac{A}{B}$.

Multiplication by logarithms

Since logarithms are the exponents of the powers to which a constant number is to be raised, it follows that:

PRINCIPLE. *The logarithm of the product of two or more numbers is equal to the sum of their logarithms.*

Let a be any base.

Then $\log_a (mn) = \log_a m + \log_a n$

For, let $a^x = m$ and $a^y = n$

Then $\log_a m = x$ and $\log_a n = y$

$$mn = a^{x+y}$$

Hence $\log_a (mn) = x + y$
 $= \log_a m + \log_a n$

EXERCISES

1. Multiply .0381 by 77.

SOLUTION

$$\begin{array}{rcl}
 \text{Let} & n & = .0381 \times 77 \\
 \text{Then} & \log n & = \log .0381 + \log 77 \\
 & \log .0381 & = 8.5809 - 10 \\
 & \log 77 & = 1.8865 \\
 & \log n & = 10.4674 - 10 \\
 & & = 0.4674 \\
 & \therefore n & = 2.934
 \end{array}$$

NOTE. Three figures of a number corresponding to a logarithm may be found accurately from this table, and in most cases the fourth will be correct. In finding logarithms or antilogarithms, allowance should be made for the figures after the fourth. Whenever they express .5 or more than .5 of a unit, the figure in the fourth place should be made 1 more.

Using logarithms, multiply :

- | | | |
|---------------|-----------------|--------------------|
| 2. 3.8 by 56 | 6. 2.26 by 85 | 10. 289 by .7854 |
| 3. 72 by 39 | 7. 7.25 by 240 | 11. 42.37 by .236 |
| 4. 8.5 by 6.2 | 8. 3272 by 75 | 12. 2912 by .7281 |
| 5. 1.64 by 35 | 9. .892 by .805 | 13. 1.414 by 2.829 |

Division by logarithms

Since the logarithms of two numbers to a common base represent exponents of the same number, it follows that :

PRINCIPLE. *The logarithm of the quotient of two numbers is equal to the logarithm of the dividend minus the logarithm of the divisor.*

Let a be any base.

$$\begin{array}{rcl}
 \text{Then} & \log_a (m \div n) & = \log_a m - \log_a n \\
 \text{For, let} & a^x = m \text{ and } a^y = n & \\
 \text{Then} & \log_a m = x \text{ and } \log_a n = y & \\
 & m \div n = a^{x-y} & \\
 \text{Hence} & \log_a (m \div n) = x - y & \\
 & = \log_a m - \log_a n &
 \end{array}$$

EXERCISES

1. Divide .00468 by 73.4.

SOLUTION

Let

$$n = .00468 \div 73.4$$

Then

$$\log n = \log .00468 - \log 73.4$$

$$\log .00468 = 7.6702 - 10$$

$$\log 73.4 = 1.8657$$

$$\log n = 5.8045 - 10$$

$$\therefore n = .00006376$$

2. Divide 12.4 by 16.

SOLUTION

Let

$$n = 12.4 \div 16$$

Then

$$\log n = \log 12.4 - \log 16$$

$$\log 12.4 = 1.0934 = 11.0934 - 10$$

$$\log 16 = 1.2041$$

$$\log n = 9.8893 - 10$$

$$9.8893 - 10 = \log .775$$

$$\therefore n = .775$$

NOTE. The positive part of the logarithm of the dividend may be made to exceed that of the divisor, if necessary to avoid subtracting a larger number from a smaller one, as in the above solution, by adding $10 - 10$ or $20 - 20$, etc.

Using logarithms, divide :

3. 3025 by 55

8. 10 by 3.14

13. 1 by 40

4. 4090 by 32

9. .6911 by .7854

14. 1 by 75

5. 3250 by 57

10. 2.816 by 22.5

15. 200 by .5236

6. .2601 by .68

11. 4 by .00521

16. 300 by 17.32

7. 3950 by .250

12. 26 by .06771

17. .220 by .3183

Extended operations in multiplication and division

Though *negative* numbers have no common logarithms, operations involving negative numbers may be performed by considering only their *absolute values* and then giving to the result the proper sign without regard to the logarithmic work.

Since dividing by a number is equivalent to multiplying by its reciprocal, for every operation of division an operation of multiplication may be substituted. In extended operations in multiplication and division with the aid of logarithms, the latter method of dividing is the more convenient.

The logarithm of the reciprocal of a number is called the **cologarithm** of the number.

The cologarithm of 100 is the logarithm of $\frac{1}{100}$, which is -2 . It is written $\text{colog } 100 = -2$.

Since the logarithm of 1 is 0 and the logarithm of a quotient is obtained by subtracting the logarithm of the divisor from that of the dividend, it is evident that the cologarithm of a number is 0 minus the logarithm of the number, or the logarithm of the number with the sign of the logarithm changed; that is, if $\log_a m = x$, then $\text{colog}_a m = -x$.

Since subtracting a number is equivalent to adding it with its sign changed, it follows that:

PRINCIPLE. *Instead of subtracting the logarithm of the divisor from that of the dividend, the cologarithm of the divisor may be added to the logarithm of the dividend.* That is,

To any base, $\log(m \div n) = \log m + \text{colog } n$

EXERCISES

- Find by logarithms the value of $\frac{18.5}{9.3}$.

SOLUTION

$$\text{Let } n = \frac{18.5}{9.3} = 18.5 \times \frac{1}{9.3}$$

Then

$$\log n = \log 18.5 + \text{colog } 9.3$$

$$\log 18.5 = 1.2672$$

$$\text{colog } 9.3 = \frac{9.0315 - 10}{10}$$

$$\log n = \frac{10.2987 - 10}{10}$$

$$= .2987$$

$$\therefore n = 1.989$$

The *cologarithm* of a number is found by subtracting the logarithm of the number from 0 or from 10 - 10, etc.

$$\text{Thus:} \quad 10.0000 - 10$$

$$\log 9.3 = \frac{0.9685}{10}$$

$$\text{colog } 9.3 = \frac{9.0315 - 10}{10}$$

2. Find by logarithms the value of $\frac{.063 \times 58.5 \times 799}{458 \times 15.6 \times .029}$.

SOLUTION

Let
$$n = \frac{.063 \times 58.5 \times 799}{458 \times 15.6 \times .029}$$

Then

$$\log n = \log .063 + \log 58.5 + \log 799 + \text{colog } 458 + \text{colog } 15.6 + \text{colog } .029$$

$\log .063 = 8.7993 - 10$	$10.0000 - 10$
$\log 58.5 = 1.7672$	$\log 458 = 2.6609$
$\log 799 = 2.9025$	$\text{colog } 458 = 7.3391 - 10$
$\text{colog } 458 = 7.3391 - 10$	$10.0000 - 10$
$\text{colog } 15.6 = 8.8069 - 10$	$\log 15.6 = 1.1931$
$\text{colog } .029 = 1.5376$	$\text{colog } 15.6 = 8.8069 - 10$
$\log n = 31.1526 - 30$	$10.0000 - 10$
$= 1.1526$	$\log .029 = 8.4624 - 10$
$\therefore n = 14.21$	$\text{colog } .029 = 1.5376$

Find by logarithms the value of :

- | | |
|--|--|
| <p>3. $\frac{110 \times 3.1 \times .653}{33 \times 7.854 \times 1.7}$</p> | <p>9. $\frac{3.516 \times 485 \times 65}{3.33 \times 17 \times 18 \times 73}$</p> |
| <p>4. $\frac{15 \times .37 \times 26.16}{11 \times 8 \times .18 \times 6.67}$</p> | <p>10. $\frac{.4051 \times (-12.45)}{(-8.988) \times .01442}$</p> |
| <p>5. $\frac{(-3.04) \times .2608}{2.046 \times .06219}$</p> | <p>11. $\frac{78 \times 52 \times 1605}{338 \times 767 \times 431}$</p> |
| <p>6. $\frac{600 \times 5 \times 29}{.7854 \times 25000 \times 81.7}$</p> | <p>12. $\frac{.5 \times .315 \times 428}{.317 \times .973 \times 43.7}$</p> |
| <p>7. $\frac{500 \times 30 \times 50}{7000 \times .16 \times 3.4}$</p> | <p>13. $\frac{3.14 \times 5.28 \times 10.5}{.407 \times .621 \times 4.5}$</p> |
| <p>8. $\frac{(-5.02) \times .347}{30.55 \times .4318}$</p> | <p>14. $\frac{.8562 \times (-4.5)}{(-3.25) \times .008}$</p> |

Finding powers by logarithms

Since logarithms are simply exponents, it follows that :

PRINCIPLE. *The logarithm of a power of a number is equal to the logarithm of the number multiplied by the index of the power.*

Let a be any base.

Then $\log_a m^n = n \log_a m$

For, let $a^x = m$ and let n be any number.

Then $\log_a m = x$

$$a^{nx} = m^n$$

Hence $\log_a m^n = nx = n \log_a m$

EXERCISES

1. Find the value of $.25^2$.

SOLUTION

Let $n = .25^2$

Then $\log n = 2 \log .25$

$$\log .25 = 9.3979 - 10$$

$$2 \log .25 = 18.7958 - 20$$

$$= 8.7958 - 10$$

$$\therefore n = .06249$$

NOTE. By actual multiplication it is found that $.25^2 = .0625$, whereas the result obtained by the use of the table is $.06249$.

Also, by multiplication it is found that $18^2 = 324$, whereas the result obtained by the use of the table is 324.1 .

Such inaccuracies as these must be expected when a four-place table is used.

Find by logarithms the value of :

- | | | | |
|--------------|---------------|---------------|----------------------------|
| 2. 7^2 | 7. $.78^5$ | 12. 4.07^3 | 17. $(\frac{3}{20})^2$ |
| 3. 11^2 | 8. 8.05^2 | 13. $.543^3$ | 18. $(\frac{1}{1.05})^5$ |
| 4. $(-47)^2$ | 9. 1.04^5 | 14. $(-7)^4$ | 19. $(\frac{128}{1738})^2$ |
| 5. 4.9^3 | 10. 1.025^7 | 15. 1.02^5 | 20. $(\frac{675}{4121})^3$ |
| 6. 5.2^4 | 11. $.714^2$ | 16. 1.738^3 | 21. $(\frac{1}{243})^4$ |

Finding roots by logarithms

Since logarithms are simply exponents, it follows that :

PRINCIPLE. *The logarithm of a root of a number is equal to the logarithm of the number divided by the index of the required root.*

Let a be any base.

Then $\log_a \sqrt[n]{m} = \frac{\log_a m}{n}$

For, let $a^x = m$ and let n be any number.

Then $\log_a m = x$

$$a^{x \div n} = \sqrt[n]{m}$$

Hence $\log_a \sqrt[n]{m} = x \div n = \frac{\log_a m}{n}$

EXERCISES

1. Find by logarithms the square root of .1296.

SOLUTION

$$\text{Let } n = \sqrt{.1296}$$

$$\log n = \frac{1}{2} \log .1296$$

$$\log .1296 = 19.1126 - 20$$

$$\frac{1}{2} \log .1296 = 9.5563 - 10$$

$$\therefore n = .36$$

Find by logarithms the value of :

- | | | | |
|---|----------------------------|--|----------------------------|
| 2. $225^{\frac{1}{2}}$ | 8. $(-1331)^{\frac{1}{3}}$ | 14. $\sqrt{2}$ | 20. $\sqrt[5]{-2}$ |
| 3. $12.25^{\frac{1}{2}}$ | 9. $1024^{\frac{7}{10}}$ | 15. $\sqrt[4]{3}$ | 21. $\sqrt[3]{.027}$ |
| 4. $.2023^{\frac{1}{2}}$ | 10. $.6724^{\frac{1}{2}}$ | 16. $\sqrt[3]{5}$ | 22. $\sqrt{30\frac{4}{5}}$ |
| 5. $326^{\frac{1}{2}}$ | 11. $5.929^{\frac{1}{3}}$ | 17. $\sqrt[5]{6}$ | 23. $\sqrt[3]{.90}$ |
| 6. $.512^{\frac{1}{3}}$ | 12. $.4624^{\frac{2}{3}}$ | 18. $\sqrt[3]{2}$ | 24. $\sqrt[4]{.52}$ |
| 7. $.1182^{\frac{1}{2}}$ | 13. $1.4641^{\frac{1}{4}}$ | 19. $\sqrt[4]{4}$ | 25. $\sqrt[5]{.032}$ |
| 26. $\frac{176}{15 \times 3.1416}$ | | 28. $\frac{14.5\sqrt[3]{-1.6}}{11}$ | |
| 27. $\frac{(-100)^2}{48 \times 64 \times 11}$ | | 29. $\sqrt{\frac{.434 \times 96^4}{64 \times 1500}}$ | |

30. Applying the formula $A = \pi r^2$, find the area (A) of a circle whose radius (r) is 12.35 meters. ($\pi = 3.1416$.)

31. Applying the formula $V = \frac{4}{3} \pi r^3$, find the volume (V) of a sphere whose radius (r) is 40.11 centimeters.

32. The formula $V = .7854 d^2 l$ gives the volume of a right cylinder d units in diameter and l units long, V , d , and l being corresponding units. How many feet of wire which has a diameter of .3648 inch can be made from a cubic foot of copper?

33. Solve for x : $10 = 4^x$.

SOLUTION

$$\log 10 = x \log 4$$

$$x = \frac{\log 10}{\log 4} = \frac{1}{.6021} = 1.6608$$

Solve for x :

34. $5.6 = (1.03)^{1+x}$

37. $4^{x+1} \cdot 5^{x-1} = 100$

35. $12^x = 220$

38. $\sqrt[3]{12} = 3$

36. $\begin{cases} 2^{x+y} = 6 \\ 2^{x+1} = 3^y \end{cases}$

39. $\begin{cases} 3^x = 2^y \\ 4^x = 20^y \end{cases}$

40. What is the amount of \$475 for 10 years at 6% compound interest?

SOLUTION

$$a = p(1 + r)^n = 475(1 + .06)^{10}$$

$$\log a = \log 475 + 10 \log 1.06$$

$$\log 475 = 2.6767$$

$$10 \log 1.06 = 0.2530$$

$$\log a = 2.9297$$

$$\therefore a = \$850.60$$

Find the amount, at compound interest, of:

41. \$225 for 5 years at 8%

43. \$400 for 10 years at 3%

42. \$700 for 5 years at 6%

44. \$1200 for 20 years at 4%

NOTE. When interest is compounded semiannually, there are twice as many periods as years at half the annual rate. When it is compounded quarterly, there are 4 times as many periods at $\frac{1}{4}$ the annual rate, etc.

45. In how many years will \$2000 amount to \$4250 at 6% if interest is compounded annually? semiannually?

46. How long will it be necessary to leave \$500 in a savings bank paying 4%, compounded semiannually, to have it amount to \$750?

47. At what annual rate will \$800 in 5 years amount to \$1025 if interest is compounded quarterly?

CHAPTER XXI

PROGRESSIONS

A succession of numbers, each of which after the first is derived from the preceding number or numbers according to some fixed law, is called a **series**.

The successive numbers are called the **terms** of the series. The first and last terms are called the **extremes**, and all the others, the **means**.

In the series 2, 4, 6, 8, 10, 12, 14, each term after the first is greater by 2 than the preceding term. This is the **law** of the series. Also since the first term $= 2 \cdot 1$, the second term $= 2 \cdot 2$, the third term $= 2 \cdot 3$, etc., the law of the series may be expressed thus:
$$nth \text{ term} = 2n$$

In the series 2, 4, 8, 16, 32, 64, 128, each term after the first is twice the preceding term; or, expressing the law of the series by an equation or formula,

$$nth \text{ term} = 2^n$$

ARITHMETIC PROGRESSION

A series each term of which after the first is derived from the preceding by the addition of a constant number is called an **arithmetic series**, or an **arithmetic progression**.

The number that is added to any term to produce the next is called the **common difference**.

2, 4, 6, 8, ... and 15, 12, 9, 6, ... are arithmetic progressions. In the first, the common difference is 2 and the series is ascending; in the second, the common difference is -3 and the series is descending.

It is evident that the terms of an arithmetic progression *increase* or *decrease* numerically, according as the common difference, d , is *positive* or *negative*.

To find the n th, or last, term of an arithmetic series

In the arithmetic series

$$1, 3, 5, 7, 9, 11, 13, 15, 17, 19$$

the common difference is 2, or $d = 2$. This difference appears *once* in the *second* term, for $3 = 1 + d$; *twice* in the *third* term, for $5 = 1 + 2d$; *three* times in the *fourth* term, for $7 = 1 + 3d$; and so on to the *tenth*, or last, term, which equals $1 + 9d$.

In $a, a + d, a + 2d, a + 3d, \dots$

which is the general form of an arithmetic progression, a representing the first term and d the common difference, observe that the coefficient of d in the expression for any term is *one less* than the number of the term.

Then if the n th, or last, term is represented by l ,

$$l = a + (n - 1)d \quad (\text{I})$$

A.P. is an abbreviation of the words *arithmetic progression*.

EXERCISES

1. What is the 10th term of the series 6, 9, 12, ...?

SOLUTION

$$\begin{aligned} l &= a + (n - 1)d \\ &= 6 + (10 - 1)3 \\ &= 33 \end{aligned}$$

EXPLANATION. Since the series 6, 9, 12, ... is an A.P., the common difference of whose terms is 3, substitute 6 for a , 10 for n , and 3 for d in the formula for the n th term. The 10th term is thus found to be 33.

2. Find the 20th term of the series 7, 11, 15, ...
3. Find the 16th term of the series 2, 7, 12, ...
4. Find the 24th term of the series 1, 16, 31, ...
5. Find the 18th term of the series 1, 8, 15, ...
6. Find the 13th term of the series $-3, 1, 5, \dots$
7. Find the 49th term of the series $1, 1\frac{1}{3}, 1\frac{2}{3}, \dots$

8. Find the 15th term of the series 45, 43, 41, ...

SUGGESTION. The common difference is -2 .

9. Find the 10th term of the series 5, 1, -3 , ...

10. Find the 16th term of the series a , $3a$, $5a$, ...

11. Find the 7th term of the series $x - 3y$, $x - 2y$, $(x - y)$, ...

12. A body falls $16\frac{1}{2}$ feet the first second, 3 times as far the second second, 5 times as far the third second, etc. How far will it fall during the tenth second?

To find the sum of n terms of an arithmetic series

Let a represent the first term of an A.P.; d , the common difference; l , the last term; n , the number of terms; s , the sum of the terms. Write the sum of n terms in the usual order and in the reverse order, and add the two equal series.

$$\begin{aligned} s &= a + (a+d) + (a+2d) + \cdots + (l-d) + l \\ s &= l + (l-d) + (l-2d) + \cdots + (a+d) + a \\ \hline 2s &= (a+l) + (a+l) + (a+l) + \cdots + (a+l) + (a+l) \end{aligned}$$

$$\text{or } 2s = n(a+l)$$

$$\therefore s = \frac{n}{2}(a+l), \quad \text{or} \quad n\left(\frac{a+l}{2}\right) \quad (\text{II})$$

Handwritten note: $s = \frac{n}{2}(2a + (n-1)d)$

EXERCISES

1. Find the sum of 20 terms of the series 2, 5, 8, ...

SOLUTION

$$l = a + (n-1)d = 2 + (20-1) \times 3 = 59$$

$$s = n\left(\frac{a+l}{2}\right) = 20\left(\frac{2+59}{2}\right) = 610$$

EXPLANATION. Since the last term is not given, it is found by formula I and substituted for l in the formula for the sum.

Find the sum of :

2. The first 16 terms of the series 1, 5, 9, ...

3. The first 10 terms of the series -2 , 0 , 2 , ...

Find the sum of :

4. The first 6 terms of the series $1, 3\frac{1}{2}, 6, \dots$.
5. The first 8 terms of the series $a, 3a, 5a, \dots$.
6. The first n terms of the series $1, 7, 13, \dots$.
7. The first a terms of the series $x, x + 2a, x + 4a, \dots$.
8. The first 7 terms of the series $4, 11, 18, \dots$.
9. The first 10 terms of the series $1, -1, -3, \dots$.
10. The first 10 terms of the series $1, \frac{1}{2}, 0, \dots$.
11. How many strokes does a common clock, striking hours, make in 12 hours?
12. A man was hired to dig a well 20 feet deep. He received \$.50 for the first foot, \$.60 for the second foot, \$.70 for the third foot, and so on. How much did he receive for the job?
13. A body falls $16\frac{1}{2}$ feet the first second, 3 times as far the second second, 5 times as far the third second, etc. How far will it fall in 10 seconds?
14. Thirty flowerpots are arranged 4 feet apart in a straight line. How far must the gardener walk if, after watering each plant, he must return to a well 4 feet from the first plant and in line with the plants, assuming that he starts at the well?
15. How long is a toboggan slide if it takes 12 seconds to reach the bottom on a toboggan going 4 feet the first second and increasing in velocity 2 feet each second?
16. Starting from rest, a train went .18 of a foot the first second, .54 of a foot the next second, .90 of a foot the third second, and so on, reaching its highest speed in 3 minutes 40 seconds. How far did the train go before reaching top speed?
17. In a potato race each contestant has to start from a mark and bring back, one at a time, 8 potatoes, the first of which is 6 feet from the mark and each of the others 6 feet farther than the preceding potato. How far must each contestant go in order to finish the race?

18. A man purchases a \$1200 automobile by paying \$300 down and monthly installments of \$100 and interest on the unpaid balance. If the yearly rate is 6%, what is the total amount of interest?

SUGGESTION. The monthly balances and interest due are evidently as follows. The interest payments form an arithmetic progression.

BALANCE	INTEREST
\$900.00	\$4.50
800.00	4.00
700.00	3.50
.....	...
100.00	.50

19. A man bought a house for \$8500, paying \$2500 cash, and giving a first mortgage for \$3500 and a second mortgage for the balance, bearing 6% interest. He paid off the second mortgage in installments of \$100 every three months. Find the total amount of interest paid on the second mortgage.

20. Each member of a Christmas club deposits in a bank 5 cents the first week and each succeeding week increases the deposit by 5 cents. What is the fiftieth deposit of each member and what is the total number of dollars deposited by n members after each has made the fiftieth deposit?

21. A firm offered to sell a radio set listed at \$150 for \$30 down and \$10 a month, with 6% interest on unpaid balances, or for \$135 cash. How much would a purchaser save by taking advantage of the cash offer?

22. A piano company offered to sell a piano for \$200 cash and 20 monthly payments of \$40, with interest at the rate of $\frac{1}{2}\%$ a month on all unpaid amounts, or for a cash payment of \$900. How much would the purchaser save if he borrowed \$700 for a year and 8 months at 6% and took advantage of the cash offer?

The two fundamental formulas,

$$(I) \ l = a + (n - 1)d \quad \text{and} \quad (II) \ s = \frac{n}{2}(a + l)$$

contain *five elements*, a , d , l , n , and s . Since these formulas are independent simultaneous equations, if they contain but two unknown elements, they can be solved. Hence if any *three* of the five elements are known, the other *two* can be found.

EXERCISES

1. Given $d = 3$, $l = 58$, $s = 260$, find a and n .

SOLUTION

Substitute the given values in (I) and (II):

$$58 = a + (n - 1)3, \quad \text{or} \quad (1) \quad a + 3n = 61$$

$$260 = \frac{1}{2}n(a + 58), \quad \text{or} \quad (2) \quad an + 58n = 520$$

Solving (1) and (2) for a and n and rejecting the value $\frac{104}{3}$ for n , we have $a = 46$ and $n = 5$.

NOTE. Since the number of terms must be a positive integer, fractional or negative values of n are rejected whenever they occur.

2. Given $a = 11$, $d = -2$, $s = 27$, find the series.

SOLUTION

Substitute the given values in (I) and (II):

$$l = 11 + (n - 1)(-2), \quad \text{or} \quad (1) \quad l = 13 - 2n$$

$$27 = \frac{1}{2}n(11 + l), \quad \text{or} \quad (2) \quad 54 = 11n + ln$$

Solving (1) and (2) for l and n , we have $l = 7$ or -5 and $n = 3$ or 9 . Hence the series is 11, 9, 7 or 11, 9, 7, 5, 3, 1, -1 , -3 , -5 .

3. How many terms are there in the series 2, 6, 10, ..., 66?
4. What is the sum of the series 1, 6, 11, ..., 61?
5. How many terms are there in the series $-1, 2, 5, \dots$, if the sum is 221?
6. Find n and s in the series 2, 9, 16, ..., 86.
7. Find l and s in $-10, -8\frac{1}{2}, -7, \dots$ to 10 terms.
8. The sum of the series $\dots, 22, 27, 32, \dots$ is 714. If there are 17 terms, what are the first and last terms?

9. If $s = 113\frac{2}{3}$, $a = \frac{1}{3}$, and $d = 2$, find n .
10. What is the sum of the series $\dots, -1, 3, 7, \dots, 23$, if the number of terms is 16?
11. What are the extremes of the series $\dots, 8, 10, 12, \dots$, if $s = 300$ and $n = 20$?
12. Find an A.P. of 14 terms having 10 for its 6th term, and 0 for its 11th term.
13. Find an A.P. of 15 terms such that the sum of the 5th, 6th, and 7th terms is 60, and the sum of the last three terms is 132.

From (I) and (II) derive the formula for :

14. l in terms of a, n, s 16. d in terms of a, n, s
 15. s in terms of a, d, n 17. a in terms of d, n, s

To insert arithmetic means

EXERCISES

1. Insert 5 arithmetic means between 1 and 31.

SOLUTION

Since there are 5 means, there must be 7 terms. Hence in the formula $l = a + (n - 1)d$, $l = 31$, $a = 1$, $n = 7$, and d is unknown.

Solving, we have $d = 5$.

Hence 1, 6, 11, 16, 21, 26, 31 is the series.

2. Insert 9 arithmetic means between 1 and 6.
 3. Insert 10 arithmetic means between 24 and 2.
 4. Insert 7 arithmetic means between 10 and -14 .
 5. Insert 3 arithmetic means between $a - b$ and $a + b$.

If A is the arithmetic mean between a and b in the series

$$\begin{aligned} & a, A, b \\ A - a &= b - A \\ \therefore A &= \frac{a + b}{2} \end{aligned}$$

That is,

PRINCIPLE. *The arithmetic mean between two numbers is equal to half their sum.*

Find the arithmetic mean between :

6. $\frac{2}{3}$ and $\frac{1}{2}$

9. $\frac{m}{n}$ and $\frac{n}{m}$

7. $a + b$ and $a - b$

8. $(a + b)^2$ and $(a - b)^2$ 10. $a + b + c$ and $a - b + c$

Problems

Problems in arithmetic progression involving two unknown elements commonly suggest series of the form

$$a, a + d, a + 2d, a + 3d, \text{ etc.}$$

Frequently, however, the solution of problems is more readily accomplished by representing the series as follows :

1. When there are *three* terms, the series may be written,

$$a - d, a, a + d$$

2. When there are *five* terms, the series may be written,

$$a - 2d, a - d, a, a + d, a + 2d$$

3. When there are *four* terms, the series may be written,

$$a - 3d, a - d, a + d, a + 3d$$

The sum of the terms of a series represented as above evidently contains but one unknown number.

1. The sum of three numbers in arithmetic progression is 30 and the sum of their squares is 462. What are the numbers?

SOLUTION

Let the series be $a - d, a, a + d$.

$$(1) \quad (a - d) + a + (a + d) = 30$$

$$(2) \quad (a - d)^2 + a^2 + (a + d)^2 = 462$$

$$(3) \quad 3a = 30 \quad \text{From (1)}$$

$$(4) \quad a = 10$$

$$(5) \quad 3a^2 + 2d^2 = 462 \quad \text{From (2)}$$

$$(6) \quad d = \pm 9$$

Forming the series from $a = 10$ and $d = \pm 9$, we have for the terms

$$1, 10, 19 \quad \text{or} \quad 19, 10, 1$$

2. The sum of three numbers in arithmetic progression is 18 and their product is 120. What are the numbers?
3. The sum of three numbers in arithmetic progression is 21 and the sum of their squares is 155. Find the numbers.
4. There are three numbers in arithmetic progression and the sum of their squares is 93. If the third number is 4 times as large as the first, what are the numbers?
5. Find the sum of the odd numbers from 1 to 99, inclusive.
6. The product of the extremes of an arithmetic progression of 10 terms is 70 and the sum of the series is 95. What are the extremes?
7. Fifty-five logs are to be piled so that the top layer shall consist of 1 log, the next layer of 2 logs, the next layer of 3 logs, etc. How many logs must be placed in the bottom layer?
8. It cost Mr. Smith \$84 to have a well dug. If the cost of digging was \$1.50 for the first yard, \$1.75 for the second, \$2.00 for the third, etc., how deep was the well?
9. How many arithmetic means must be inserted between 4 and 25, so that the sum of the series may be 116?
10. Prove that the sum of n consecutive odd integers, beginning with 1, is n^2 .

GEOMETRIC PROGRESSION

A series of numbers each of which after the first is derived by multiplying the preceding number by some constant multiplier is called a **geometric series**, or a **geometric progression**.

2, 4, 8, 16, 32 and a^4 , a^3 , a^2 , a are geometric progressions.

In the first series the constant multiplier is 2; in the second it is $\frac{1}{a}$.

G.P. is an abbreviation of the words *geometric progression*.

The constant multiplier is called the **ratio**.

It is evident that the terms of a geometric progression *increase* or *decrease* in absolute value, according as the ratio is numerically *greater* or *less* than 1.

To find the n th, or last, term of a geometric series

Let a represent the first term of a G.P., r the ratio, n the number of terms, and l the last, or n th, term.

Then the series is $a, ar, ar^2, ar^3, ar^4, \dots$.

Observe that the exponent of r is one less than the number of the term; that is,

$$l = ar^{n-1} \quad (I)$$

EXERCISES

- Find the 9th term of the series, 1, 3, 9, $\dots$.

SOLUTION

$$\begin{aligned} l &= ar^{n-1} \\ &= 1 \times 3^8 \\ &= 6561 \end{aligned}$$

EXPLANATION. In this exercise $a = 1$, $r = 3$, and $n = 9$.

Substituting these values in the formula for l , we have for the 9th term 6561.

NOTE. The use of logarithms will sometimes shorten the arithmetic work.

- Find the 10th term of the series 1, 2, 4, $\dots$.
- Find the 8th term of the series $\frac{1}{4}, \frac{1}{2}, 1, \dots$.
- Find the 9th term of the series 6, 12, 24, $\dots$.
- Find the 11th term of the series $\frac{1}{2}, 1, 2, \dots$.
- Find the 7th term of the series 2, -6, 18, $\dots$.
- Find the 6th term of the series 4, 20, 100, $\dots$.
- Find the 6th term of the series 6, -18, 54, $\dots$.
- Find the 10th term of the series $1, \frac{1}{2}, \frac{1}{4}, \dots$.
- Find the 10th term of the series $1, \frac{2}{3}, \frac{4}{9}, \dots$.
- Find the 8th term of the series $\frac{1}{2}, -\frac{1}{3}, \frac{2}{9}, \dots$.
- Find the 11th term of the series $a^9b, a^8b^2, \dots$.
- Find the n th term of the series $2, \sqrt{2}, 1, \dots$.

14. If a man begins business with a capital of \$2000 and doubles it every year for 6 years, how much is his capital at the end of the sixth year? ($n = 7$. Why?)

15. The population of the United States was 76.3 millions in 1900. If it doubles itself every 25 years, what will it be in the year 2000?

16. A man's salary was raised $\frac{1}{4}$ every year for 5 years. If his salary was \$1600 the first year, what was it the sixth year?

17. The population of a city, which at a certain time was 20,736, increased in geometric progression 25% each decade. What was the population at the end of 40 years?

18. A sum of money invested at 6% compound interest doubles itself in 12 years. How much will \$1.00 invested at 6% compound interest amount to in 48 years?

19. How much will \$200 amount to in 4 years with interest at 4%, compounded annually? compounded semiannually?

20. A man who wanted 10 bushels of wheat thought \$1.25 a bushel too high a price; but he agreed to pay \$.03 for the first bushel, \$.06 for the second, \$.12 for the third, and so on. How much did the last bushel cost him?

21. The machinery in a manufacturing establishment is valued at \$20,000. If its value depreciates each year to the extent of 10% of its value at the beginning of that year, how much will the machinery be worth at the end of 5 years?

22. From a grain of corn there grew a stalk that produced an ear of 150 grains. These grains were planted, and each produced an ear of 150 grains. This process was repeated until there were 4 harvestings. If 75 ears of corn make 1 bushel, how many bushels were there the fourth year?

A series consisting of a limited number of terms is called a **finite series**.

A series consisting of an unlimited number of terms is called an **infinite series**.

To find the sum of a finite geometric series

Let a represent the first term, r the ratio, n the number of terms, l the n th, or last, term, and s the sum of the terms.

$$(1) \quad s = a + ar + ar^2 + ar^3 + \dots + ar^{n-1}$$

$$(2) \quad rs = ar + ar^2 + ar^3 + \dots + ar^{n-1} + ar^n \quad r \times (1)$$

$$(3) \quad rs - s = ar^n - a \quad (2) - (1)$$

$$\therefore s = \frac{ar^n - a}{r - 1}, \quad \text{or} \quad \frac{a - ar^n}{1 - r} \quad (II)$$

We have seen (page 375) that $a^n - b^n$ is always divisible by $a - b$, when n represents a positive integral exponent.

Hence we may rewrite formula (II) as follows:

$$s = \frac{ar^n - a}{r - 1}$$

$$s = \frac{a(r^n - 1)}{r - 1}$$

$$s = a(r^{n-1} + r^{n-2} + \dots \text{to } n \text{ terms})$$

which, when expanded, becomes (1) above.

Since $ar^{n-1} = l$, $ar^n = rl$

Substituting rl for ar^n in (II), we have:

$$s = \frac{rl - a}{r - 1}, \quad \text{or} \quad \frac{a - rl}{1 - r} \quad (III)$$

EXERCISES

1. Find the sum of 6 terms of the series 3, 9, 27, ...

SOLUTION

$$\begin{aligned} s &= \frac{ar^n - a}{r - 1} \\ &= \frac{3 \times 3^6 - 3}{3 - 1} = 1092 \end{aligned}$$

EXPLANATION. Since the first term a , the ratio r , and the number of terms n are known, formula II, which gives the sum in terms of a , r , and n , is used.

2. Find the sum of 8 terms of the series 1, 2, 4, ...

3. Find the sum of 8 terms of the series $1, \frac{1}{2}, \frac{1}{4}, \dots$

4. Find the sum of 10 terms of the series $1, 1\frac{1}{2}, 2\frac{1}{4}, \dots$

5. Find the sum of 7 terms of the series $2, -\frac{2}{3}, \frac{2}{9}, \dots$

6. Find the sum of 12 terms of the series $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \dots$.
7. Find the sum of 7 terms of the series $1, 2x, 4x^2, \dots$.
8. Find the sum of 7 terms of the series $1, -2x, 4x^2, \dots$.
9. Find the sum of n terms of the series $1, x^2, x^4, \dots$.
10. Find the sum of n terms of the series $1, 2, 4, \dots$.
11. Find the sum of n terms of the series $1, \frac{1}{3}, \frac{1}{9}, \dots$.
12. The extremes of a geometric series are 1 and 729, and the ratio is 3. What is the sum of the series?
13. What is the sum of the series $3, 6, 12, \dots, 192$?
14. What is the sum of the series $7, \dots, -56, 112, -224$?
15. What is the first term of a series if $s = 315$, $r = 2$, and $n = 6$?

NOTE. Use logarithms in examples 16 to 22.

16. How much would a man have to his credit at the end of a month of 30 days if he were able to set aside 1¢ the first day, 2¢ the second day, 4¢ the third day, and so on throughout the month?

17. A man deposited \$200 every 6 months in a savings bank which paid 4% interest, compounded twice a year. How much was there to his credit immediately after the fourth deposit?

SUGGESTION

First \$200 receives interest for 3 periods and amounts to $200(1.02)^3$

Second \$200 receives interest for 2 periods and amounts to $200(1.02)^2$

Third \$200 receives interest for 1 period and amounts to $200(1.02)$

Fourth \$200 receives no interest and amounts to 200

The sum will be $200(1 + 1.02 + 1.02^2 + 1.02^3) = 200 \left[\frac{1.02^4 - 1}{1.02 - 1} \right]$

18. A company sets aside \$2500 at the end of each year. These amounts accumulate as a sinking fund. If the company receives interest at the rate of 5%, compounded annually, how much is there in the sinking fund at the end of 15 years?

19. A man wishing to buy a house invests in a Building and Loan Association which allows him $5\frac{1}{2}\%$ interest, compounded yearly. If he is able to set aside \$500 at the end of each year, how much has he toward the house at the end of 10 years?

20. A man paid a life insurance premium of \$46.50 at the beginning of each year for 20 years. If the insurance company gets 6% interest, compounded annually, how much will this premium amount to in 20 years?

SUGGESTION

The first payment bears interest for 20 years and amounts to $46.50(1.06)^{20}$. The other payments amount respectively to $46.50(1.06)^{19}$, $46.50(1.06)^{18}$, . . . $46.50(1.06)$. The sum of these amounts is

$$46.50 \left[\frac{1.06^{21} - 1.06}{.06} \right]$$

21. A young man took out a twenty-payment life insurance policy for \$1000, paying a yearly premium of \$41.20. If he had deposited \$41.20 a year in a bank which paid 5% interest, compounded annually, how much would he have had to his credit at the end of the 20 years?

22. The owner of a house deposited the income of the monthly rent of \$125 the first of each month in a bank which allowed 3% interest, compounded monthly. How much did he have to his credit at the end of 2 years?

To find the sum of an infinite geometric series

If the ratio r is numerically less than 1, it is evident that the successive terms of a geometric series become numerically less and less. Hence, in an infinite decreasing geometric series, the n th term (l or ar^{n-1}), and consequently ar^n , can be made less than any assignable number, though not absolutely equal to zero.

Formula (II) may be written :

$$s = \frac{a - ar^n}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}$$

Since by taking enough terms ar^n can be made less than any assignable number, the second fraction may be neglected.

Hence the formula for the sum of an infinite decreasing geometric series is

$$s = \frac{a}{1 - r} \quad (\text{IV})$$

EXERCISES

1. Find the sum of the series $1, \frac{1}{10}, \frac{1}{100}, \dots$

SOLUTION

Substituting 1 for a and $\frac{1}{10}$ for r in (IV), we have :

$$s = \frac{1}{1 - \frac{1}{10}} = \frac{1}{\frac{9}{10}} = \frac{10}{9}$$

Find the value of :

- | | |
|---|--|
| 2. $1 + \frac{1}{2} + \frac{1}{4} + \dots$ | 5. $-4 - 1 - \frac{1}{4} - \dots$ |
| 3. $3 + \frac{3}{4} + \frac{3}{16} + \dots$ | 6. $-2 + \frac{2}{5} - \frac{2}{25} + \dots$ |
| 4. $1 - \frac{1}{3} + \frac{1}{9} - \dots$ | 7. $100 - 10 + 1 - \dots$ |
| 8. $1 + x + x^2 + x^3 + \dots$, when $x = .9$ | |
| 9. $1 - x + x^2 - x^3 + \dots$, when $x = \frac{3}{8}$ | |

10. Find the value of the repeating decimal $.185185185 \dots$

NOTE. Such a number is sometimes written $.18\dot{5}$.

SOLUTION

Since $.185185185 \dots = .185 + .000185 + .000000185 + \dots$, $a = .185$ and $r = .001$.

$$\text{Substitute in (IV): } .185185185 \dots = s = \frac{.185}{1 - .001} = \frac{5}{27}$$

Find the value of :

- | | | | |
|---------------------|---------------------|----------------------|-------------------------|
| 11. $.407407 \dots$ | 13. $1.94444 \dots$ | 15. $.007007 \dots$ | 17. $2.3\dot{5}$ |
| 12. $.363636 \dots$ | 14. $.020303 \dots$ | 16. $5.032828 \dots$ | 18. $3.5\dot{1}\dot{3}$ |

To insert geometric means between two terms

EXERCISES

1. Insert 3 geometric means between 2 and 162.

SOLUTION

$$l = ar^{n-1}$$

$$162 = 2r^4$$

$$r = \pm 3$$

EXPLANATION. Since there are 3 means, there are 5 terms, and $n - 1 = 4$. Solving for r and neglecting imaginary values, we have $r = \pm 3$.

Therefore the series is either 2, 6, 18, 54, 162 or 2, - 6, 18, - 54, 162.

2. Insert 3 geometric means between 1 and 625.

3. Insert 5 geometric means between $4\frac{1}{2}$ and $\frac{2048}{81}$.

4. Insert 4 geometric means between $\frac{343}{16}$ and $\frac{64}{9}$.

5. Insert 4 geometric means between 5120 and 5.

6. Insert 4 geometric means between $4\sqrt{2}$ and 1.

7. Insert 5 geometric means between a^6 and b^6 .

8. Insert 4 geometric means between x and $-y$.

If G is the geometric mean between a and b in the series a, G, b , then.

$$\frac{a}{G} = \frac{G}{b}$$

That is,  $G = \pm \sqrt{ab}$

PRINCIPLE. The geometric mean between two numbers is equal to the square root of their product.

Observe that the geometric mean between two numbers is also their mean proportional.

Find the geometric mean between :

9. 8 and 50

12. $(a + b)^2$ and $(a - b)^2$

10. $\frac{1}{2}$ and $3\frac{5}{9}$

13. $\frac{a^2 + ab}{a^2 - ab}$ and $\frac{ab + b^2}{ab - b^2}$

11. $1\frac{1}{6}$ and $\frac{3}{4}$

Since formula I (page 415) with formula II (page 417), or III, which is equivalent to II, forms a system of two independent simultaneous equations containing five elements, if *three elements are known*, the other *two* can be found by elimination.

NOTE. Solving for n , since it is an exponent, requires the use of logarithms, except in cases where its value may be determined by inspection.

Problems

1. Given r , l , and s , find a .
2. The ratio of a geometric progression is 5, the last term is 625, and the sum is 775. What is the first term?
3. The ratio of a geometric progression is $\frac{1}{10}$, the sum is $\frac{1}{3}$, and the series is infinite. What is the first term?
4. Find l in terms of a , r , and s .
5. Find the last term of the series 5, 10, 20, $\dots$, the sum of whose terms is 155.
6. If $\frac{1}{8} + \frac{1}{8}\sqrt{2} + \frac{1}{4} + \dots = 1\frac{7}{8}(1 + \sqrt{2})$, what is the last term and the number of terms?
7. Deduce the formula for r in terms of a , l , and s .
8. If the sum of the geometric progression 32, $\dots$, 243 is 665, what is the ratio? Write the series.
9. The sum of a geometric progression is 700 greater than the first term and 525 greater than the last term. What is the ratio? If the first term is 81, what is the progression?
10. Deduce the formula for r in terms of a , n , and l .
11. The first term of a geometric progression is 3, the last term is 729, and the number of terms is 6. What is the ratio? Write the series.
12. Find l in terms of r , n , and s .
13. A sled went 100 feet the first second after reaching the foot of a hill. How far did it go on the level, if its velocity decreased each second $\frac{1}{5}$ of that of the previous second?

14. A man sent out 2 chain letters with the request that each person receiving a letter should copy it and send it to 2 others. If everybody complied with the request, how many letters had been written when 12 sets had been sent out?

15. A ball thrown vertically into the air 100 feet falls and rebounds 40 feet the first time, 16 feet the second time, and so on. What is the whole distance through which the ball will pass?

16. A man planted a garden containing 275.6 square rods. How long did it take him if he planted 3 square rods the first day, 4 square rods the second day, $5\frac{1}{3}$ square rods the third day, and so on?

17. Show that equal multiples of numbers in geometric progression are also in geometric progression.

18. The sum of three numbers in geometric progression is 19, and the sum of their squares is 133. Find the numbers.

SUGGESTION. When there are but three terms in the series, they may be represented by a^2 , ar , r^2 , or by a , $\sqrt{ar}$, r .

19. The product of three numbers in geometric progression is 8 and the sum of their squares is 21. Find the numbers.

20. The sum of the first and second of four numbers in geometric progression is 15, and the sum of the third and fourth is 60. What are the numbers?

SUGGESTION. Four unknown numbers in geometric progression may be represented by $\frac{a^2}{r}$, a , r , $\frac{r^2}{a}$.

21. From a cask of vinegar $\frac{1}{3}$ is drawn off and the cask is filled by pouring in water. Show that if this is done 6 times, the contents of the cask will be more than $\frac{9}{10}$ water.

22. If the quantity, and the corresponding pressure, of the air in the receiver of an air pump is diminished by $\frac{1}{10}$ of itself at each stroke of the piston, and if the initial pressure is 14.7 pounds a square inch, find, to the nearest tenth of a pound, what the pressure will be after 6 strokes.

CHAPTER XXII

NUMERICAL TRIGONOMETRY

MEASUREMENT OF ANGLES

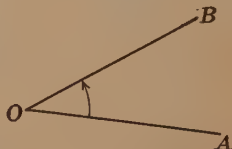
Idea of relationship

We have seen that the equation is a valuable tool for investigating relations between quantities. One of the practical uses is to be found in the study of the right triangle.

Angles

In the figure below, lines AO and BO form the angle AOB . OA and OB are called the sides or the rays of this angle. O , the point of intersection of OA and OB , is called the **vertex** of the angle.

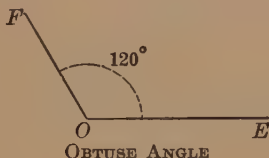
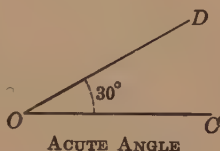
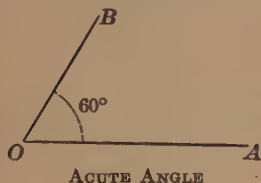
An **angle** may be defined as the **amount of rotation** made by a line rotating in a plane about a fixed point. The size of the angle does not depend on the length of its sides, but upon the amount of rotation of one of its sides. Hold your pencil along the line OA shown in the figure. Turn the pencil, keeping O fixed, until the pencil falls on the line OB . The pencil is said to have turned through the angle AOB . In reading an angle, the vertex is read as the *middle* letter.



Unit of measurement of angles

The circumference of a circle has been divided for convenience of measurement into 360 parts, called **degrees**. 1° is taken as a unit of measurement of an angle. This represents $\frac{1}{360}$ of a complete turn. Each degree is divided into 60 minutes, and each minute into 60 seconds.

The following figures illustrate angles of various sizes.



A **straight angle** is one whose sides form a straight line ; it contains 180° . A **right angle** is half a straight angle and contains 90° . An **acute angle** contains less than 90° . An **obtuse angle** contains more than 90° and less than 180° .

The sign for *angle* is $\angle$.

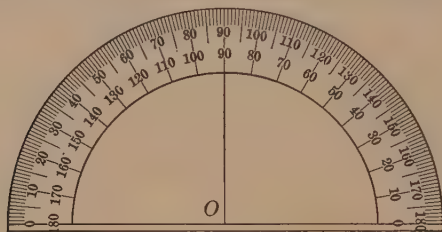
The sum of $\angle AOB$, $\angle COD$, and $\angle MON = 180^\circ$, a **straight angle**.

EXERCISES

1. Read each angle in the figures above.
2. If the number of degrees in angles AOB and COD are added, what kind of angle do we obtain? How many degrees are there in their sum?
3. Angle $EOF +$ angle AOB is equal to how many degrees?
4. $\frac{1}{2}$ angle $AOB + \frac{1}{2}$ angle MON is how many degrees?

The protractor

A protractor is a tool used in measuring angles.

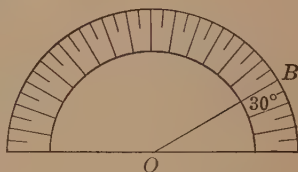


PROTRACTOR

NOTE. Each pupil should procure an inexpensive protractor and use it in his work.

EXERCISES

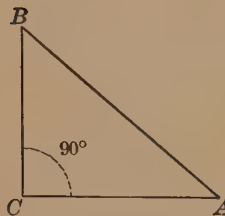
1. This figure will illustrate the method of measuring angles by the use of a protractor. Draw several angles of different sizes and find the number of degrees in each.



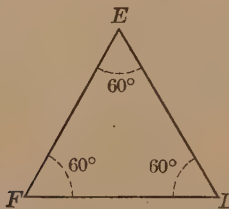
MEASURING ANGLES

2. Draw angles of 60° , 30° , 45° , 80° , and 10° .

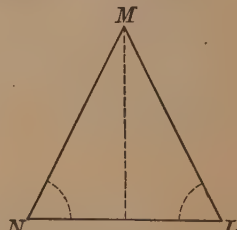
In a **right triangle** (ABC), one of the angles is a *right angle*.



RIGHT TRIANGLE



EQUILATERAL TRIANGLE



ISOSCELES TRIANGLE

In an **equilateral triangle** (DEF), the three sides are equal in length and the three angles are equal in size.

In an **isosceles triangle** (LMN), *two* sides are equal in length and *two* angles are equal in size.

EXERCISES

1. Using a protractor, measure the angles of each of the triangles on page 426.

If you have done your work carefully, you will find that *the sum of the three angles of any triangle is equal to 180° .*

2. How many degrees are there in the sum of the two acute angles of a right triangle?

3. How many degrees are there in each angle of an equilateral triangle?

4. If two angles of a triangle are equal to 65° and 35° , how many degrees are there in the third angle?

5. In the following table are given two angles of a triangle ABC . Find the third angle.

	$\angle A$	$\angle B$	$\angle C$
(1)	30°	60°	?
(2)	45°	45°	?
(3)	60°	?	60°
(4)	90°	30°	?
(5)	30°	70°	?
(6)	50°	?	60°
(7)	?	50°	40°
(8)	?	65°	70°
(9)	80°	?	70°

On page 71 we learned that :

The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.

If c represents the hypotenuse and a and b represent the legs, we have :

$$c^2 = a^2 + b^2 \text{ and } c = \sqrt{a^2 + b^2}$$

Also $a^2 = c^2 - b^2 \text{ and } a = \sqrt{c^2 - b^2}$

Also $b^2 = c^2 - a^2 \text{ and } b = \sqrt{c^2 - a^2}$

6. If $a = 5$ and $b = 12$, find c .

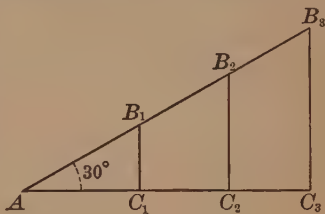
7. If $c = 26$ and $b = 10$, find a .

8. If $c = 17$ and $a = 15$, find b .

TRIGONOMETRIC RATIOS

Similar figures have the same *shape* (page 187). All circles, regardless of size, are similar figures; also all squares.

In the triangles at the right the sides AC_3 , AC_2 , and AC_1 are called *corresponding sides*; also AB_3 , AB_2 , and AB_1 ; also B_3C_3 , B_2C_2 , and B_1C_1 . Similarly the angles AC_1B_1 , AC_2B_2 , and AC_3B_3 are called *corresponding angles*; also the angles AB_1C_1 , AB_2C_2 , and AB_3C_3 .



Similar triangles have the same shape; their corresponding angles are equal and their corresponding sides are proportional. That is, the ratios of the corresponding sides are equal.

Thus, in the similar triangles AB_1C_1 and AB_2C_2 and AB_3C_3 ,

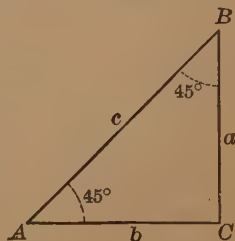
$$\frac{B_1C_1}{AC_1} = \frac{B_2C_2}{AC_2} = \frac{B_3C_3}{AC_3}$$

The ratio found is true in any right triangle which has an angle equal to angle A , regardless of the size of the triangle. This fact enables us to obtain a table of ratios that are true in any right triangle.

Tangent of an angle

In this figure we have a right triangle with angles A and B equal. The side opposite angle A is lettered a . The side b , which is opposite angle B , is said to be *adjacent to angle A*.

The ratio of the side opposite angle A to the side adjacent to angle A , that is, the ratio $\frac{a}{b}$, is called the **tangent** of angle A .



$\therefore$ *Tangent of angle A* = $\frac{\text{side opposite}}{\text{side adjacent}} = \frac{a}{b}$ or, using the abbreviation "tan" for tangent,

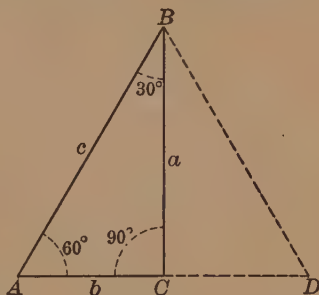
$$(1) \quad \tan A = \frac{a}{b}$$

Since in the figure on page 428, $a = b$, triangle ABC is an isosceles triangle and angle $A = \text{angle } B$. But angle $A = 45^\circ$. (Why?) Therefore, substituting in (1) above, we have

$$(2) \quad \tan 45^\circ = \frac{a}{b} = \frac{a}{a} = 1$$

$$\text{Hence } \tan 45^\circ = 1$$

In the figure at the right, ABD is an equilateral triangle and C is the mid-point of the side AD .



Angle $A = 60^\circ$, angle $ABC = 30^\circ$, and angle $ACB = 90^\circ$. The distance b is one half the length of AD , since C is the mid-point of AD . Hence b is one half of c , for c and AD are equal in length, being sides of an equilateral triangle. Therefore the length of $AB = 2b$.

$$\text{Then } a^2 = (2b)^2 - b^2 \quad (\text{Why?})$$

$$= 3b^2$$

$$a = b\sqrt{3}$$

(See also page 263.) Hence:

$$\tan 30^\circ = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{b}{a} = \frac{b}{b\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3} = .577$$

$$\tan 60^\circ = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{a}{b} = \frac{b\sqrt{3}}{b} = \sqrt{3} = 1.732$$

EXERCISES

1. Using a protractor, draw a right triangle having one angle equal to 40° . Find the ratio of the side opposite to the side adjacent to this angle, to two decimal places, thus obtaining an approximate value of $\tan 40^\circ$.

In like manner find:

2. $\tan 20^\circ$

3. $\tan 50^\circ$

4. $\tan 10^\circ$

The results should equal approximately the values of the tangents given in the following table:

TABLE OF TANGENTS

ANGLE	TANGENT	ANGLE	TANGENT
0°	.000	50°	1.192
10°	.176	60°	1.732
20°	.364	70°	2.748
30°	.577	80°	5.671
40°	.839	90°	infinite

In the same manner as in exercise 1, find:

5. $\tan 15^\circ$

6. $\tan 75^\circ$

7. $\tan 65^\circ$

Problems

Using the facts above established, solve the following:

1. In the right triangle ABC , if $b = 120$ feet and angle $A = 20^\circ$, find a .

SOLUTION

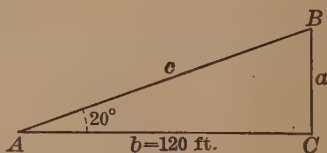
$$\tan 20^\circ = \frac{a}{120}$$

$$120 \tan 20^\circ = a$$

$$120 \times .364 = a$$

$$43.68 = a$$

$$a = 43.7 \text{ ft.}$$



2. An engineer wishes to measure the height of a high chimney. He measures the angle at A and finds it to be 30° . He measures AC and finds it to be 200 feet. How high is the chimney BC ?

SOLUTION

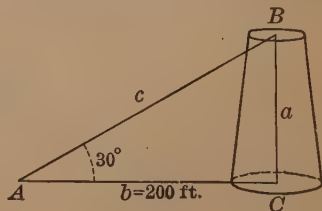
$$\tan 30^\circ = \frac{a}{200}$$

$$200 \tan 30^\circ = a$$

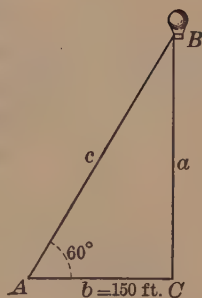
$$200 \times .577 = a$$

$$a = 115.4$$

The chimney is 115 feet high.

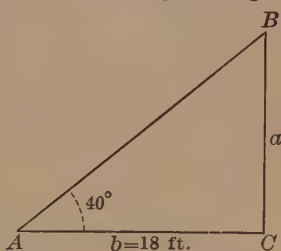


3. A balloon, B , is anchored at point A by a rope. The rope makes an angle of 60° with the ground. The point C on the ground directly under the balloon is 150 feet from A . If the rope is straight, find the height of the balloon.



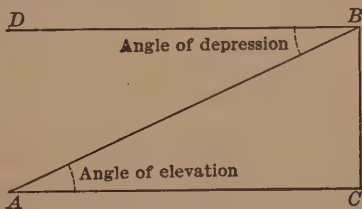
4. In a triangle lettered like the one at the right, angle $A = 30^\circ$ and $b = 60$ feet. How long is a ? Sketch the figure.

5. How many feet does a represent in the second figure on this page?



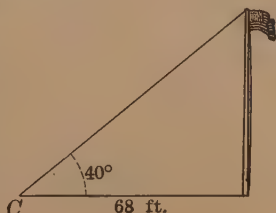
The **angle of elevation** of an object above the point of observation is the angle which a line from the eye to the object makes with a horizontal line in the same vertical plane. Thus, in the figure below, angle CAB is the *angle of elevation* of the point B above the point A .

The **angle of depression** of an object below the point of observation is the angle which a line from the eye to the object makes with a horizontal line in the same plane. Thus, in the figure, angle DBA is the *angle of depression* of the point A below the point B .

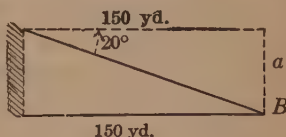


The *angle of elevation*, CAB , equals the *angle of depression*, DBA .

6. The angle of elevation of the top of a flagpole from C , a point on the ground, is 40° . If the distance from C to the foot of the pole is 68 feet, find the height of the pole.



7. A man is standing on a vertical cliff looking out upon the ocean. The angle of depression of a sailboat is 20° . If the boat, B , is 150 yards from the foot of the cliff, find a , the height of the cliff.



Sine of an angle

The ratio of the side *opposite* an angle to the *hypotenuse* of a right triangle is called the **sine** of that angle.

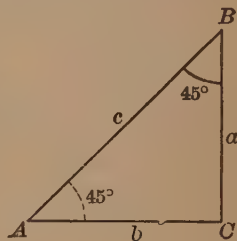
The triangle at the right is an isosceles right triangle, sides a and b being equal. $\angle A = \angle B = 45^\circ$. Why?

The sine of angle A is abbreviated $\sin A$.

$$\sin A = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{a}{c}$$

$$\text{Since } a = b, c = \sqrt{a^2 + a^2} = \sqrt{2} a^2 \quad \therefore c = a\sqrt{2}$$

$$\sin 45^\circ = \frac{a}{c} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2} = .707$$



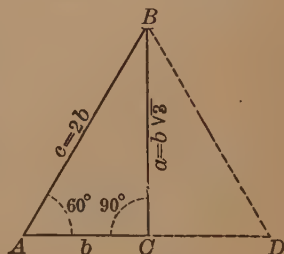
The figure at the right is an equilateral triangle and C is the midpoint of the side AD . Angle $A = 60^\circ$, angle $ABC = 30^\circ$, and angle $ACB = 90^\circ$. The distance b is one half of AB . (Why?)

$$\text{Since } AD = AB, c = 2b$$

$$\therefore a = b\sqrt{3} \quad (\text{Why?})$$

$$\sin 60^\circ = \frac{a}{c} = \frac{b\sqrt{3}}{2b} = \frac{\sqrt{3}}{2} = .866$$

$$\sin 30^\circ = \frac{b}{c} = \frac{b}{2b} = \frac{1}{2}$$



EXERCISES

1. Draw a right triangle having an angle of 10° . Find the ratio of the side opposite this angle to the hypotenuse, thus obtaining an approximate value of $\sin 10^\circ$.

In like manner find :

2. $\sin 20^\circ$ 3. $\sin 80^\circ$ 4. $\sin 50^\circ$ 5. $\sin 70^\circ$

The results should equal approximately the values in the following table :

TABLE OF SINES

ANGLE	SINE	ANGLE	SINE
0°	.000	50°	.766
10°	.174	60°	.866
20°	.342	70°	.940
30°	.500	80°	.985
40°	.643	90°	1.000

6. In the same manner find (a) $\sin 35^\circ$; (b) $\sin 75^\circ$.

Problems

1. In the triangle at the right find side a .

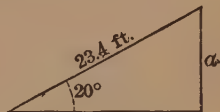
SOLUTION

$$\frac{a}{23.4} = \sin 20^\circ$$

$$a = 23.4 \times .342$$

$$a = 8.00$$

Hence $a = 8$ feet.



2. The diagram below represents a 150-foot rope reaching the ground from the top of a pole. It makes an angle of 50° with the ground. How high is the pole?

SOLUTION

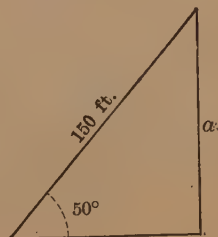
$$\frac{a}{150} = \sin 50^\circ$$

$$\frac{a}{150} = .766$$

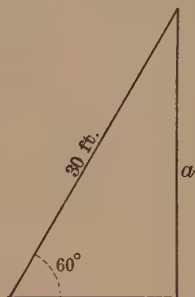
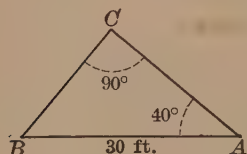
$$a = .766 \times 150$$

$$a = 114.9$$

Hence the pole is 115 feet high.



3. How long is BC , shown in the figure at the left below? How long is AC ?



4. A 30-foot ladder is leaning against the side of a house (see right-hand figure above) and makes an angle of 60° with the ground. What is the vertical distance from the top of the ladder to the ground?

5. In the right triangle ABC shown below:

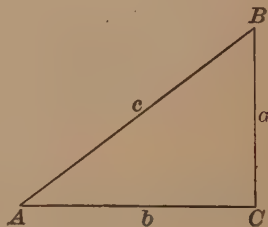
(a) Given angle $A = 20^\circ$;
 $c = 120$ feet; find a .

(b) Given angle $A = 40^\circ$;
 $c = 60$ feet; find a .

(c) Given angle $A = 70^\circ$;
 $c = 76.5$ feet; find a .

(d) Given angle $A = 60^\circ$;
 $a = 48.5$ feet; find c .

(e) Given angle $A = 80^\circ$;
 $a = 29.4$ feet; find c .



Cosine of an angle

The ratio of the side *adjacent* to an angle to the *hypotenuse* of the right triangle is called the **cosine** of that angle.

The cosine of angle A is abbreviated $\cos A$.

In the figure above,

$$\cos A = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{b}{c}$$

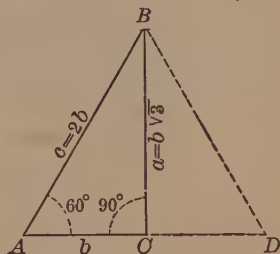
In the figure below, since angle $A = 60^\circ$ and $c = 2b$, we have

$$\cos 60^\circ = \frac{b}{c} = \frac{b}{2b} = \frac{1}{2} = .50$$

$$\cos \angle ABC = \frac{a}{c}$$

In this figure, since $\angle ABC = 30^\circ$ and $a = b\sqrt{3}$ (why?), we have

$$\cos 30^\circ = \frac{b\sqrt{3}}{2b} = \frac{\sqrt{3}}{2} = \frac{1.732}{2} = .866$$



In the isosceles triangle on page 432, $\sin 45^\circ = \frac{a}{c}$ and $\cos 45^\circ = \frac{b}{c}$. Since $a = b$ (why?), $\cos 45^\circ = \sin 45^\circ = \frac{1}{2}\sqrt{2} = .707$.

EXERCISES

1. By the use of a protractor, draw a right triangle having an angle of 40° . Find the ratio of the *side adjacent* to the given angle to the *hypotenuse* of the right triangle. The ratio will be the approximate value of the cosine of 40° .

In like manner find :

2. $\cos 10^\circ$ 3. $\cos 20^\circ$ 4. $\cos 70^\circ$ 5. $\cos 80^\circ$

The results should equal approximately the values of the cosines given in the following table :

TABLE OF COSINES

ANGLE	COSINE	ANGLE	COSINE
0°	1.000	50°	.643
10°	.985	60°	.500
20°	.940	70°	.342
30°	.866	80°	.174
40°	.766	90°	.000

6. In the same manner find (a) $\cos 35^\circ$; (b) $\cos 65^\circ$.

Problems

1. In the figure below, c is the anchor wire of a pole. The wire meets the ground 30 feet from the foot of the pole. Find the length of the anchor wire.

SOLUTION

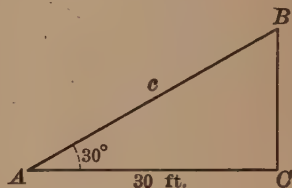
$$\frac{30}{c} = \cos 30^\circ$$

$$\frac{30}{c} = .866$$

$$30 = .866 c$$

$$\frac{30}{.866} = c$$

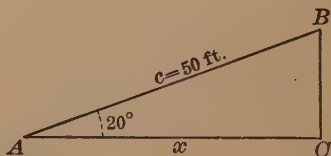
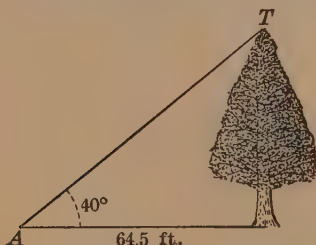
$$34.6 = c$$



Hence the anchor wire is 35 feet long.

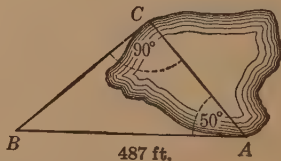
2. The rope of a flagpole makes an angle of 20° with the ground at a point 32 feet from the foot of the pole. Find the length of the rope. (First draw diagram.)

3. Find the distance from A , a point on the ground, to T , the top of a tree, if A is 64.5 feet from the foot of the tree.



4. How long is x in triangle ABC above if $c = 50$ feet and angle $A = 20^\circ$?

5. A troop of boy scouts wished to find the distance CA across a pond. They marked off CB , perpendicular to CA . Next they laid off AB to make an angle of 50° with CA . They measured AB and found it to be 487 feet. What was the distance CA ?



Summary of trigonometric ratios

You have been using some of the fundamental principles of *trigonometry*, a branch of mathematics which is especially concerned with the sides and angles of triangles. It is useful for *indirect measurement*, that is, measurement by computation instead of by direct application of a unit of measure like the foot rule. It is widely used in engineering and navigation. The basic mathematics of surveying is trigonometry.

The three ratios — **sine**, **cosine**, and **tangent** — are called **trigonometric ratios** or **functions**. The use of these ratios simplifies the solution of problems involving indirect measurement.

By the use of these three ratios any part of a right triangle may be found if any two parts, of which at least one must be a side, in addition to the right angle are known. The known parts may be :

- (1) An acute angle and any side of the triangle.
- (2) The hypotenuse and either leg of the triangle.
- (3) Both legs of the triangle.

NOTE. Three other functions sometimes used are the **cotangent** (cot), the **secant** (sec), and the **cosecant** (csc). (See last diagram, page 434.)

$$\cot A = \frac{b}{a} = \frac{\text{side adjacent}}{\text{side opposite}}$$

$$\sec A = \frac{c}{b} = \frac{\text{hypotenuse}}{\text{side adjacent}}$$

$$\csc A = \frac{c}{a} = \frac{\text{hypotenuse}}{\text{side opposite}}$$

The finding of unknown parts by applying the trigonometric ratios and algebraic principles is called *solving the triangle*.

When the sum of two angles is a right angle, the angles are called *complementary*. The *complement* of an angle of $30^\circ = (90^\circ - 30^\circ) = 60^\circ$ and the complement of an angle of $60^\circ = 30^\circ$. What are the complements of the following angles : 45° ; 28° ; 67° ; 40° ; 85° ; d° ; $(2n - s)^\circ$; $(n + 8)^\circ$; $40^\circ 15'$?

The following table gives a summary of the values of the three functions of 30° , 45° , and 60° , found on pages 429, 432, and 435:

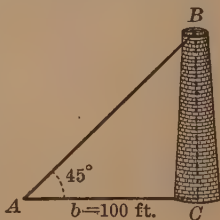
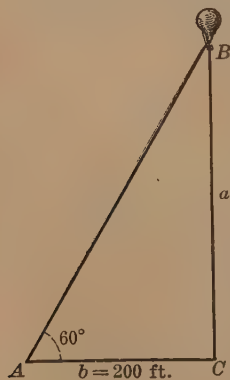
ANGLE	30°	45°	60°
sin	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$
tan	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$

By reference to the table it is evident that the sine of an angle is equal to the *cosine of its complement*; likewise the cosine of an angle is equal to the *sine of its complement*. This can be seen in a more general way by referring to the figure on page 439.

EXERCISES

Using the ratios indicated above, solve the following:

1. In the triangle ABC , $A = 60^\circ$, $b = 120$ feet. Find a .
2. An engineer wished to measure the length of a high chimney. He measured the angle at A and found it to be 45° . He measured AC and found it to be 100 feet. How high was the chimney?



3. Two persons, A and C , at the same time make observations of the position of a balloon, B . A finds that the angle of elevation of the balloon is 60° , while C observes that the balloon is directly above him. If the distance between A and C is 200 feet, find the height of the balloon.

4. In a triangle lettered as in the figures on page 438, angle $A = 30^\circ$, $b = 75$ feet. How long is a ? Sketch the figure.

5. One leg of a carpenter's steel square is 24 inches and the other leg is 18 inches. Tell how with the use of this square the carpenter can lay out angles of 30° , 45° , and 60° .

6. What relation does $\cot 60^\circ$ bear to $\tan 60^\circ$? $\csc 60^\circ$ to $\sin 60^\circ$? $\sec 60^\circ$ to $\cos 60^\circ$?

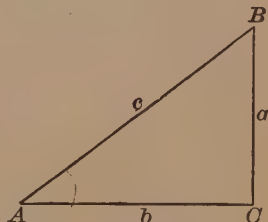
Using the ratios

In solving problems, keep in mind this summary and figure:

$$\sin A = \frac{a}{c} = \frac{\text{side opposite}}{\text{hypotenuse}}$$

$$\cos A = \frac{b}{c} = \frac{\text{side adjacent}}{\text{hypotenuse}}$$

$$\tan A = \frac{a}{b} = \frac{\text{side opposite}}{\text{side adjacent}}$$



Always draw, first of all, a sketch of the figure representing the problem.

Write the one of the three ratios which applies to the conditions of the problem, using a table to complete the equation. Solve the equation.

Table of trigonometric ratios. Interpolation

It will be useful to become familiar with a more complete table of ratios, as shown on page 440. The ratios given are, in most cases, only approximate, but are accurate enough for the work of this chapter.

The table on page 440 gives the values of functions only for degrees. If the given angle contains minutes, its functions cannot be read directly from the table.

In such cases we make use of the approximation process of **interpolation**. The exercises on page 441 illustrate this process.

TABLE OF TRIGONOMETRIC RATIOS

The abbreviation *hyp* means *hypotenuse*; *adj* means the *side adjacent* to the angle; *opp* means the *side opposite* the angle.

ANGLE	SINE (opp/hyp)	COSINE (adj/hyp)	TANGENT (opp/adj)	ANGLE	SINE (opp/hyp)	COSINE (adj/hyp)	TANGENT (opp/adj)
0°	.0000	1.0000	.0000	45°	.7071	.7071	1.0000
1°	.0175	.9998	.0175	46°	.7193	.6947	1.0355
2°	.0349	.9994	.0349	47°	.7314	.6820	1.0724
3°	.0523	.9986	.0524	48°	.7431	.6691	1.1106
4°	.0698	.9976	.0699	49°	.7547	.6561	1.1504
5°	.0872	.9962	.0875	50°	.7660	.6428	1.1918
6°	.1045	.9945	.1051	51°	.7771	.6293	1.2349
7°	.1219	.9925	.1228	52°	.7880	.6157	1.2799
8°	.1392	.9903	.1405	53°	.7986	.6018	1.3270
9°	.1564	.9877	.1584	54°	.8090	.5878	1.3764
10°	.1736	.9848	.1763	55°	.8192	.5736	1.4281
11°	.1908	.9816	.1944	56°	.8290	.5592	1.4826
12°	.2079	.9781	.2126	57°	.8387	.5446	1.5399
13°	.2250	.9744	.2309	58°	.8480	.5299	1.6003
14°	.2419	.9703	.2493	59°	.8572	.5150	1.6643
15°	.2588	.9659	.2679	60°	.8660	.5000	1.7321
16°	.2756	.9613	.2867	61°	.8746	.4848	1.8040
17°	.2924	.9563	.3057	62°	.8829	.4695	1.8807
18°	.3090	.9511	.3249	63°	.8910	.4540	1.9626
19°	.3256	.9455	.3443	64°	.8988	.4384	2.0503
20°	.3420	.9397	.3640	65°	.9063	.4226	2.1445
21°	.3584	.9336	.3839	66°	.9135	.4067	2.2460
22°	.3746	.9272	.4040	67°	.9205	.3907	2.3559
23°	.3907	.9205	.4245	68°	.9272	.3746	2.4751
24°	.4067	.9135	.4452	69°	.9336	.3584	2.6051
25°	.4226	.9063	.4663	70°	.9397	.3420	2.7475
26°	.4384	.8988	.4877	71°	.9455	.3256	2.9042
27°	.4540	.8910	.5095	72°	.9511	.3090	3.0777
28°	.4695	.8829	.5317	73°	.9563	.2924	3.2709
29°	.4848	.8746	.5543	74°	.9613	.2756	3.4874
30°	.5000	.8660	.5774	75°	.9659	.2588	3.7321
31°	.5150	.8572	.6009	76°	.9703	.2419	4.0108
32°	.5299	.8480	.6249	77°	.9744	.2250	4.3315
33°	.5446	.8387	.6494	78°	.9781	.2079	4.7046
34°	.5592	.8290	.6745	79°	.9816	.1908	5.1446
35°	.5736	.8192	.7002	80°	.9848	.1736	5.6713
36°	.5878	.8090	.7265	81°	.9877	.1564	6.3138
37°	.6018	.7986	.7536	82°	.9903	.1392	7.1154
38°	.6157	.7880	.7813	83°	.9925	.1219	8.1443
39°	.6293	.7771	.8098	84°	.9945	.1045	9.5144
40°	.6428	.7660	.8391	85°	.9962	.0872	11.4301
41°	.6561	.7547	.8693	86°	.9976	.0698	14.3007
42°	.6691	.7431	.9004	87°	.9986	.0523	19.0811
43°	.6820	.7314	.9325	88°	.9994	.0349	28.6363
44°	.6947	.7193	.9657	89°	.9998	.0175	57.2900
45°	.7071	.7071	1.0000	90°	1.0000	.0000	infinite

EXERCISES

1. Find the sine of 12.5° .

SOLUTION

This angle is between 12° and 13° .

$$\sin 13^\circ = .2250$$

$$\sin 12^\circ = .2079$$

$$\text{The difference, } d, = .0171$$

$$.5 d = .00855, \text{ or } .0086$$

$$\sin 12.5^\circ = .2079 + .0086 = .2165$$

2. Find the cosine of 41.8° .

SOLUTION

This angle is between 41° and 42° . You will observe that as the angle *increases* the cosine *decreases*.

$$\cos 41^\circ = .7547$$

$$\cos 42^\circ = .7431$$

$$d = .0116$$

$$.8 d = .0093$$

$$\cos 41.8^\circ = .7547 - .0093 = .7454$$

Using the table and interpolating, find the following:

3. $\sin 36.5^\circ$

8. $\cos 41.7^\circ$

4. $\sin 62.6^\circ$

9. $\tan 72.1^\circ$

5. $\sin 19.2^\circ$

10. $\tan 31.4^\circ$

6. $\cos 25.5^\circ$

11. $\tan 18.3^\circ$

7. $\cos 27.8^\circ$

12. $\cos 80.9^\circ$

NOTE. The parts of a degree are sometimes expressed in minutes ($'$). There are 60 minutes ($60'$) in 1 degree (1°). Therefore

$$42.5^\circ = 42^\circ 30'$$

$$35^\circ 20' = 35.3^\circ$$

$$36.2^\circ = 36^\circ 12'$$

$$26^\circ 50' = 26.8^\circ$$

$$16.7^\circ = 16^\circ 42'$$

$$19^\circ 22' = 19.4^\circ$$

$$55.4^\circ = 55^\circ 24'$$

$$78^\circ 30' = 78.5^\circ$$

To interpolate angles expressed in degrees and minutes, change to degrees and decimal parts of a degree and continue as in exercises 1 and 2 above.

Using the table and interpolating, find:

13. $\sin 20^\circ 30'$

16. $\cos 48^\circ 2'$

14. $\sin 31^\circ 20'$

17. $\tan 84^\circ 40'$

15. $\cos 24^\circ 12'$

18. $\tan 86^\circ 45'$

NOTE. If the function of the angle is given, we may find the angle by a similar process, as shown by the following illustrations.

19. Of what angle is .8241 the sine?

SOLUTION

.8241 is between $\sin 55^\circ$ and $\sin 56^\circ$.

$$\sin 56^\circ = .8290$$

$$\sin 55^\circ = .8192$$

$$\hline d = .0098$$

$$\sin a^\circ = .8241$$

$$\sin 55^\circ = .8192$$

$$\hline d' = .0049$$

$$\frac{d'}{d} = \frac{.0049}{.0098} = \frac{1}{2} = .5$$

The difference is .5 of a degree. Therefore .8241 is the sine of 55.5° ($55^\circ 30'$).

20. Of what angle is .4985 the cosine?

SOLUTION

.4985 is between $\cos 60^\circ$ and $\cos 61^\circ$.

$$\cos 60^\circ = .5000$$

$$\cos 61^\circ = .4848$$

$$\hline d = .0152$$

$$\cos 60^\circ = .5000$$

$$\cos a^\circ = .4985$$

$$\hline d' = .0015$$

$$\frac{d'}{d} = \frac{.0015}{.0152} = .1$$

Angle a is .1 the distance from 60° to 61° , or 60.1° ($60^\circ 6'$).

21. .2672 is the sine of what angle?

22. .7697 is the sine of what angle?

23. .8704 is the cosine of what angle?

24. .8100 is the cosine of what angle?

25. .4552 is the tangent of what angle?

26. 1.772 is the tangent of what angle?

27. A rectangular field 25 rods long and 20 rods wide has a diagonal path from one corner to the opposite one. Find the length of this path and the angles it makes with the sides of the field.

28. The hypotenuse of a right triangle is 624 feet, and one acute angle is 23.2° . Find the remaining parts of the triangle.

29. In a right triangle ABC , $A = 16^\circ 30'$, $c = 24.6$. Find a .

30. In a right triangle ABC , $B = 72^\circ 45'$, $b = 185$. Find a .

Logarithms of trigonometric ratios

The use of logarithms will facilitate computation in trigonometry. In order that it may not be necessary to compute the logarithm of each natural function used, a table of logarithms of these ratios is given.

For example, the natural tangent of 52° is 1.2799. The equivalent logarithmic ratio, or $\log \tan 52^\circ$, is 0.1072, which is the logarithm of 1.2799.

The natural $\sin 25^\circ$ is .4226; the $\log \sin 25^\circ$ is $9.6259 - 10$, written in the table as 9.6259.

The natural functions, given in the table on page 440, are decimal fractions less than 1, except for the tangents of angles between 45° and 90° , and cosine 0° and sine 90° . Therefore, with these exceptions, the characteristics of the logarithms of the trigonometric functions are negative. In order to avoid printing the negative characteristics, the characteristic of every logarithm, with the exceptions noted above, is printed 10 too large. Hence 10 must be subtracted from all such characteristics. The -10 is not printed, but it must be understood.

For illustration, $\log \cos 67^\circ 30'$, which equals $9.5828 - 10$, is given in the table as 9.5828, while $\log \tan 67^\circ 30'$ is given as 0.3828.

TABLE OF LOGARITHMS OF TRIGONOMETRIC RATIOS

ANGLE	LOG SIN	LOG COS	LOG TAN
0.00°		0.0000	
0.50	7.9408	0.0000	7.9409
1.00	8.2419	9.9999	8.2419
1.50	8.4179	9.9999	8.4181
2.00	8.5428	9.9997	8.5431
2.50	8.6397	9.9996	8.6401
3.00	8.7188	9.9994	8.7194
3.50	8.7857	9.9992	8.7865
4.00	8.8436	9.9989	8.8446
4.50	8.8946	9.9987	8.8960
5.00	8.9403	9.9983	8.9420
5.50	8.9816	9.9980	8.9836
6.00	9.0192	9.9976	9.0216
6.50	9.0539	9.9972	9.0567
7.00	9.0859	9.9968	9.0891
7.50	9.1157	9.9963	9.1194
8.00	9.1436	9.9958	9.1478
8.50	9.1697	9.9952	9.1745
9.00	9.1943	9.9946	9.1997
9.50	9.2176	9.9940	9.2236
10.00	9.2397	9.9934	9.2463
10.50	9.2606	9.9927	9.2680
11.00	9.2806	9.9919	9.2887
11.50	9.2997	9.9912	9.3085
12.00	9.3179	9.9904	9.3275
12.50	9.3353	9.9896	9.3458
13.00	9.3521	9.9887	9.3634
13.50	9.3682	9.9878	9.3804
14.00	9.3837	9.9869	9.3968
14.50	9.3986	9.9859	9.4127
15.00	9.4130	9.9849	9.4281
15.50	9.4269	9.9839	9.4430
16.00	9.4403	9.9828	9.4575
16.50	9.4533	9.9817	9.4716
17.00	9.4659	9.9806	9.4853
17.50	9.4781	9.9794	9.4987
18.00	9.4900	9.9782	9.5118
18.50	9.5015	9.9770	9.5245
19.00	9.5126	9.9757	9.5370
19.50	9.5235	9.9743	9.5491
20.00	9.5341	9.9730	9.5611
20.50	9.5443	9.9716	9.5727
21.00	9.5543	9.9702	9.5842
21.50	9.5641	9.9687	9.5954
22.00	9.5736	9.9672	9.6064
22.50	9.5828	9.9656	9.6172
23.00	9.5919	9.9640	9.6279
23.50	9.6007	9.9624	9.6383
24.00	9.6093	9.9607	9.6486
24.50	9.6177	9.9590	9.6587
25.00	9.6259	9.9573	9.6687
25.50	9.6340	9.9555	9.6785

TABLE OF LOGARITHMS OF TRIGONOMETRIC RATIOS (*Continued*)

ANGLE	LOG SIN	LOG COS	LOG TAN
26.00°	9.6418	9.9537	9.6882
26.50	9.6495	9.9518	9.6977
27.00	9.6570	9.9499	9.7072
27.50	9.6644	9.9479	9.7165
28.00	9.6716	9.9459	9.7257
28.50	9.6787	9.9439	9.7348
29.00	9.6856	9.9418	9.7438
29.50	9.6923	9.9397	9.7526
30.00	9.6990	9.9375	9.7614
30.50	9.7055	9.9353	9.7701
31.00	9.7118	9.9331	9.7788
31.50	9.7181	9.9308	9.7873
32.00	9.7242	9.9284	9.7958
32.50	9.7302	9.9260	9.8042
33.00	9.7361	9.9236	9.8125
33.50	9.7419	9.9211	9.8208
34.00	9.7476	9.9186	9.8290
34.50	9.7531	9.9160	9.8371
35.00	9.7586	9.9134	9.8452
35.50	9.7640	9.9107	9.8533
36.00	9.7692	9.9080	9.8613
36.50	9.7744	9.9052	9.8692
37.00	9.7795	9.9023	9.8771
37.50	9.7844	9.8995	9.8850
38.00	9.7893	9.8965	9.8928
38.50	9.7941	9.8935	9.9006
39.00	9.7989	9.8905	9.9084
39.50	9.8035	9.8874	9.9161
40.00	9.8081	9.8843	9.9238
40.50	9.8125	9.8810	9.9315
41.00	9.8169	9.8778	9.9392
41.50	9.8213	9.8745	9.9468
42.00	9.8255	9.8711	9.9544
42.50	9.8297	9.8676	9.9621
43.00	9.8338	9.8641	9.9697
43.50	9.8378	9.8606	9.9772
44.00	9.8418	9.8569	9.9848
44.50	9.8457	9.8532	9.9924
45.00	9.8495	9.8495	0.0000
45.50	9.8532	9.8457	0.0076
46.00	9.8569	9.8418	0.0152
46.50	9.8606	9.8378	0.0228
47.00	9.8641	9.8338	0.0303
47.50	9.8676	9.8297	0.0379
48.00	9.8711	9.8255	0.0456
48.50	9.8745	9.8213	0.0532
49.00	9.8778	9.8169	0.0608
49.50	9.8810	9.8125	0.0685
50.00	9.8843	9.8081	0.0762
50.50	9.8874	9.8035	0.0839
51.00	9.8905	9.7989	0.0916
51.50	9.8935	9.7941	0.0994

TABLE OF LOGARITHMS OF TRIGONOMETRIC RATIOS (*Continued*)

ANGLE	LOG SIN	LOG COS	LOG TAN
52.00°	9.8965	9.7893	0.1072
52.50	9.8995	9.7844	0.1150
53.00	9.9023	9.7795	0.1229
53.50	9.9052	9.7744	0.1308
54.00	9.9080	9.7692	0.1387
54.50	9.9107	9.7640	0.1467
55.00	9.9134	9.7586	0.1548
55.50	9.9160	9.7531	0.1629
56.00	9.9186	9.7476	0.1710
56.50	9.9211	9.7419	0.1792
57.00	9.9236	9.7361	0.1875
57.50	9.9260	9.7302	0.1958
58.00	9.9284	9.7242	0.2042
58.50	9.9308	9.7181	0.2127
59.00	9.9331	9.7118	0.2212
59.50	9.9353	9.7055	0.2299
60.00	9.9375	9.6990	0.2386
60.50	9.9397	9.6923	0.2474
61.00	9.9418	9.6856	0.2562
61.50	9.9439	9.6787	0.2652
62.00	9.9459	9.6716	0.2743
62.50	9.9479	9.6644	0.2835
63.00	9.9499	9.6570	0.2928
63.50	9.9518	9.6495	0.3023
64.00	9.9537	9.6418	0.3118
64.50	9.9555	9.6340	0.3215
65.00	9.9573	9.6259	0.3313
65.50	9.9590	9.6177	0.3413
66.00	9.9607	9.6093	0.3514
66.50	9.9624	9.6007	0.3617
67.00	9.9640	9.5919	0.3721
67.50	9.9656	9.5828	0.3828
68.00	9.9672	9.5736	0.3936
68.50	9.9687	9.5641	0.4046
69.00	9.9702	9.5543	0.4158
69.50	9.9716	9.5443	0.4273
70.00	9.9730	9.5341	0.4389
70.50	9.9743	9.5235	0.4509
71.00	9.9757	9.5126	0.4630
71.50	9.9770	9.5015	0.4755
72.00	9.9782	9.4900	0.4882
72.50	9.9794	9.4781	0.5013
73.00	9.9806	9.4659	0.5147
73.50	9.9817	9.4533	0.5284
74.00	9.9828	9.4403	0.5425
74.50	9.9839	9.4269	0.5570
75.00	9.9849	9.4130	0.5719
75.50	9.9859	9.3986	0.5873
76.00	9.9869	9.3837	0.6032
76.50	9.9878	9.3682	0.6196
77.00	9.9887	9.3521	0.6366
77.50	9.9896	9.3353	0.6542

TABLE OF LOGARITHMS OF TRIGONOMETRIC RATIOS (*Continued*)

ANGLE	LOG SIN	LOG COS	LOG TAN
78.00°	9.9904	9.3179	0.6725
78.50	9.9912	9.2997	0.6915
79.00	9.9919	9.2806	0.7113
79.50	9.9927	9.2606	0.7320
80.00	9.9934	9.2397	0.7537
80.50	9.9940	9.2176	0.7764
81.00	9.9946	9.1943	0.8003
81.50	9.9952	9.1697	0.8255
82.00	9.9958	9.1436	0.8522
82.50	9.9963	9.1157	0.8806
83.00	9.9968	9.0859	0.9109
83.50	9.9972	9.0539	0.9433
84.00	9.9976	9.0192	0.9784
84.50	9.9980	8.9816	1.0164
85.00	9.9983	8.9403	1.0580
85.50	9.9987	8.8946	1.1040
86.00	9.9989	8.8436	1.1554
86.50	9.9992	8.7857	1.2135
87.00	9.9994	8.7188	1.2806
87.50	9.9996	8.6397	1.3599
88.00	9.9997	8.5428	1.4569
88.50	9.9999	8.4179	1.5819
89.00	9.9999	8.2419	1.7581
89.50	10.0000	7.9408	2.0591
90.00	10.0000		

Use of logarithms in solving right triangles

EXERCISES

In the following exercises use the table of logarithms (pages 396, 397) and the table of logarithms of trigonometric ratios (pages 444–447).

1. In right triangle ABC , $A = 75^\circ$ and $b = 21.43$ feet. Find a .

SOLUTION

$$\frac{a}{b} = \tan A$$

$$a = b \tan A$$

$$\begin{aligned} \log a &= \log b + \log \tan A \\ &= \log 21.43 + \log \tan 75^\circ \end{aligned}$$

$$\log b = 1.3310$$

$$\log \tan 75^\circ = 0.5719$$

$$\log a = 1.9029$$

$$\therefore a = 79.97 \text{ feet}$$



2. In right triangle ABC below, $B = 43^\circ 48'$ and $b = 50.94$ feet. Find c .

SOLUTION

$$\frac{b}{c} = \sin B$$

$$b = c \sin B$$

$$\frac{b}{\sin B} = c$$

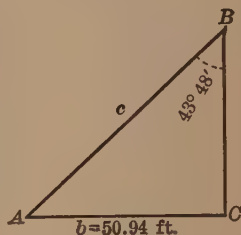
$$\begin{aligned} \log c &= \log b + \text{colog} \sin B \\ &= \log 50.94 + \text{colog} \sin 43^\circ 48' \end{aligned}$$

$$\log b = 1.7071$$

$$\text{colog} \sin 43^\circ 48' = 0.1598$$

$$\log c = 1.8669$$

$$\therefore c = 73.60 \text{ feet}$$

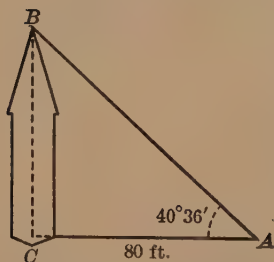


In right triangle ABC , with the right angle at C , find, from the parts given, the parts required:

GIVEN

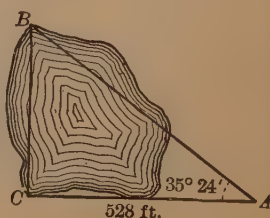
REQUIRED

- | | | |
|-----------------------|--------------------|------------------|
| 3. $A = 41^\circ 30'$ | $b = 164$ feet | a, c , and B |
| 4. $B = 35^\circ 48'$ | $a = 24.65$ feet | b, c , and A |
| 5. $c = 500$ feet | $A = 18^\circ 18'$ | a, b , and B |
| 6. $c = 17.3$ feet | $B = 25^\circ 45'$ | a, b , and A |
| 7. $A = 42^\circ 16'$ | $a = 24.53$ feet | b, c , and B |



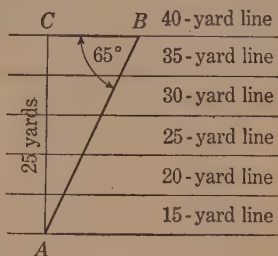
8. The angle of elevation of the top of a church steeple, 80 feet on level ground from a point directly beneath the top of the steeple, is $40^\circ 36'$. Find the height of the steeple.

9. Two points, B and C , are on opposite sides of a pond. The line CA is at right angles to the line of sight from C to B . If the angle BAC is $35^\circ 24'$ and CA is 528 feet, find the distance BC .



10. A tent pole is 20.5 feet high, and the ropes from its top to stakes in the ground are 34.75 feet long. What angle does a rope make with the level ground?

11. In a game between two high-school football teams, a player ran in a straight line from A , on the 15-yard line, to B on the 40-yard line. The line AB makes an angle of 65° with the line BC . Find the distance from A to B .



In the following let ABC be the right triangle, with C the right angle. Find, from the given parts, the parts required.

GIVEN

REQUIRED

12. $A = 50^\circ 30'$, $a = 25$

 b , c , and B

13. $B = 35^\circ 20'$, $b = 48$

 a , c , and A

14. $a = 76$, $B = 57^\circ$

 b , c , and A

15. $A = 25^\circ 15'$, $b = 85$

 a , c , and B

16. $B = 67^\circ 10'$, $b = 100$

 a , c , and A

17. $c = 250$, $A = 63^\circ 30'$

 a , b , and B

18. $B = 76^\circ 20'$, $c = 340$

 a , b , and A

19. $a = 124$, $A = 72^\circ 45'$

 b , c , and B

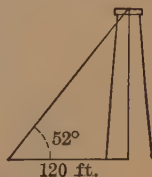
20. $A = 71^\circ 50'$, $c = 32.6$

 a , b , and B

21. $b = 14.5$, $B = 68^\circ 12'$

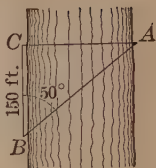
 a , c , and A

22. The angle of elevation of the top of a high chimney, as observed from a point 120 feet from the foot of the chimney, is 52° . Find the height of the chimney.

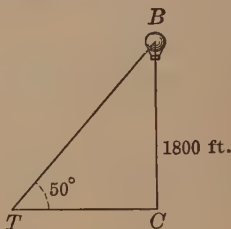
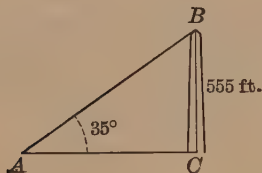


23. The angle of elevation of an airplane at a point P on level ground is $48^\circ 12'$. The point Q on the ground directly under the airplane is 506.3 yards from P . Find the height of the airplane.

24. In order to find the width of a river, a distance BC is measured along the bank and the angle B is found. The point A is directly opposite C on the other bank, so that ABC is a right triangle. BC is 150 feet. $B = 50^\circ$. Find the width of the river.



25. The Washington Monument is 555 feet in height. A man is standing in a place where the angle of elevation of the top of the monument is 35° . How far from the base of the monument is he standing?



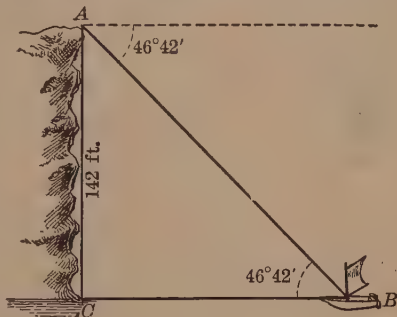
26. B represents a balloon 1800 feet directly over camp C . The angle of elevation of the balloon from trench T is 50° . Find the distance from the trench to the camp.

27. A tree 66.25 feet high stands at the edge of a level swamp. At a point directly across the swamp on the opposite edge, the angle of elevation of the top of the tree is $10^\circ 42'$. Find the width of the swamp.

28. A wire cable 71.5 feet long is attached to the top of a derrick. The other end of the cable is fastened to a stake driven into the ground. The cable makes an angle of $69^\circ 30'$ with the ground. Find the height of the derrick.

29. A pole outside a radio station is broken by the wind. Its top strikes the level ground $25\frac{1}{2}$ feet from the foot of the pole, and the upper part makes an angle of $43^\circ 16'$ with the ground. Find the entire length of the pole.

30. From A , the top of a vertical cliff at the water's edge, 142 feet above the level of the ocean, the angle of depression of a sailboat, B , is $46^\circ 42'$. Find the distance from C , the foot of the cliff, to the sailboat.



31. A surveyor wishes to obtain the distance between two points, P and Q , on opposite sides of a swamp. From the point P to point Q a line is sighted. A line PR , 350 yards long, is laid off at right angles to line of sight PQ . The angle PRQ is found to be $32^\circ 18'$. Find the distance from P to Q .

32. An airplane, A , is directly above a signal station, S . From P , a point 440 yards from S on a level plane, the angle of elevation of the airplane is $20^\circ 24'$. At what height above the signal station is the airplane?

33. From a point on shore the angle of elevation of the top of a harbor lighthouse, 65 feet high, is $15^\circ 30'$.

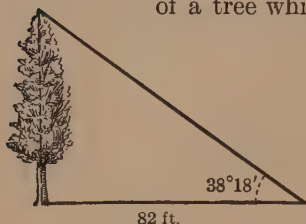


Find the distance from the point on shore to the foot of the lighthouse.

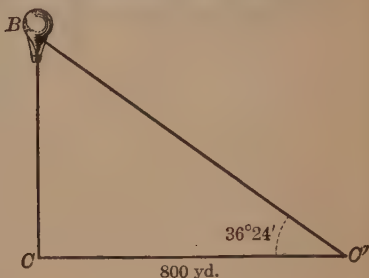
34. The Washington Monument is 555 feet high. What is the angle of elevation of the top of the monument from a point 300 feet from the foot, on the same horizontal plane?

35. A ladder 25 feet long strikes the side of the building 21 feet from the level ground. Find the angle which the ladder makes with the side of the building.

36. If the altitude of the sun is $38^\circ 18'$, what is the height of a tree which casts a shadow 82 feet long?



37. Two army camps, C and C' , situated on a level plane, are 800 yards apart. Above C is a balloon B , anchored to the ground. If the angle of elevation of the balloon B from C' is $36^\circ 24'$, find the height of the balloon.



38. A rope 83 feet long is attached to the top of a building and makes an angle of $62^\circ 20'$ with the level ground. Find the height of the building.

39. From the top of a lighthouse 75 feet high, on the shore of Lake Erie, the angle of depression of a steamer is $21^\circ 48'$. How far from the lighthouse is the steamer?

40. From two cruisers A and B , so situated that a straight line joining them passes exactly beneath a lighthouse, the



angles of elevation of C , the top of the lighthouse, are respectively $15^\circ 18'$ and $20^\circ 48'$. If the lighthouse is 85 feet high, find the distance between the cruisers.

41. Two signal towers on a level plane are each 100 feet in height. From a point on a straight line joining them the angle of elevation of the first is $12^\circ 36'$ and of the second, $19^\circ 40'$. Find the distance between the signal towers.

CHAPTER XXIII

GENERAL REVIEW

DEFINITIONS AND PRINCIPLES

1. How does the graph of a quadratic equation show that the roots are real and equal? real and unequal? imaginary?
2. What is the form of the graph of a simple equation? of two inconsistent equations?
3. What is the form of the graph of an equation like $ax^2 + bx + c = 0$?
4. (a) What is meant by the minimum point of a graph?
(b) Solve graphically the following equation and indicate the minimum point of the graph:

$$x^2 + 6x + 5 = 0$$

5. What powers of $\sqrt{-1}$ are real? imaginary?
6. Illustrate how an extraneous root may be introduced in the solution of an equation; how a root may be lost. In the latter case, what precautions should be observed?
7. Define and illustrate homogeneous equations; symmetric equations.
8. Explain two methods of forming a quadratic equation when the roots are given. Illustrate by forming the equations whose roots are 5 and $-\frac{2}{5}$; $2 + 3\sqrt{3}$ and $2 - 3\sqrt{3}$; $\sqrt{-2} + 1$ and $\sqrt{-2} - 1$; $3 \pm 2\sqrt{5}$.
9. What is the discriminant of a quadratic equation? Illustrate in $5x^2 - 8 = 7x$. What can we tell about the roots of the equation by inspecting the discriminant?

10. Define and illustrate a variable; a constant; a function; dependence; variation.

11. Without solving a quadratic equation, how can we tell what is the sum of its roots? the product of its roots? Illustrate in the equation $2x^2 + 8x = 7$.

12. What is the factor theorem? Illustrate its application in factoring:

$$x^3 - 8x^2 + 19x - 12 \quad x^4 - ax^3 - a^2x^2 - a^3x - 2a^4$$

13. Using the factor theorem, show when $x^n - y^n$ can be exactly divided by $x - y$; by $x + y$.

14. What is the binomial theorem? Pascal's triangle? Give the r th term of the expansion of $(a + b)^n$.

15. What will be the sum of the coefficients of the expansion of $(a - b)^{10}$? of $(a + b)^5$? Give a practical application of the use of the binomial expansion.

16. Define logarithm; base; mantissa; characteristic.

17. What does $\log ax$ equal? $\log \frac{a}{x}$? $\log x^a$? $\log \sqrt[a]{x}$?

18. Define antilogarithm; interpolation.

19. Define and illustrate two different progressions. Name the progression to which each of the following series belongs:

(a) 3, 8, 13, ...

(c) 12, 8, 4, ...

(b) $3, 4\frac{1}{2}, 6\frac{3}{4}, \dots$

(d) $12, 8, 5\frac{1}{3}, \dots$

20. What does each of the following formulas represent?

(a) $\frac{n}{2}(a + l)$

(c) $\frac{ar^n - a}{r - 1}$

(e) ar^{n-1}

(b) $\frac{rl - a}{r - 1}$

(d) $a + (n - 1)d$

(f) $\frac{a}{1 - r}$

21. Define and illustrate: sine; cosine; tangent. Name and define three other angular functions.

22. Draw a diagram and show that the tangent of an angle equals the sine divided by the cosine.

23. Distinguish between power and root. Illustrate each.

24. Give the law of exponents in finding powers; roots.
25. For what values of n between 1 and 12 is $(-2)^n$ positive? negative? What is the sign for any power of a positive number?
26. How is a fraction raised to a power? How is its root found? Raise $\frac{9}{16}$ to the second power. $\sqrt{\frac{9}{16}} = ?$
27. How is the power of a product found? the root?
28. What operation is indicated by a radical sign? In what other way may this operation be indicated? Illustrate.
29. How many values has $\sqrt{25}$? What is the principal square root of 25? What is the principal cube root of -8 ?
30. What is the index of a root? What index is meant when none is expressed?
31. What is the sign of an odd root of a number? of an even root of a number?
32. What is the meaning of x^0 ? of x^{-2} ?
33. When a number has a fractional exponent, what does the numerator of the exponent show? the denominator?
34. How may any factor be transferred from one term of a fraction to the other? Write $\frac{x^2}{a^{-3}b}$ without a denominator.
35. Define and illustrate: radical, radicand, entire surd, mixed surd, and conjugate surds.
36. What is meant by the order of a radical? Illustrate by giving radicals of different orders.
37. How may a radical of the second order be represented graphically? Represent graphically $\sqrt{26}$; $\sqrt{3}$; $2\sqrt{2}$.
38. What is a rational number? an irrational number? From the following select the rational numbers:
- 8 $\frac{2}{9}$ $\sqrt{3}$ $\sqrt[3]{8}$ $-7\frac{2}{3}$ $\sqrt{25}$ $5^{\frac{1}{2}}$
39. Are $\sqrt[3]{15}$ and $\sqrt{1 + \sqrt{2}}$ radicals? surds? Are all radicals surds? Are all surds radicals?

40. How may the coefficient of a radical be placed under the radical sign? Express as entire surds:

$$\frac{1}{3}\sqrt{2} \quad 9\sqrt{bc} \quad \frac{2}{7}\sqrt[3]{x^2y}$$

41. How can a rational factor be written as the coefficient of an irrational factor? Illustrate by reducing $\sqrt{40b^2c}$, $\sqrt{72mn^2}$, and $\sqrt[3]{24}$ to this form.

42. What are similar radicals? When numbers have fractional exponents with different denominators, how may they be multiplied?

43. Give the relation between the sides of a right triangle.

44. Define rationalization; rationalizing factor; surd; conjugate surds.

45. Give the rationalizing factors for the following expressions:

$$(a) x + \sqrt{y} \qquad (c) \sqrt{x} - \sqrt{y} + \sqrt{z}$$

$$(b) \sqrt{x} - \sqrt{y} \qquad (d) \sqrt{x} - \sqrt{y} - \sqrt{z}$$

46. Why do equations involving radical expressions sometimes have no root?

47. Define and illustrate a real number; an imaginary number; a complex number. Tell what kind of number each of the following is:

$$\sqrt{-4} \quad \sqrt[3]{-8} \quad 3 + \sqrt[3]{-5} \quad 3\sqrt{-3} \quad 7 + \sqrt{-6}$$

FUNDAMENTAL OPERATIONS

1. Add $5a + 7b$, $9a - 11b + c$, $-5a + 2b - 3c$, and $a + 3b + c$. Check when $a = 8.5$, $b = 2.5$, and $c = 3$.

2. Add $5a^2 + 3ab + 2b^2$, $-3a^2 + 7ab$, and $a^2 - 7b^2$, and subtract the sum from $10a^2 + ab - b^2$.

3. What must be added to $ax^2 - by^2 + cz^2$ to give $px^2 + qxy - ry^2$?

4. Subtract $3a^2 - 7a + 10$ from $5a^2 + 6a - 7$ and check by the substitution of 2 for a .

5. From the sum of $2x^2 + 3xy - 7y^2$ and $3.8x^2 - 2.5xy + 6.1y^2$ subtract the difference between $4.2x^2 - 2.3xy$ and $3.7x^2 + 5.4y^2$.

6. What expression added to $15a^2 - 7ab + 5b^2$ will give $20a^2 + 5ab$? Check.

7. What expression subtracted from the sum of $5xy - 7xz + 3yz$ and $8xy + 5xz - yz$ will give $xy + xz - yz$? Perform the indicated operations and simplify each result:

8. $(x^3 - x - 2)(2x^3 + 3x + 4)$ Check with $x = 2$.

9. $(2a^n + b^{-n})(3a^n - 2b^{-n}) + (3a^n - b^{-n})^2$

10. $(3a^n - 2b)(2a^n + b) - (3a^n + 5b)^2$

11. $(x^2 + 5 - 3x^{-2})(2x^2 - 3 + 5x^{-2})$

12. $(3x^{n+2} - 4x^{n+1} + 6x^n)(2x^{n+1} - 3x^n)$

13. $(\frac{1}{2}a^2 + \frac{1}{3}a + \frac{1}{4})(\frac{1}{3}a + \frac{1}{4}) - (\frac{1}{4}a^2 - \frac{1}{3}a + \frac{1}{2})(\frac{1}{3}a - \frac{1}{4})$

14. $(2x^2 - 3x + 5)(3x^{-2} + 2x^{-1} - 6)$

Find the quotients:

15. $(4x^6 - 9x^4 + 25 - 14x^3 - x^2) \div (3x^2 - x - 5 + 2x^3)$ Check by multiplication.

16. $(x^5 + 50 - 70x + 37x^2) \div (x^2 - 2x + 10)$ Check with $x = 2$.

17. $(a^3 - 3abc + b^3 + c^3) \div (a + b + c)$ Check.

18. $(x^{-3} - 8y^6) \div (x^{-1} - 2y^2)$ Check.

19. $(a^{\frac{3}{2}} - ab^{\frac{1}{2}} + a^{\frac{1}{2}}b - b^{\frac{3}{2}}) \div (a^{\frac{1}{2}} - b^{\frac{1}{2}})$ Check.

20. $(6x^{-3} + 10x^{-2} - 4x^{-1} - 10 - 2x) \div (x^{-2} - 1)$

21. $(e^{2x} + 2 + e^{-2x}) \div (e^x + e^{-x})$ Check.

In the next four exercises use the method of synthetic division:

22. $(a^3 - 14a^2 + 28a - 42) \div (a - 6)$

23. $(x^3 - 19x^2 + 18x - 15) \div (x - 5)$

24. $(2x^3 + x^2 + 7x + 4) \div (x + 1)$

25. $(2x^3 + 7x + 20) \div (x - 2)$

26. What value must m have in order that $9x^2 - 30x - 4x^4 + m$ may be exactly divisible by $5 - 2x^2 - 3x$?

FACTORING

Factor:

1. $c^2 + c - 72$
2. $3y^2 - 7y - 6$
3. $9x^{4a} + 30x^{2a} + 25$
4. $3mr^2 - 363m^3$
5. $m^2 - 8mn + 16n^2 - 4R^2$
6. $64a^9 + 8$
7. $x^5 - y^5$
8. $x^{12} + y^{12}$
9. $a^6 - 64$
10. $m^2 + 16n^2 - 16 + 8mn$
11. $y^4 - 9y^3 + 21y^2 + y - 30$
12. $.14x^{2a} - .11x^ny^n - .15y^{2n}$
13. $2(m-n)^2 - 4(m-n) + 2$
14. $x^4 + 2x^3 - x - 2$
15. $x^3 + 3x^2 - 30x + 36$
16. $x^4 - 6x^2y^2 + y^4$
17. $10a - 7a^2 - 6a^3$
18. $x^6 + y^3$
19. $15a^2 - 77a + 10$
20. $x^3 - x^2 - x + 1$
21. $x^4 - 4x^2 + 3$
22. $4a^2 - 6b - 9 - b^2$
23. $4(3x - 2)^2 - y^2$
24. $9x^3(2a + b)^2 - x$
25. $x^4 - 16(x^2 - 2)^2$
26. $x^2 + 2y - 4y^2 - x$
27. $14x^2 - 5x - 24$
28. $x^4 - 4x^3 - x^2 + 16x - 12$
29. $(x + y)^3 + x + y$
30. $4m^2 + 9n^2 - 16 + 12mn$
31. $acx^2 - bcx + adx - bd$
32. $x^5 + 32$
33. $x^4 + 4$
34. $a^3 + 8$
35. $m^2 - 2mn + n^2 + 5m - 5n$
36. $4x^4 - 21x^2y^2 + 9y^4$
37. $a^4 - 4a^3 - a^2 + 16a - 12$
38. $x^{3m} - x^{2m} + x^{2m+1} - x^{m+1}$
39. $x^2 + .1x - .06$
40. $x^{6a} - y^6$
41. $x^3 + 5x^2 - 2x - 24$
42. $2ac + 6ad + bc + 3bd$
43. $3x^{3a} + 3x^{2a} - 18x^a$
44. $25a^{2n} - (2b + c)^2$
45. $9a^4 - 15a^2b + 6.25b^2$
46. $21a^2 - 29ab^2 - 10b^4$
47. $3a^3 + 6a^2 - 15a - 18$
48. $24a^2 - 22ab - 35b^2$
49. $6a^3 - 13a^2b + ab^2 + 2b^3$
50. $25x^2 + 12yz - 9y^2 - 4z^2$
51. $x^3 - 8x^2 + 17x - 10$
52. $x^5 + 7x^4 - 7x^3 - 49x^2 + 6x + 42$
53. $4(a - b)^2 - 12(a - b)c + 9c^2$

FRACTIONS

Perform the indicated operations and express each result in the simplest form :

$$1. \frac{x^4 - 9x^3 - 36x^2}{a^2x^2 - 10a^2x + 9a^2} \times \frac{x^4 - 4x^2 + 3}{x^4 - 7x^2 - 18} \div \frac{x^3 + x^2}{a^2x^2 + 2a^2}$$

$$2. \frac{c^2 - 22}{c^2 - 2c - 8} + \frac{c - 5}{4 - c}$$

$$4. \left(\frac{ab}{a^2 + b^2} \right) \left(1 - \frac{a^4}{b^4} \right)$$

$$3. \frac{\frac{3}{4} + \frac{2}{3} - 1}{\frac{5}{6} + \frac{1}{2}}$$

$$5. \frac{\frac{2m + n}{m + n} - 1}{1 - \frac{n}{m + n}}$$

$$6. \frac{x^2 - 4x + 4}{10x - 21} \div \frac{x^2 + x - 6}{10x^2 + 9x - 63}$$

$$7. \frac{2 - \frac{2x - 3}{x + 1}}{3 - \frac{3x^2 - 13}{x + 1}}$$

$$8. \frac{\frac{a - 3}{3b} - \frac{3b}{a - 3}}{\frac{1}{3b} - \frac{1}{a - 3}}$$

$$9. \left(\frac{4a - 10}{a^2 - 5a + 6} + \frac{3}{3 - a} \right) \div \left(1 - \frac{a + 1}{a^2 - 2a - 3} \right)$$

$$10. \left(\frac{a^3}{4b} + \frac{a^2}{3} \right) \div \left(\frac{a^3}{5b} - \frac{a^2}{2} \right) \quad 11. \frac{x^4 - y^4}{x^2 - 2xy + y^2} \div \frac{x^2 + xy}{x - y}$$

$$12. \left(\frac{a + 2}{a} + \frac{2}{a - 3} \right) \times \left(\frac{a}{a - 2} - \frac{3}{a + 3} \right)$$

$$13. \frac{1 - 2x}{1 - x^3} \div \frac{1 - 2x + x^2 - 2x^3}{1 + 2x + 2x^2 + x^3}$$

$$14. \frac{3ax^2 + 3b}{6x - 3} + \frac{2(bx + ax^2)}{1 - 4x^2} - \frac{ax^2 - b}{2x + 1}$$

$$15. \left(\frac{3a^4 - 75a^2}{3a - 7} \right) \left(6a - \frac{7}{a} - 11 \right) \div \left(2 + \frac{5}{a^2} + \frac{11}{a} \right)$$

$$16. \frac{5x^2 - 9x + 11}{2x^2 + x - 6} - \frac{x - 4}{3 - 2x} \div \frac{3x - 1}{x + 2}$$

$$17. \left(\frac{6x}{x^2 - 4} + \frac{3}{2 - x} \right) \div \frac{3}{x^2 - x - 6}$$

Perform the indicated operations and express each result in the simplest form:

$$18. \left(\frac{a^2}{b-c} + \frac{b^2}{c-b} \right) \left(\frac{b}{a^2-b^2} + \frac{c}{b^2-a^2} \right)$$

$$19. \frac{1-x^2}{x^2-4x-5} \div \frac{x^2+2x-3}{25-x^2}$$

$$20. \frac{4x^2-1}{6x^2-13x+6} \times \frac{4x^2-12x+9}{4x^2-4x-3} \div \frac{2(ax-bx)-a+b}{3ax+3bx-2b-2a}$$

$$21. \frac{4}{a-3} - \frac{a+1}{a^2+3a+9} + \frac{a^2-38a-3}{a^3-27}$$

$$22. \frac{\frac{a-5}{2} - 7 + \frac{24}{a}}{\frac{9-3a}{a}}$$

$$23. \frac{1 - \frac{5x-7}{x^2-1}}{2x-11 + \frac{26-2x}{x+1}}$$

EVALUATION OF RADICALS AND EXPONENTIAL EXPRESSIONS

Find the value of:

$$1. 8^{-\frac{2}{3}} \times 16^{\frac{3}{4}} \times 2^0 \times (\sqrt{3})^4 \times (\sqrt{-2})^2$$

$$2. \frac{3a\sqrt{a^2-2ab}}{4b} \text{ (to nearest tenth) if } a = 8.5 \text{ and } b = 3.2$$

$$3. \sqrt{98x^2} - \sqrt{\frac{9}{8}} - \frac{1}{8}\sqrt{2(4x-6)^2} \text{ if } x = 3$$

$$4. \frac{\left(-\frac{27}{4a}\right)^{\frac{1}{3}} - 7 + \frac{24}{a}}{9a^{-1} - 3} \text{ if } a = 2$$

$$7. 64^{-\frac{2}{3}} - 3(13)^0 + 12(2)^{-2}$$

$$8. (\sqrt{6} - 2\sqrt{5})^2$$

$$9. \frac{1}{8}\sqrt{90} - 4\sqrt{\frac{5}{8}} - \frac{\sqrt{2}}{\sqrt{5}}$$

$$5. \frac{2+2^{-1}}{5} + (-8)^0 - 4^{\frac{3}{2}}$$

$$10. 2\sqrt{150} - 4\sqrt{54} + 6\sqrt{24}$$

$$6. (.125)^{-\frac{2}{3}} + \frac{3}{2+2^{-1}}$$

$$11. \frac{2 - \sqrt{3} + \sqrt{5}}{2 + \sqrt{3} - \sqrt{5}}$$

12. $6\sqrt{4\frac{1}{8}} - \sqrt{96} - \sqrt[3]{\frac{2}{9}} - \frac{4}{3}\sqrt{\frac{27}{8}} + \frac{1}{3}\sqrt[3]{-48}$

13. $x^3 - 3x^2 + x + 1$ if $x = 1 - \sqrt{2}$

14. $\frac{a^2 - ab + b^2}{a^3 + b^3}$ if $a = \sqrt{2} + 1$, $b = \sqrt{2} - 1$

15. $x^2 + 2x - 4$ if $x = 2\sqrt{5} - 1$

16. Rationalize the denominator of:

(a) $\frac{3\sqrt{5} - 2\sqrt{2}}{2\sqrt{5} + 4\sqrt{2}}$ (b) $\frac{3 + 2\sqrt{-1}}{3 - 2\sqrt{-1}}$ (c) $\frac{2 + \sqrt{-3}}{3 + \sqrt{-2}}$

17. Find the square root of:

(a) $33 + 20\sqrt{2}$ (b) $14 + 3\sqrt{3}$ (c) $35 - 3\sqrt{96}$

18. Simplify:

(a) $\frac{a^{\frac{3}{2}} + ab}{ab - b^3} - \frac{\sqrt{a}}{\sqrt{a} - b}$ (b) $\frac{a^{-1}b\sqrt{c}}{a^{\frac{2}{3}}} \div \sqrt{\frac{a^2b^{-1}}{c^{-3}}}$

Find the value of:

19. $4\sqrt{24} + 2\sqrt{54} - \sqrt{6} + 3\sqrt{96} - 5\sqrt{150}$

20. $3\sqrt{75} - 3\sqrt{12} + 2\sqrt{48} - 2\sqrt{192}$

21. $(3\sqrt{2} - 2\sqrt{3})(4\sqrt{2} + 3\sqrt{3})$

22. $\sqrt{16a^3 - 48a^2b} + \sqrt{9ab^2 - 27b^3}$ if $a = 5$, $b = \frac{1}{3}$

23. $\frac{3^0x + 4x^{-1}}{x^{-\frac{2}{3}}}$ if $x = 8$

26. $\sqrt[5]{4} \times \sqrt[5]{16}$

27. $\sqrt[5]{3} \div \sqrt[5]{24}$

24. $(1 - \sqrt{3} + \sqrt{5})(\sqrt{3} - \sqrt{5})$

28. $3\sqrt{14}\sqrt{21} - \frac{\sqrt{12}}{\sqrt{2}}$

25. $(3\sqrt{6} - 2\sqrt{5})^2$

29. $(3\sqrt{5} - 2\sqrt{2})(2\sqrt{5} + 4\sqrt{2})$

30. $\frac{5 - \sqrt{3}}{3\sqrt{3}}$ (to nearest hundredth)

31. $.6 \times 32^0 + .8 \times 4^{-2} - 12 \times 9^{-\frac{1}{2}} + 5 \times 8^{\frac{1}{3}}$

32. $\frac{2\sqrt{3} + 3\sqrt{2}}{\sqrt{6} + \sqrt{3}}$ (to nearest tenth)

33. $\frac{\sqrt{5} - \sqrt{2}}{\sqrt{5} + \sqrt{2}}$ (to nearest hundredth)

Find the value of:

34. $2\sqrt{3} - \frac{1}{2}\sqrt{12} + 4\sqrt{27} + \sqrt{\frac{1}{3}} - \sqrt{108}$
35. $8^{\frac{3}{2}} + 8^0 + 8^{-1} + 3.72 \times 10^{-4} + 0 \times 16^{\frac{3}{2}}$
36. $\frac{2\sqrt{5} - 6}{-3\sqrt{5}}$ (to nearest tenth)
37. $25^0 + .25^{\frac{1}{2}} - 8^{\frac{1}{3}} \times 4^{-\frac{1}{2}} + .027^{\frac{1}{3}}$
38. $\frac{2}{3 - \sqrt{5}} + \frac{\sqrt{3} + \sqrt{5}}{\sqrt{3} - \sqrt{5}}$ (to nearest tenth)
39. $(\sqrt{6} - 2\sqrt{5})^2 - (2\sqrt{6} + 3\sqrt{5})(\sqrt{6} - \sqrt{5})$
40. $\frac{(2\sqrt{3} - 3\sqrt{5})(\sqrt{3} + 2\sqrt{5})}{\sqrt{3} - \sqrt{5}}$
41. $\frac{2\sqrt{10} - 5}{\sqrt{5} + 2\sqrt{2}}$ (to nearest tenth)

LINEAR EQUATIONS

Solve:

1. $\frac{5}{x-7} - \frac{7}{x+3} = \frac{5x+1}{x^2-4x-21}$ Check.
2. $\frac{1}{4}(3x-5) - \frac{1}{12}(9x-7) = \frac{2}{3}x$
3. $\frac{2x}{6} - \frac{x-3}{3} = 12 - \frac{x+4}{2} - x$
4. $\frac{x}{2} - 2\left(\frac{4x}{5} - 3\right) = 4 - \frac{3}{2}\left(\frac{x}{2} + 1\right)$ Check.
5. $\frac{7k}{5} - \frac{1}{14}(k-11) = \frac{3}{7}(k-25) + 34$
6. $\frac{16x+3}{10} - \frac{2x-5}{5x-1} = \frac{8x-1}{5}$
7. $.25x + 17 = .2x - 15$ Check.
8. $3.73y - 4.26 = 2.89y - 1.74$
9. $(x+2)^2 + x^2 = (x-2)(x-1) + (x+1)^2$
10. $x - .2x + \frac{x+1}{.4} = 7.12$

11. Two partners in business agree to divide their profits of \$5290 so that A shall receive 30% more than B. Find the share of each.

12. A man who had 6 hours at his disposal rode out into the country at the rate of 12 miles an hour and returned on foot at the rate of 3 miles an hour. Find the distance he rode.

13. Separate 42 into two parts such that the greater part divided by the less equals 2, with a remainder of 3.

14. A man has $3\frac{1}{2}$ hours at his disposal. How far can he ride out of town in an automobile at 27 miles an hour and return in time by a trolley car at 18 miles an hour?

15. A schoolroom seating 40 pupils has some single seats and some double seats. If there are twice as many double seats as single seats, find the number of each.

16. Find two numbers differing by 12, such that 7 times the smaller exceeds the greater by 30.

17. A starts from a certain place and walks $3\frac{1}{2}$ miles an hour; $2\frac{1}{2}$ hours later B starts from the same place and rides in the same direction $8\frac{1}{2}$ miles an hour. In how many hours will B overtake A?

18. In a purse containing \$2.50 there are one half as many quarters as nickels and one third as many dimes as nickels. How many pieces of each kind are there?

19. The admission tickets for an entertainment were \$.25 each for adults and \$.10 each for children. The turnstile showed that 385 persons entered and the gate receipts were \$62.65. How many children entered?

20. One machine can make 100 dozen buttons in 48 minutes; a smaller machine can make the same number in 1 hour and 12 minutes. How long will it take both machines together to make 100 dozen buttons?

21. Find three consecutive numbers such that if they are divided by 2, 3, and 4 respectively, the sum of the quotients will equal the next higher consecutive number.

RATIO, PROPORTION, AND VARIATION

1. If $x^{-\frac{2}{3}}:2 = 1:x^{\frac{1}{3}}$, find x .

2. If $a:b = c:d$, prove that

$$a(a+b+c+d) = (a+b)(a+c).$$

3. Show that the mean proportional between two numbers is the geometric mean between these numbers.

4. Mr. Ash and Mr. Bond were in partnership, their profits being divided in the ratio of 4 to 5 in favor of Mr. Bond. Mr. Cox and Mr. Dunn also were in partnership, their profits being divided in the ratio of 4 to 7 in favor of Mr. Dunn. The two partnerships combined into one with a capital of \$80,000, the first men putting in \$9 to every \$11 of the second partners. How should a gain of \$12,000 be divided among the four men?

Write a formula to represent each of the relations expressed in the following examples 5 to 12:

5. The distance an automobile travels varies with the number of hours it goes and with the speed.

6. The number of days it takes to do a certain piece of work varies inversely with the number of hours worked each day.

7. The distance a body falls varies as the square of the number of seconds it falls.

8. The volume of a rectangular box varies with its length, its width, and its thickness.

9. The pitch of a musical note varies with the number of vibrations made per second.

10. The force required to balance a given weight with a lever varies inversely as its distance of application from the fulcrum.

11. The per capita cost of instruction for students in a school varies directly as the number of teachers employed and inversely as the number of students taught.

12. The number of seconds required for the vibration of a pendulum varies as the square root of the length of the pendulum.

13. Two numbers are in the ratio of 3 to 4. If 7 is subtracted from each of them, the remainders are in the ratio of 2 to 3. Find the numbers.

14. Two numbers are in the ratio 3 : 5. If 4 is subtracted from the smaller number and 8 is added to the larger number, the ratio becomes 5 : 12. Find the numbers.

15. A dairyman wishes to combine milk and cream to make 40 gallons of a mixture which shall contain 50% butter fat. If the milk contains 5% butter fat and the cream contains 75% butter fat, how many gallons of each must be used?

16. If 3 quarts of gasoline are mixed with 5 quarts of kerosene, how many quarts of kerosene must be added to make the mixture $\frac{3}{4}$ kerosene?

17. How much water must be added to a gallon of alcohol which is 90% pure to reduce it to 80% pure?

SIMULTANEOUS LINEAR EQUATIONS

Solve:

$$1. \quad \begin{cases} .3x - .2y = .7 \\ .5x - 8.5 = -1.5y \end{cases}$$

$$5. \quad \begin{cases} 3x + 4y = 12 \\ 7x + 2y = 17 \end{cases}$$

$$2. \quad \begin{cases} 5x - 3y = 3.5 \\ 4x + 7y = 31 \end{cases}$$

$$6. \quad \begin{cases} 6x - y = .9 \\ x + 2y = .8 \end{cases}$$

$$3. \quad \begin{cases} ax + by = b \\ cx - dy = d \end{cases}$$

$$7. \quad \begin{cases} .3x + .2y = 9.5 \\ .2x + .3y = 10.5 \end{cases}$$

$$4. \quad \begin{cases} x - 2y + 3z = 2 \\ 2x - 3y + z = 1 \\ 3x - y + 2z = 9 \end{cases}$$

$$8. \quad \begin{cases} x - 2y + 3z = 6 \\ 2x + 3y - 4z = 20 \\ 3x - 2y + 5z = 26 \end{cases}$$

9. A report states that in a certain factory 1200 men and women are employed. The average daily wage is \$3.40 for a man and \$1.80 for a woman. If the labor cost is \$3376 a day, how many men and how many women are employed?

10. On one occasion a man buys 3 tons of hard coal and 2 tons of soft coal, paying \$51. At the same prices he later buys 2 tons of hard coal and 6 tons of soft coal, paying \$62. What is the price of each per ton?

11. In sending a telegram there is a fixed rate for the first 10 words and a fixed rate for each additional word. If a message of 31 words costs \$.98 and a message of 45 words costs \$1.40, what are these two fixed rates?

12. A coffee mixture is made up of two brands differing in cost by \$.04 a pound. If the dealer makes 25% profit and if he sells his mixture of 4 pounds of the cheaper coffee and 3 pounds of the better coffee for \$2.60, find the cost per pound of each brand.

13. One pound of coffee and 5 pounds of sugar together cost \$.70. After the price of sugar has advanced $12\frac{1}{2}\%$ and the price of coffee 20%, 3 pounds of sugar and 2 pounds of coffee together cost \$.99. Find the price of each per pound.

14. If the greater of two numbers is divided by the less, the quotient is 4 and the remainder is 1. If the less is divided by the greater, the quotient is .24 and the remainder is 2. Find the numbers.

15. If I lend a certain sum of money for a specified time at 6% simple interest, the interest exceeds the loan by \$160. If the rate is 4% and the time $\frac{1}{2}$ as long, the loan exceeds the interest by \$480. Find the sum of money lent.

16. A man had \$4.50 in 24 coins, consisting of nickels, quarters, and half dollars. The number of nickels equaled the number of quarters increased by twice the number of half dollars. How many coins of each kind did he have?

FORMULAS

1. Solve for n the formula $c = \frac{en}{R + nr}$. Solve for R .

What is the value of R when $e = 1.5$, $n = 10$, $c = 8$, $r = .05$?

- (2) Solve for S the formula $C = \frac{S \times E}{S \times b + R}$. Find the

value of C to the nearest hundredth when $S = 4$, $E = 1.07$, $b = 2.4$, $R = 27$.

3. Evaluate $A = \sqrt{3.6\pi v^3}$ when $v = 6.281$.

4. Solve for d the formula $V = \sqrt{\frac{H \times 2500}{l \times \frac{13.9}{d}}}$. Find the

value of d when $H = 120$, $l = 1000$, $V = 10$.

5. Solve for L the formula $C = \frac{E}{\sqrt{R^2 + (2\pi nL)^2}}$.

6. If $P = \frac{v}{608.3} \sqrt{\frac{w}{d}} - .14d$, find P when $d = 12.5$, $w = 1250$, and $v = 2016$.

- (7) Find the diameter of a circle circumscribed about a triangle whose sides are 5, 7, and 8, using the formula

$d = \frac{abc}{2\sqrt{s(s-a)(s-b)(s-c)}}$, where d is the diameter, a ,

b , and c represent the sides, and s represents $\frac{a+b+c}{2}$.

Find the result correct to two decimal places.

8. Solve for v the formula $h = \frac{v^2}{2g}$. Find the value of v when $h = 25$ and $g = 32$.

9. Solve for c the formula $w = 59.6\sqrt{Lc}$. Find c when $w = 492$ and $L = 55,000$.

10. Evaluate $\frac{1}{t} \left(\frac{T^2}{R^2} - 1 \right)$ when $T = 8$, $R = 2$, $t = 10$.

11. Solve for M the formula $L = \frac{Mt - g}{t}$.
12. Solve for t the formula $s = \frac{a}{2}(2t - 1)$. Find the value of t when $a = 5.65$ and $s = 73.45$.
13. Solve for l the formula $s = \frac{a - rl}{1 - r}$. Find the value of l when $a = 3$, $r = 2$, and $s = 765$.
14. Solve for F the formula $\frac{1}{P} + \frac{1}{Q} = \frac{1}{F}$. Find F if $Q = 15.2$, $P = 3.3$.
15. Find the value of h from $r = \frac{s^2 + h^2}{2h}$. What does h equal when $s = 8$ and $r = 10$?
16. If $a = \frac{2b - 3}{b^2}$ and $C = \frac{a + 1}{a + 3}$, find C when $b = -1$.
17. Solve for r the formula $C = \frac{E}{R + r}$. Find the value of r to the nearest tenth when $C = 15.6$, $R = 4.3$, $E = 110.7$.
18. Eliminate a from the two formulas $l = ar^{n-1}$ and $s = \frac{rl - a}{r - 1}$ and derive a formula for l .

POWERS AND ROOTS

Expand the following:

- | | | |
|---|--------------------------------------|----------------------------------|
| 1. $(3a - 7b)^4$ | 5. $(2 - \frac{x}{3})^7$ | 9. $(7a^2 - 5b^{\frac{2}{3}})^4$ |
| 2. $(2x^2 + \frac{y}{x})^5$ | 6. $(\frac{2x}{y} - \frac{y}{3x})^5$ | 10. $(3a^2 + 5b)^5$ |
| 3. $(a^{\frac{1}{2}} - 2b^{\frac{1}{3}})^3$ | 7. $(\frac{5a}{3} - \frac{2}{a})^6$ | 11. $(4x - 3y^2)^6$ |
| 4. $(5a^{-\frac{1}{2}} + \frac{3}{5}b^{\frac{2}{3}})^4$ | 8. $(2a^2 + \frac{b}{2a})^8$ | 12. $(6r^2 + 3r^{-3})^5$ |
13. Write the first four terms of the expansion of:
- (a) $(a^2 - 5b)^{10}$ (b) $(a + 2b)^{15}$ (c) $(x^3 - 3y)^{20}$

14. Find the 7th term of the expansion of each of the following:

$$(a) \left(\frac{2a^2}{3} - \frac{3b}{2} \right)^{12} \quad (b) \left(\frac{2a}{b} - \frac{b}{3a} \right)^{10} \quad (c) (2a^2 - b^{\frac{1}{2}})^9$$

15. Find the value, correct to six decimal places, of each of the following:

$$(a) 1.05^5 \quad (b) 1.04^6 \quad (c) 1.03^7 \quad (d) 1.02^8$$

16. Mr. Leeds deposited \$250 in a savings bank paying 4% interest, compounded quarterly. How much should be credited to his account at the expiration of 3 years?

17. Find the interest which should be credited to a deposit of \$300 left in a bank for 5 years if the rate is 5%, compounded semiannually.

Find the square root of:

$$18. x^4 + \frac{3}{2}x^2 + \frac{1}{16} - 2x^3 - \frac{x}{2}$$

$$19. 4x^4 + y^4 + \frac{4}{3}xy^3 - \frac{8}{3}x^3y - \frac{32}{9}x^2y^2$$

$$20. x^2 + 25 + y^2 - 2xy + 10x - 10y$$

$$21. 5x^2 + 12x + 16x^6 + 4 - 23x^4 - 22x^3 + 8x^5$$

22. If $A = \sqrt{s(s-a)(s-b)(s-c)}$ represents the area of a triangle in terms of its sides and $s = \frac{a+b+c}{2}$, find the area when the sides are: 26, 28, 30; 27, 30, 51.

QUADRATIC EQUATIONS

1. Find, to the nearest tenth, the roots of $\frac{x^2}{4} - \frac{3x}{2} = \frac{5}{2}$.

2. Find, to the nearest hundredth, the roots of the equation $3m^2 + .5m = .07$.

3. For what values of k will the roots of the equation $y^2 - \frac{ky}{2} + 6\frac{1}{4} = 0$ be equal? real and unequal? imaginary?

4. Find the value of r if the difference of the roots of the equation $x^2 = 5x - r$ is 11. Find r if the roots are equal.

5. Without solving the equation, determine the nature of the roots of the equation $6x^2 + 7x - 6 = 0$.

6. Without solving or substituting, determine whether 4 and -3 are roots of the equation $x^2 - x + 10 = 0$.

7. Find, to the nearest hundredth, the value of x from $x^2 - 2.2x - 1.16 = 0$.

Solve:

$$8. \frac{5x}{3} = x + \frac{3-x}{2+x}$$

$$16. \frac{1}{2}(x+1) - \frac{x}{3}(2x-1) = \frac{2}{3}$$

$$9. x^2 + \sqrt{x^2 + 16} = 14$$

$$17. \frac{3x + \sqrt{3x - 3}}{3x - \sqrt{3x - 3}} = 1$$

$$10. \sqrt{x} + \sqrt{4+3x} = 6$$

$$11. \sqrt{2x+1} + \sqrt{3x+4} = 7$$

$$18. x - \sqrt{2x^2 - 7} = -3$$

$$12. \sqrt{5x-4} - \sqrt{2x-4} = 2$$

$$19. \sqrt{2+x} = 4 - \sqrt{10-3x}$$

$$13. \left(x - \frac{1}{x}\right)^2 + \frac{5}{6}\left(x - \frac{1}{x}\right) = 1$$

$$20. \sqrt{1 + \frac{x}{a}} - \sqrt{1 - \frac{a}{x}} = 1$$

$$14. (x-1)^{\frac{4}{5}} + (x-1)^{\frac{2}{5}} = 2$$

$$21. 2x^2 - 5nx = 3n^2$$

$$15. 4x^4 - 5x^2 + 1 = 0$$

$$22. 2x^{\frac{1}{3}} = x^{\frac{1}{6}} + 1$$

$$23. \sqrt{x-1} + \sqrt{2x-1} = \sqrt{4x+5}$$

$$24. \sqrt{x^2 - a^2} = 2 - \frac{1}{\sqrt{x^2 - a^2}}$$

$$25. \sqrt{2+x} - \sqrt{14} + \sqrt{5-x} = 0$$

$$26. 2\sqrt{x^2 - 5x + 8} = 2x^2 - 10x + 12$$

$$27. \sqrt{a^2 - x} - \sqrt{b^2 + x} = a - b$$

$$28. 3x^2 - 7 + 3\sqrt{3x^2 - 16x + 21} = 16x$$

29. Write with integral coefficients the quadratic equation whose roots are $\frac{2 + \sqrt{-11}}{3}$ and $\frac{2 - \sqrt{-11}}{3}$; the equation

whose roots are $-2 - \sqrt{6}$ and $-2 + \sqrt{6}$.

30. Write in simplest form the quadratic equation whose roots are $a + b \pm (a - b)\sqrt{-1}$; the equation whose roots are $\frac{m + \sqrt{n}}{2}$ and $\frac{m - \sqrt{n}}{2}$.

31. Solve the equation $ax^2 + bx + c = 0$. Show the conditions under which the roots are (a) real and equal, (b) real and unequal, (c) rational, (d) surds, (e) imaginary.

32. For what values of k will the quadratic equation $2x^2 - 3(x + k)^2 + 4x - 6(x + k) - 25 = 0$ have equal roots? Find these roots.

33. Find what values of k will make the two values of x equal in the solution of: $\begin{cases} 16x^2 + 25y^2 = 400 \\ y = \frac{4}{5}x + k \end{cases}$

Without solving, determine the nature of the roots and their product and sum in the following equations:

34. $3x^2 - 5 = 7x$

37. $6x^2 - 13x = -6$

35. $12x^2 + 3 = 6x$

38. $3x^2 + 8 = 9x$

36. $x^2 - 1.8x + .5 = 0$

39. $3x^2 - 1.2x = 4.16$

40. The so-called effective area of a chimney is given by the formula $E = A - \frac{3}{8}\sqrt{A}$, where A is measured area. Solve this equation for A in terms of E .

41. A piece of tin 9 inches by 12 inches is to be made into an open box with a base area of 60 square inches, by cutting equal squares from the four corners and then bending up the edges. What is the size, to the nearest tenth of an inch, of the square cut from each corner?

SIMULTANEOUS QUADRATIC EQUATIONS

Solve:

1. $\begin{cases} x^2 - xy + y^2 = 7 \\ 2x - 3y = 0 \end{cases}$

3. $\begin{cases} 2x^2 + 2xy + y^2 = 10 \\ 5x^2 + 4y^2 = 21 \end{cases}$

2. $\begin{cases} x^4 + x^2y^2 + y^4 = 91 \\ x^2 - xy + y^2 = 7 \end{cases}$

4. $\begin{cases} x^2 + y^2 = 10 \\ x^2 - xy + y^2 = 7 \end{cases}$

$$5. \begin{cases} xy + x^2 = 24 \\ y - 3x = -4 \end{cases}$$

$$10. \begin{cases} x^2 + y^2 = 20 \\ x + 2y = 5 \end{cases}$$

$$6. \begin{cases} \frac{1}{x} - \frac{1}{y} = \frac{1}{3} \\ \frac{1}{x^2} + \frac{1}{y^2} = \frac{5}{9} \end{cases}$$

$$11. \begin{cases} \frac{1}{x^2} + \frac{1}{y^2} = 61 \\ \frac{1}{x} - \frac{1}{y} = 1 \end{cases}$$

$$7. \begin{cases} x^2 + y^2 = 25 \\ xy - 12 = 0 \end{cases}$$

$$12. \begin{cases} x^2 - .04 y^2 = 45 \\ x + .2 y = 9 \end{cases}$$

$$8. \begin{cases} x^2 + xy = 12 \\ xy - y^2 = 2 \end{cases}$$

$$13. \begin{cases} x^2 - xy = 12 \\ xy - y^2 = 3 \end{cases}$$

$$9. \begin{cases} x^4 + 4 y^4 = 65 \\ x^2 + 2 y^2 - 2 xy = 5 \end{cases}$$

$$14. \begin{cases} x^2 + xy + 2 y^2 = 4 \\ 2 x^2 - xy + 2 y^2 = 3 \end{cases}$$

$$15. \begin{cases} x^3 + 9 xy - 3 x^2 y - 27 y^2 = -20 \\ x^2 + 9 y = 10 \end{cases}$$

16. A man sold a number of radio sets. If he had charged \$10 more for each set he would have received \$1800; but if he had sold 10 more sets at the original price he would have received \$2000. Find the price of each set and the number of sets.

17. A rectangular wheat field having an area of $2\frac{1}{2}$ acres is surrounded by a lane 1 rod wide. The area of the lane is 86 square rods. Find the dimensions of the field. (1 acre = 160 square rods.)

18. The wattage of an electric current is obtained by getting the product of the voltage and amperage. An electric flatiron draws 550 watts. Find the number of volts and amperes it draws if 3 times the number of volts increased by twice the number of amperes is 340.

19. If 63 is subtracted from a certain number expressed by two digits, its digits will be transposed. If the number is multiplied by the sum of its digits, the product will be 729. What is the number?

20. The area of a right triangle is equal to that of a square whose side is k . The hypotenuse of the triangle is h . Find the sides of the triangle in terms of h and k .

21. A certain principal with interest for 1 year amounts to \$132.50. If the principal were \$25 more and the rate of interest 1% less, the amount in 1 year would be \$157.50. Find the principal and the rate of interest.

22. Charles and John started at the same time from the same place and went at right angles to each other at constant rates. At the end of 2 hours they were 10 miles apart. Charles continued for 4 miles more, while John went 16 miles farther; then they were 26 miles apart. Find the rate at which each traveled.

23. A rectangular ceiling has a skylight 5 feet by 10 feet cut into it. If the surface exclusive of the skylight is 1950 square feet and the length is 10 feet more than the width, find the dimensions of the ceiling.

24. If \$820 is put at interest for a certain number of years at a certain rate, it amounts to \$955.30. If the time were 1 year less and the rate $\frac{1}{2}\%$ more, the amount would be \$918.40. Find the time and the rate.

PROGRESSIONS AND LOGARITHMS

1. A man desiring to pay off a debt of \$5040 in monthly payments finds that he can pay \$25 the first month and increase this payment by \$2 each succeeding month. How long will it take him to pay off the debt, if interest is not included?

2. Derive a formula for the sum of n terms of an infinite decreasing geometric progression in terms of a and r .

3. Express as an ordinary fraction the repeating decimal .1313; .85.

4. Find the sum of all the integers between 100 and 800 that are divisible by 3.

5. Find the sum of the first n terms of the following series :

(a) 1, 2, 3, 4 ... (c) 2, 4, 8, 16 ...

(b) 5, 10, 15, 20 ... (d) $a - 2b, ab^2 - 2b^3, ab^4 - 2b^5 \dots$

6. A ball is dropped to the pavement and rebounds 4 feet. It continues to rebound $\frac{1}{2}$ as far each time as it did the preceding time. What is the total distance through which the ball will pass after first striking the pavement?

7. Find by the formula the sum of the first 6 terms of the progression $2\frac{2}{5}, 3\frac{1}{5}, 4\frac{4}{15} \dots$.

8. Find the 11th term and the sum of the first 11 terms of the progression $1, \frac{3}{2}, 2, \frac{5}{2} \dots$.

9. How many arithmetic means are inserted between $-\frac{3}{2}$ and $\frac{9}{2}$ when the sum of the means is $\frac{21}{2}$?

10. The amount (a) of p dollars at r per cent for n years, when interest is compounded annually, is expressed by the formula $a = p(1 + r)^n$. By the use of logarithms find the amount of \$1600 at 6% for 12 years, interest being compounded annually.

11. If a man deposits \$200 at the end of each 6 months in a savings bank which pays 4% interest, compounded semi-annually, how much should there be to his credit at the end of 10 years?

12. A certain month contained 5 Fridays. The sum of the numbers representing the corresponding dates was 85. On what date did the first Friday fall and on what day did the last of the month occur? (Do not obtain the answer by guess or trial.)

13. By the use of a formula, find the number of consecutive integers, beginning with 10, that must be taken in order that the sum may equal 2035.

14. A man borrowed \$1500, agreeing to pay principal with accrued interest each year at 6% in 4 equal annual installments. Find the sum to be paid at the end of each year.

15. Solve for x the equation $577 = 455(1.02)^{2x}$.

16. How long will it take \$350 to amount to \$500 in a savings bank paying $4\frac{1}{2}\%$, compounded semiannually?

17. The formula $D = \sqrt[3]{\frac{W}{.5236(A - g)}}$ gives the diameter of a spherical balloon which is to lift a given weight W . Find D if $A = .0807$, $g = .0056$, and $W = 1250$.

18. Find the time it will take a sum of money to double itself if placed at interest at 5% , compounded annually; compounded semiannually.

19. Find the value of n in the formula $n = \frac{1}{2L} \sqrt{\frac{Mg}{m}}$ when $L = 78.5$, $M = 5468$, $g = 980$, and $m = 0.0065$.

20. If $T = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$, find the value of T when $a = 54.7$, $b = 65.3$, $c = 83.4$, and $s = \frac{a+b+c}{2}$.

21. Solve $3^{x+2} = 405$.

Find the value of:

22. $\frac{716}{35 \times 3.1416}$

25. $\sqrt{\frac{400}{55 \times 3.1416}}$

23. $\sqrt{\frac{200^2}{72 \times 32 \times 11}}$

26. $\frac{14.8 \times \sqrt[3]{.729}}{111}$

24. $\frac{.73 \times 31.6}{3\sqrt[5]{241 \times 3.764}}$

27. $\sqrt[3]{\frac{.434 \times 58^2}{46 \times 1576}}$

GRAPHS

1. A grocer found one year that at \$1.25 a basket he sold 10 baskets of peaches; at \$1, he sold 30 baskets; at \$.75, 70 baskets; at \$.60, 85 baskets; and at \$.50, 90 baskets. Represent this graphically and from your graph estimate the probable demand when the price is \$1.10 a basket.

2. Below is a table showing the weekly wage at ages from 14 years to 25 years of boys who leave school at the age of 14 and boys who leave school at the age of 18:

Age in years	Weekly wage of a boy who leaves school at the age of 14					Weekly wage of a boy who leaves school at the age of 18				
14	.	.	.	.	\$10.00					
16	.	.	.	.	12.00					
18	.	.	.	.	13.00	.	.	.	.	\$15.00
20	.	.	.	.	14.00	.	.	.	.	20.00
22	.	.	.	.	15.00	.	.	.	.	25.00
24	.	.	.	.	18.00	.	.	.	.	35.00
25	.	.	.	.	21.00	.	.	.	.	40.00

On the same diagram make a graph of the two sets of data, using a broken line for each. Extend the graph and estimate the weekly wages of a man 28 years old.

3. The temperature readings for a certain town from 8 A.M. to 6 P.M. for one day are given in the following table:

	A.M.	A.M.	A.M.	A.M.	M.	P.M.	P.M.	P.M.	P.M.	P.M.	P.M.
Hour	8	9	10	11	12	1	2	3	4	5	6
Reading	-5°	-4°	-1°	4°	8°	12°	13°	13°	12°	10°	6°

(a) Make a graph of the above table.

(b) Determine from the graph the approximate temperature at 11.30 A.M.; at 2.30 P.M. Estimate it for 7 P.M.

(c) Determine from the graph the time when the reading was 11°.

(d) Determine from the graph during what hour there was the least change in temperature.

4. The average weight of boys at different ages, beginning at 6 years and continuing to 15 years, is given in the table below:

Age (in years)	6	7	8	9	10	11	12	13	14	15
Weight (in pounds)	50	53	57	62	67	72	78	85	93	105

(a) Make a graph of this table.

(b) A boy, normal in every respect, weighs 87 pounds. What is his approximate age? Show how this approximation is made from the graph.

5. The formula for the amount (a) when money is invested at simple interest is $a = p(1 + rt)$, p representing the principal, r the rate per cent, expressed fractionally, and t the time in years. Make a graph of the formula if $p = \$200$ and $r = 6\%$. From this graph estimate the number of years in which the principal will double itself.

6. A freight boat leaves New York for a European port, sailing at the rate of 15 knots an hour. Five hours later a transport leaves the same harbor, sailing over the same course at the rate of 20 knots an hour. The freight boat, 2 hours after its departure, is delayed 1 hour by a defect in its machinery. In how many hours will the transport overtake the freight boat? Solve graphically.

7. A freely falling body will, at the end of the first second, have passed over approximately 16 feet; at the end of the second second, 64 feet; at the end of the third, 144 feet; at the end of the fourth, 256 feet, etc. Represent graphically the path of a ball that falls for 4 seconds and has a uniform horizontal velocity of 40 feet a second.

8. A cubic foot of water weighs 62.5 pounds. The weight of water may therefore be expressed by the formula $W = 62.5 V$, where W represents the weight in pounds and V represents the volume in cubic feet.

(a) Complete the following table and make a graph of it; that is, make a graph of the formula $W = 62.5 V$:

V (in cubic feet)	1	2	4	6	8	10
W (in pounds)	?	?	?	?	?	?

(b) Show from the graph what the weight of 7 cubic feet of water should be.

9. The wave length range of a radio receiver is controlled by a single dial graduated from 0 to 100.

DIAL READING	WAVE LENGTH IN METERS
10	250
30	320
40	350
60	410
80	480

(a) Make a graph of the above data.

(b) Extend the graph and estimate the total range of the receiver; that is, estimate the wave lengths corresponding to the 0 and 100 settings of the dial.

10. On the same axes plot the graph of each of the following equations and from the graphs determine the common solutions:

$$\begin{cases} x^2 + y^2 = 65 \\ x + 3y + 5 = 0 \end{cases}$$

11. Represent graphically the equation $x^2 - 6x + 6 = y$ from $x = -1$ to $x = 6$ and from the graph determine, to the nearest tenth, the roots of the equation $x^2 - 6x + 6 = 0$.

12. Using the same set of axes, plot the following equations and from the graph determine the solutions they have in common:

$$\begin{cases} x^2 + 2y = 17 \\ x - y + 1 = 0 \end{cases}$$

13. Draw the graph of the equation $x^2 - 4x + 2 = y$ from $x = -1$ to $x = +5$. From this graph read the roots, to the nearest tenth, of the equation $x^2 - 4x + 2 = 0$.

14. On the same axes represent graphically the following set of equations and from the graph draw a conclusion as to the nature of the solutions to this set:

$$\begin{cases} y^2 = 8x + 1 \\ 4x - 5y = -13 \end{cases}$$

15. By the aid of graphs find one solution of the following simultaneous equations:

$$\begin{cases} x^2 + y = 7 \\ x + y^2 = 11 \end{cases}$$

NUMERICAL TRIGONOMETRY

Let a and b be the sides of a right triangle opposite the acute angles A and B , with c , the hypotenuse, opposite the right angle C . Using logarithms, find the remaining parts of the following triangles, giving the sides to four significant figures and the angles to the nearest minute.

- | | |
|-----------------------------------|-------------------------------------|
| 1. $a = 15.3$, $b = 12.7$ | 5. $b = 38$, $c = 45.5$ |
| 2. $a = 8.9$, $A = 28^\circ 30'$ | 6. $c = 33.58$, $A = 30^\circ 20'$ |
| 3. $b = 58.75$, $A = 35^\circ$ | 7. $c = 25.45$, $B = 65^\circ 40'$ |
| 4. $b = 32$, $B = 37^\circ 30'$ | 8. $a = 15.58$, $c = 20.35$ |

9. Find the acute angles of a right triangle if one of the sides is one quarter of the hypotenuse.

10. Show that the area of a right triangle is equal to :
(1) $\frac{1}{2} bc \sin A$; (2) $\frac{1}{2} ac \sin B$; (3) $\frac{1}{2} b^2 \tan A$; (4) $\frac{1}{2} a^2 \tan B$.

Using logarithms, find the area of the right triangle ABC if :

- | | |
|-------------------------------------|-------------------------------------|
| 11. $A = 25^\circ 50'$, $b = 15.3$ | 13. $B = 42^\circ 45'$, $b = 40.5$ |
| 12. $B = 38^\circ 30'$, $a = 28.7$ | 14. $A = 52^\circ 20'$, $a = 32.7$ |

15. How far from a tree 180 feet high is a man if he observes the angle of elevation of the top of the tree to be $10^\circ 40'$?

16. What is the angle of elevation of a road if there is a rise of 1 foot in every 35 feet of horizontal distance?

17. At a height of 70 feet above the water a man observes the angle of depression of a boat to be $15^\circ 20'$. How far distant is the boat?

18. How high is a tree if it casts a shadow 62.8 feet long when the angle of elevation of the sun is $50^\circ 40'$?

19. What is the angle of elevation of the top of a tower 128 feet high from a place 230 feet from its base?

20. A man standing 350 feet from the base of a cliff finds that the angle of elevation of the top of the cliff is $35^\circ 50'$. How high is the cliff?

EXAMINATION QUESTIONS

The following questions have appeared on recent examinations in algebra.

1. Factor each of the following:

(a) $a^{12} - b^6$

(f) $x^4 - x^2$

(b) $6x^2 + 3x - 18$

(g) $m^{2n} - 11m^n + 18$

(c) $x^4 - 1.8x^2 + .81$

(h) $6p^2 - pq - q^2$

(d) $y^3 - 5y^2 + 2y + 8$

(i) $9a^2 + 6ab + b^2 - c^2$

(e) $a^{6+m} + 2a^{3+m}b^2 + a^mb^4$

(j) $a^3 + a^2b + a + b$

2. (a) Simplify as far as possible each of the following:

$$-16^{-\frac{3}{4}}$$

$$\frac{2^0 - 2^{-2}}{2 - 3(4)^{-1}}$$

$$\sqrt{-16}$$

(b) Rationalize the denominator in $\frac{\sqrt{6} - \sqrt{3}}{4 - 2\sqrt{2}}$ and ex-

press the resulting fraction in its simplest form.

3. Reduce to a simple fraction in its lowest terms and check, letting $x = 3$:

$$\frac{\frac{x+2}{x-2} + \frac{x-2}{x+2}}{\frac{x+2}{x-2} - \frac{x-2}{x+2}}$$

4. (a) Solve the equation $ax^2 + bx + c = 0$ by the method of completing the square.

(b) Express in simplest form the sum of the roots and the product of the roots in terms of a , b , and c .

(c) Form the equation whose roots are -7 and 3 .

5. It cost \$74.25 to have a well dug. If the cost of digging was \$3 for the first yard, \$3.75 for the second yard, \$4.50 for the third yard, etc., how deep was the well? (Solve by formulas.)

6. Find, to the nearest tenth, the roots of the equation $.3x^2 = 4x + 1.01$.

7. The sum to which a given principal will amount if placed at compound interest is expressed by the formula

$a = p(1 + r)^n$, in which a = the amount, p = the principal, r = the rate of interest, and n = the number of years. Find the principal that will amount to \$575 in 20 years at $4\frac{1}{2}\%$, interest compounded annually. (Use logarithms to shorten the computation.)

8. The sum of the volumes of two cubes is 43 cubic feet. Find the length of an edge of each cube if the sum of these two lengths is 4 feet.

9. A dealer bought a number of sheep for \$440. After 5 sheep had died he sold the remainder at a profit of \$2 each, thereby making \$60 on the whole transaction. How many sheep did he buy?

10. Solve, and check one root:

$$\sqrt{2x + 3} - \sqrt{x + 1} = \sqrt{5x - 14}$$

11. On the same axes represent graphically each equation in the following set and from the graph read the solutions common to the two equations:

$$\begin{cases} x^2 + y^2 = 100 \\ x - 2y = -10 \end{cases}$$

12. Find the prime factors of each of the following:

(a) $3x^2 + 14x - 24$

(d) $x^{6a} - y^6$

(b) $x^5 + 32$

(e) $x^3 + 5x^2 - 2x - 24$

(c) $x^2 + .1x - .06$

(f) $135a^6 + 40b^6$

13. Divide $6x^{-3} + 10x^{-2} - 4x^{-1} - 10 - 2x$ by $x^{-2} - 1$ and express the result with positive exponents in descending powers of x .

14. (a) Simplify $\left[\frac{\sqrt{72}y^{2n}}{3} \times 9^0\right](2y^{n+2})^{-1}$.

(b) Rationalize the denominators in the following fractions and simplify the results: $\frac{6a}{\sqrt[3]{12}}$; $\frac{3 + \sqrt{2}}{2\sqrt{3} - \sqrt{2}}$.

15. Solve for x to the nearest hundredth:

$$x^2 - 2.2x - 1.16 = 0$$

16. (a) How many feet are there in the length of a stick that is q yards and s inches long?

(b) To what sum will d dollars amount if placed at simple interest for n years at $4\frac{1}{2}\%$?

(c) A pays for a bill of goods amounting to d dollars and c cents by giving a check for s dollars. If the amount of the check exceeds the amount of the bill, what change in cents is due A?

17. Solve the following and correctly group your answers:

$$\begin{cases} x^2 + y^2 = 10 \\ x^2 - xy + y^2 = 7 \end{cases}$$

18. Solve the following and check your answer:

$$\sqrt{2x+2} + \sqrt{2x-1} - \sqrt{8x+1} = 0$$

19. By the use of logarithms find the value of $\frac{1.976 \times \sqrt[5]{31}}{.8642}$.

20. (a) Without solving the equation, determine the nature of the roots of $6x^2 + 7x - 6 = 0$.

(b) Without solving or substituting, determine whether 4 and -3 are roots of the equation $x^2 - x + 10 = 0$.

21. A can do a piece of work in $4\frac{1}{2}$ days. After he has worked 3 days B joins him and they finish the work in $\frac{7}{8}$ of a day. In what time can B do the work alone?

22. Plot the graph of $x^2 - 2x - 2 = y$ between the values $x = -3$ and $x = +4$ and estimate from the graph the roots of $x^2 - 2x - 2 = 0$.

23. (a) Rationalize the denominator of $\frac{\sqrt{10}}{\sqrt{5} - \sqrt{2}}$.

(b) Solve and check: $\sqrt{x} - \sqrt{x+5} + 1 = 0$.

24. Find, to the nearest hundredth, the roots of the equation $.3x^2 - .02x = .5$.

25. Solve for x and y , correctly group your answers, and check one set:

$$\begin{cases} x^3 + y^3 = 35 \\ x^2 - xy + y^2 = 7 \end{cases}$$

26. (a) Multiply $x^{\frac{2}{3}} - 3x^{\frac{1}{3}} + 5$ by $x^{-\frac{2}{3}} + 2x^{-\frac{1}{3}} - 1$.

(b) Write the product with positive exponents in ascending powers of x .

(c) Write the answer to (b) in radical form.

27. (a) Without solving, determine the nature of the roots of the equation $3x^2 + 5x = 12$.

(b) Form the quadratic equation whose roots are $\frac{1}{2}$ and $\frac{4}{3}$.

(c) For what value of k will the roots of $3x^2 - 2x + k = 0$ be equal?

28. (a) Deduce the fundamental formula for the sum of an arithmetic progression in terms of n , a , and l .

(b) Insert four geometric means between $\frac{2}{3}$ and $-\frac{1}{48}$.

29. By the use of logarithms find the value of:

$$(a) \frac{(8.54)^3 \times \sqrt[5]{.148}}{.9532}$$

$$(b) \frac{.03472 \times \sqrt[3]{47.2}}{2.08}$$

30. A boat can travel 8 miles an hour in still water. If it can travel 20 miles downstream in the same time that it can travel 12 miles upstream, what is the rate of the stream?

31. If 20 rods are added to the length of a certain rectangular field and 4 rods are subtracted from the width of this field, the area will be increased 40 square rods. If 10 rods are subtracted from the length and 5 rods are added to the width, the area will be diminished 50 square rods. Find the dimensions of the field.

32. Using the same set of axes, represent graphically the following set of equations and from the graphs determine the solutions that the two equations have in common:

$$\begin{cases} x + 3y = 12 \\ x^2 = 6y \end{cases}$$

33. Find the prime factors of each of the following:

$$(a) 5x^{4a} - 15x^{3a} - 90x^{2a} \quad (d) y^3 - .125$$

$$(b) x^3 - 7x^2 + 4x + 12 \quad (e) x^4 - (5x - 6)^2$$

$$(c) 9x^{6a} - 30x^{3a} + 25 \quad (f) .0016x^4 - 1$$

34. (a) Rationalize the denominator of $\frac{2\sqrt{3} + 3\sqrt{2}}{5 + 2\sqrt{6}}$.

(b) Find the value of each of the following expressions:

$$(-64)^{-\frac{2}{3}} \qquad \frac{3^{-2} + 5(2)^0}{3 - 4(3)^{-1}}$$

35. Solve for x : $3\sqrt{x} - \sqrt{9x - 32} = \frac{8}{\sqrt{9x - 32}}$.

36. In any quadratic equation in one unknown number, what is the test for real roots? for equal roots? for imaginary roots? for rational roots? Determine, without solving the equations, the nature of the roots of:

(a) $3x^2 = -8 + 5x$

(c) $4\sqrt{3}x = 4x^2 + 3$

(b) $2x^2 - 5 = 3x$

(d) $3x(x - 5) = 3x - 48$

37. (a) Solve for h the formula $\frac{a}{g} = \frac{h - L}{h + L}$.

(b) Find, to the nearest tenth, the value of h when $a = 5.28$, $g = 16.08$, $L = 4.70$.

38. Deduce the fundamental formula for the sum of a geometric progression in terms of the first term, the ratio, and the numbers of terms.

39. Show how you would find the logarithm of each of the following numbers, given the logarithm of 2 and the logarithm of 3:

(a) 6

(b) $\frac{2}{3}$

(c) $\sqrt[3]{12}$

40. By the use of logarithms find the value of:

(a) $\frac{.0192(82.17)^2}{\sqrt[3]{.938}}$

(b) $\frac{5.276^4}{\sqrt[3]{.046}}$

41. Solve for x and y and correctly group your answers:

$$\begin{cases} x^3 - y^3 = 26 \\ x^2 + xy + y^2 = 13 \end{cases}$$

42. Goods cost a merchant \$72. At what price should he mark them so that he may sell them at a discount of 10% from this marked price and still make a profit of 20% of the selling price? (Give algebraic solution.)

43. Find, to the nearest hundredth, the side of a square whose area is doubled when each side is increased by 2.

44. (a) Represent graphically the equation $x^2 - 2x = y$.

(b) From this graph determine the roots of $x^2 - 2x = 0$.

(c) Find, to the nearest tenth, the roots of $x^2 - 2x = 6$.

45. Simplify: $\left(1 - \frac{b^2}{a^2}\right)\left(1 - \frac{ab - b^2}{a^2}\right)\frac{a^4}{a^3 + b^3} \cdot \frac{a - b}{a^2 + b^2}$.

46. Find the value of the expression $x + \sqrt{1 + x^2}$ when

$$x = \frac{1}{2}\left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right).$$

47. Solve the equations:

$$\begin{cases} \frac{7x + 6}{11} + y - 16 = \frac{5x - 13}{2} - \frac{8y - x}{5} \\ 3(3x + 4) = 10y - 15 \end{cases}$$

48. (a) Solve: $\begin{cases} x^2 + y^2 = 17 \\ xy = 4 \end{cases}$

(b) Construct the graphs of the two equations on the same set of axes.

49. The sum of three numbers in geometric progression is 28 and their product is 512. Find the numbers.

50. A man sold a number of barrels of apples for \$50. If he had sold 5 more barrels for \$.50 less a barrel, he would have received the same amount. Find the number of barrels and the price per barrel.

51. Mr. Hart bought a certain article for m dollars and sold it at a profit of m per cent. Mr. Burns bought another article for $2m$ dollars and sold it at a loss of $2m$ per cent. Both received the same sum. Find m .

52. (a) The m th term in a geometric progression is p , and the n th term is q . Find the first term.

(b) Write the first m terms of the series if $m = 6$, $n = 3$, $p = 8$, and $q = 27$.

53. Find the prime factors of each of the following :

(a) $6x^2 - 5x - 4$ (d) $8a^4 - 125ac^6$

(b) $12xy + 25 - 4x^2 - 9y^2$ (e) $36x^{6a} - 60x^{4a} + 25x^{2a}$

(c) $x^3 - 9x^2 + 2x + 48$ (f) $1 - y - z + yz$

54. Multiply $3a^{-\frac{2}{3}} - 4a^{-\frac{1}{3}} + 3$ by $3a^{\frac{2}{3}} + 4a^{\frac{1}{3}} + 3$ and find the value of the product when $a = 8$.

55. (a) Form the quadratic equation whose roots are $3 - \sqrt{2}$ and $3 + \sqrt{2}$.

(b) Without solving, determine the nature of the roots of the equation $12x^2 + 44x = 45$.

56. Find the roots of the following equation to the nearest tenth: $3x^2 - 15x + 11 = 0$.

57. Solve the following equation and check one root :

$$\sqrt{3x} = \frac{28}{\sqrt{2x+10}} - \sqrt{2x+10}$$

58. For boring a well 180 yards deep a contractor receives \$2 for the first yard, and for each yard thereafter \$.05 more than for the preceding one. By formula find the cost of the job.

59. The third term of a geometric progression is $\frac{8}{3}$ and the sixth term is $-\frac{64}{27}$. Find the eighth term.

60. By the use of logarithms, find the value of :

$$\frac{\sqrt[3]{978} \times (.52)^4}{71.42 \times .032}$$

61. In the formula $a = p(1 + r)^n$, find the value of n if $a = \$8575$, $p = \$5000$, and $r = 4\frac{1}{2}\%$.

62. Solve, correctly group your answers, and check :

$$\begin{cases} x^3 + y^3 = 208 \\ x + y = 4 \end{cases}$$

63. A man bought a number of articles at equal cost for \$500. He sold all but two for \$540, at an advance of \$5 on the original cost of each. How many articles did he buy?

64. Plot the graph of each of the following equations and from the graphs determine the common solutions:

$$\begin{cases} y = x^2 - 6x + 5 & (\text{Plot from } x = -1 \text{ to } x = +7) \\ 3x - 2y = 12 \end{cases}$$

65. (a) Write the middle term of the expansion of $(a - b)^{14}$ by the binomial theorem.

(b) Find the value of a^7b^7 if $a = x^{\frac{2}{3}}y^{-\frac{3}{2}}$ and $b = \frac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{3}}$, and reduce the result to a form having only positive exponents.

66. Expand by the binomial formula $(a^2 + 2b)^5$. Check the expansion if $a = 1$ and $b = 1$.

67. Expand by the binomial formula $\left(1 - \frac{5}{2x}\right)^6$.

68. Using the binomial theorem, write the first 5 terms of $(m^2 - n^3)^6$.

69. By solving the equation $x^3 - 1 = 0$, find the three cube roots of 1. Show that the square of one of the irrational roots is equal to the other irrational root.

70. Given $l = ar^{n-1}$ and $s = \frac{rl - a}{r - 1}$, find the value of l in terms of r , n , and s .

71. Simplify $(x^{-\frac{3}{2}} - y^{\frac{3}{2}})^2 + 2\left(\sqrt{\frac{y}{x}}\right)^3$.

72. A man is now 21 years old. How much money will he have to his credit when he reaches his 40th birthday if he sets aside \$250 every 6 months, beginning 6 months from now and continuing up to and including his 40th birthday, interest being compounded semiannually at 5%?

Solve, and check the results:

$$73. \begin{cases} 2x + y + z = 0 \\ 3x - 2y + 3z = 4 \\ 5x + 7y + 2z = 2 \end{cases} \quad 74. \begin{cases} 6x + 2y - 5z = 13 \\ 3x + 3y - 2z = 13 \\ 7x + 5y - 3z = 26 \end{cases}$$

75. Simplify: (a) $\frac{c\sqrt{ab} - ac}{bc - c\sqrt{ab}}$; (b) $\frac{1 - x(5 + x^2)^{-\frac{1}{2}}}{x - \sqrt{5 + x^2}}$.

76. Simplify: $\sqrt{48} - \frac{6}{\sqrt{12}} - \frac{4}{3}\sqrt[4]{729} + 2\sqrt[6]{27}$.

77. When $x = a - \frac{1}{a}$, evaluate $\frac{\sqrt{4+x^2} - 2a}{x}$.

78. Solve the simultaneous equations:
$$\begin{cases} x^3 + y^3 = 57 \\ x + y = 3 \end{cases}$$

79. By solving the equation $x^3 + 8 = 0$, find the three cube roots of -8 . Check.

80. A man walked to a station at the rate of 4 miles an hour and traveled by train at the rate of 30 miles an hour, reaching his destination in 20 hours. If he had walked 3 miles an hour and had ridden 35 miles an hour, he would have made the journey in 18 hours. Find the total distance he traveled.

81. The three dimensions of a rectangular box are in the ratio $1:3:5$ and the total outside surface of the box is 1150 square inches. Find the dimensions of the box and the length of its diagonal to two decimal places.

82. In anticipation of a holiday a merchant makes an outlay of \$50, which will be a total loss in case of rain, but which will bring him a clear profit of \$150 above the outlay if the day is pleasant. To insure against loss he takes out an insurance policy against rain for a certain sum of money, for which he has to pay a certain percentage. He then finds that because of this insurance, in either event, whether the day is pleasant or rainy he will make \$80 clear. What is the amount of the policy and what rate does the company charge him?

83. Find, to the nearest hundredth, the roots of the equation $3x^2 - 16x = 32$.

84. A number of two digits is equal to 4 times the sum of its digits, and the digits are reversed if 27 is added to the number. Find the number.

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- laws of, 86, 87, 107, 114, 236, 237, 242, 375
- of a fraction, 152-154

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- of roots, 354
- quality, 78

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- elimination by, 216, 217

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 y -axis, 207, 208
 Zero, powers of, 38
 Zero exponent, 114, 253
 Zero factor, 32, 38, 284

ANSWERS TO MILNE-DOWNEY'S STANDARD ALGEBRA

Page 10. — 1. 110. 2. 15 cents. 3. 15 oz. 4. 16; 4.
5. (2) 12; 18. 6. 40.

Page 11. — 7. 2; 6. 8. 2; 4. 9. 4; 3. 10. $(m + n)$ marbles; 10 marbles. 11. 1900. 12. 175 da. 13. 13,000 sheets.
14. 132 oranges. 15. 70; 45. 16. 19; 54. 17. 6500 sheets.

Page 12. — 18. $(14 - 4)$ yr., or 10 yr.; $(14 - a)$ yr.; $(14 + 3)$ yr., or 17 yr.; $(14 + y)$ yr. 19. 5 *c* cents. 20. $(t + 2)$ sec.

21. $(p - 10)$ lb. 22. $\frac{d}{10}$ dollars; 10 *d* cents. 23. 2 *m* mi.

24. $\frac{1}{2}c$ cents. 25. 8, 10; $x - 1, x + 1; a - 1, a + 1$. 26. $y - 2, y + 2; y - 1, y + 1$.

Page 13. — 1. (a) $s + 5 = 18$; (b) $n + 8 = 15$; (c) $m + 7 = 22$; (d) $b + 2 = 12$; (e) $n + 6 = 25$; (f) $4n = 28$; (g) $5n = 35$; (h) $n + 50 = 180$; (i) $p + 60 = 120$.

Page 14. — 1. (j) $2n + 1 = 21$; (k) $3n - 1 = 20$; (l) $n - 7 = 8$.
3. (a) $2 + m = 28$; (b) $2m = 28$. 7. $\frac{3}{4}n = 20$. 8. $\frac{1}{2}n = 16$.

Page 15. — 1. $x = 10$. 2. $n = 16$. 3. $w = 3$. 4. $y = 3$.
5. $b = 3$. 6. $a = 7$. 7. $x = 14$. 8. $x = 20$. 9. $x = 6$. 10. $r = 5$.
11. $x = 6$. 12. $x = 12$. 13. $a = 4$. 14. $a = 4$. 15. $x = 6$.
16. $x = 20$. 17. $z = 15$. 18. $r = 16$. 19. $x = 4$. 20. $x = 8$.
21. $x = 13$. 22. $x = 14$. 23. $m = 10$. 24. $x = 8$. 25. $x = 8$.
26. $a = 5$. 27. $x = 10$. 28. $x = 600$.

Page 16. — 1. $x = 3$. 2. $y = 5$. 3. $z = 7$. 4. $a = 3$.
5. $m = 25$. 6. $x = 6$. 7. $c = 10$. 8. $x = 3$. 9. $x = 6$.
10. $x = 1\frac{1}{3}$.

Page 17. — 1. $x = 5$. 2. $x = 3$. 3. $r = 13$. 4. $a = 4\frac{1}{2}$.
5. $y = 4$. 6. $b = 4.2$. 7. $n = 2$. 8. $c = 3$.

Page 18. — 6. $x = 6$. 7. $n = 10$. 8. $a = 18$. 9. $s = 3.2$.
10. $n = 10$. 11. $a = 15.5$.

Page 19. — 9. $z = 42$. 10. $a = 20$. 11. $x = 18$. 12. $a = 4$.
13. $y = 3\frac{1}{2}$. 14. $t = 3$. 15. $m = 5$. 16. $x = 5$. 17. $s = 6$.
18. $x = 4$. 19. $b = 2$. 20. $y = 5$. 21. $x = 10$. 22. $m = 8$.
23. $a = 2$. 24. $x = 6$. 25. $n = 2\frac{1}{2}$. 26. $s = 1$. 27. $s = 2$.
28. $y = 2$. 29. $y = 2$. 30. $x = 3$.

Page 20. — 5. $x = 2$. 6. $x = 3$. 7. $y = 4$. 8. $y = 15$.
 9. $a = 25$. 10. $n = 1$. 11. $p = 1$. 12. $n = 2$. 13. $x = 1$.
 14. $a = 20$. 15. $n = 3$. 16. $y = 4$.
 1. \$5.40; 6 c cents. 2. 90 mi.; 120 mi.; 30 h mi.

Page 21. — 3. 2 cents; 12 cents; $2n$ cents. 4. $(m + n)$ bats.
 5. $\frac{m}{3}$; $\frac{t}{3}$; $\frac{2t}{3}$. 6. $s + 2$. 7. $1 - d$. 8. $(c - 7)$ yr.; $(c + 10)$ yr.
 9. $(300 - n)$ dollars. 10. $(2v + 15)$ votes. 11. $2n$ yr.;
 $(2n + 5)$ yr. 12. $4n$ boys; $5n$ pupils; $5n = 30$; 24 boys.
 13. April, n books; March, $4n$ books. 14. Brother, n papers;
 Arthur, $\frac{1}{2}n$ papers. 15. $5w$ ft.; $12w$ ft.

Page 22. — 16. $4s$ in.; $4s = 48$.

Page 23. — 2. 87, 29. 3. 54, 18. 4. 36. 5. 105.

Page 24. — 8. Length, $1\frac{1}{8}$ in.; width, $\frac{3}{8}$ in. 9. Length, 9 units;
 width, 6 units. 10. 15 wk.

Page 25. — 12. 40 million ft. 13. 4 to 2. 14. $62\frac{1}{2}^\circ$, $62\frac{1}{2}^\circ$, 55° .
 15. 30 mi. 16. John's set, \$100; Henry's set, \$25. 17. 96 fresh-
 men. 18. 16 games won, 4 games lost. 19. Tickets, \$98; adver-
 tisements, \$49.

Page 26. — 20. 9 in. 21. $a = 16$ in.; $b = 16$ in.; $c = 8$ in. 22. 8 in.;
 16 in. 23. 12, 6, 3. 24. First, 20 papers; second, 10 papers; third,
 30 papers. 25. 12 in., 16 in., 9 in. 26. Plumber, \$10; each
 helper, \$2.50. 27. Pony, \$90; saddle, \$30. 28. One part, 69;
 other part, 23. 29. 100 skaters; 300 spectators.

Page 28. — 2. 16 ft. 3. 50 rd.

Page 29. — 4. 24 yd. 5. 8 in. 6. 11 ft. 7. 10.4 in. 8. 12 ft.
 9. 18 in. 10. 24 rd.
 2. 14 ft.

Page 30. — 3. 30 rd. 4. 39 yd. 5. 7 in. 6. 9.3 ft. 7. 27 in.
 8. 35.3 m. 9. 22 yd.

Page 31. — 1. $6a$. 2. $14x$. 3. $7x$. 4. 0. 5. $2a + 4b$.
 6. $8x + y$. 7. $5s + 5t$. 8. $4c$.

Page 36. — 34. (a) 9; (b) 9. 36. 43. 37. 15. 38. 22.
 39. 6. 40. 4. 41. 20. 42. 62. 43. 9. 44. 10. 45. 18.

Page 37. — 52. $5a + 3x^2$. 53. $3b - 3a^5$. 54. $ab(a - c)^2$.
 55. $\frac{2x}{5(a + b + c)}$. 56. $7xy + 3z^3$. 57. $mn - m^3$. 58. $a^5(b + c)$.
 59. $12a^2 - 5b^3$. 60. $\frac{8ab}{4c^7}$.

2. 50. 3. 30. 4. 120. 5. 180. 6. 80. 7. 450. 8. 576.
 9. 1500. 10. 6400. 11. 225. 12. 225. 13. 1,250,000. 14. 15.
 15. 6. 16. 30. 17. 4320. 18. $\frac{8}{25}$. 19. 2. 20. $2\frac{4}{5}$. 21. 1.
 22. $2\frac{9}{35}$. 23. 6. 24. $\frac{8}{15}$. 25. $1\frac{2}{30}$. 26. $\frac{2}{15}$. 27. $\frac{17}{80}$. 28. $1\frac{2}{5}$.
 29. $\frac{4}{5}$.

Page 38. — 31. 0. 32. $1\frac{11}{189}$. 33. $4\frac{25}{32}$. 34. $2\frac{11}{39}$. 35. .305.
 36. $\frac{11}{80}$. 37. 3.4. 38. $\frac{32}{33}$. 39. $1\frac{89}{320}$. 40. .16. 41. 3.925.
 42. .595. 47. 64. 48. 4. 49. 0. 50. 32. 51. 15. 52. 4.
 1. 60. 2. 70. 3. 99. 4. 100. 5. 121. 6. 365. 7. 1000.
 8. 1254. 9. 5386.

Page 39. — 10. (a) $2t + 5$; (b) $7t^2 + 3t + 2$, $5t^2 + 2t + 3$, $8t^2 + 9t + 3$, $4t^3 + 8t^2 + 6t + 7$. 11. (a) $2 \times a$, $a \times a$; (b) 2, 1; 4, 4; 6, 9; $1, \frac{1}{4}$; $\frac{2}{3}, \frac{1}{9}$; (c) $a = 0$ or 2. 12. 3^2 ; $4^2 = 2^4$; 2^5 ; 2^3 ; $(\frac{1}{2})^2$; $1^3 = 1^2$. 13. Decreased; increased. 14. 3; 4; 6; 10.
 18. 120; 12; 96; $2\frac{1}{2}$; 48. 19. 300; 240; 1200; 108; 216; 2.
 20. 4. 21. 14. 22. 5. 23. 7. 24. 354. 25. 1637. 26. 223.

Page 40. — 27. $x = 6$. 28. $n = 7$. 29. $n = 10$. 30. $n = 6\frac{1}{2}$.
 31. $a = 320$. 32. $n = 14\frac{2}{5}$. 33. $m = 6$. 34. $(50 - c + d)$ cents.
 35. $(500 - cx)$ cents.
 1. 0. 2. 84. 3. 0. 4. 10. 5. 52.

Page 41. — 6. 8. 7. 30. 8. 20. 9. 11. 10. 34. 11. 4.
 12. 60. 13. 12. 14. 64. 15. 10.
 1. $7a^2 + 7$. 2. $4x + y$. 3. $2x^3 + 6y$. 4. $5x^2 + 2$.
 5. $3xy + 21$. 6. $11a^2 + 9b$. 7. $11x^2 + 20x$. 8. $2x + 3$.
 9. $17b^2 + 7$. 10. $19ab + 8c$. 11. $6a + 6b^3$. 12. $18r + 19$.
 13. $9a^3 + 1$. 14. $5a'$

Page 56. — 2. $i = 4.75t$. 3. $i = 6.5t$. 5. George, $w = .38h$;
 John, $w = .20h + .10$. 7. \$1120. 8. \$1180. 9. \$600.
 10. $c = 8w + 1$.

Page 57. — 11. $c = 2w + 4$. 12. 41¢. 13. 49¢; 17¢; 81¢.
 14. $p = 4s$. 15. 20 in.; 24 ft. 16. 20 in.; 24 ft.; 32 yd.; 19 in.
 17. $A = s^2$. 18. 4 sq. in.; 9 sq. ft.; $6\frac{1}{4}$ sq. ft. 19. $p = 2a + 2b$.
 20. 30 in. 21. 22; 14 in. 22. $V = e^3$. 23. 8 cu. in.; 27 cu. in.;
 125 cu. in.; 216 cu. ft. 24. $A = ab$.

Page 58. — 25. $A = \frac{1}{2}ab$. 27. $V = lwh$. 28. $V = 240$.
 29. $E = 12e$. 30. 1. $c = \pi d$ or $c = \frac{2\pi}{7}d$; 2. $A = \pi r^2$ or $A = \frac{2\pi}{7}r^2$.

Page 59. — 30. 3. $A = s^2$; 4. $V = bh$; 5. $V = \frac{1}{3}bh$; 6. $c^2 = a^2 + b^2$.

Page 66. — 1. 32° ; 194° ; 140° . 2. 10° ; 20° ; 35° . 3. 1020 mi.
 4. 40 mi. an hour; $\frac{1}{5}$ mi. a minute. 5. 1680 cu. ft. 6. $81.74\frac{4}{5}$
 sq. ft.; $78\frac{4}{5}$ sq. ft.; $314\frac{4}{5}$ sq. yd. 7. 58¢. 8. \$25. 9. 540 ft.
 10. $c = 2\pi r$; 22; $17\frac{1}{2}$; $52\frac{1}{2}$.

Page 67. — 1. $t = \frac{i}{pr}$. 2. (a) $h = \frac{V}{lw}$; $w = \frac{V}{lh}$; $l = \frac{V}{wh}$.

Page 68. — 9. $t = \frac{i}{pr}$; 3 yr.

Page 71. — 8. 40 sq. in. 9. 6 in. 10. 23.9 sq. in. 11. 60 sq. in.
 12. 13 sq. ft. 13. 5. 14. 94 sq. in.

Page 72. — 15. 88 in. 16. $c = 2\pi r$; 95.5 in.

Page 73. — 8. 572 sq. in. 9. $402\frac{2}{7}$ cu. in. 10. $245\frac{1}{7}$ sq. in.
 11. $169\frac{5}{7}$ cu. in.

Page 74. — 12. 616 sq. in. 13. $179\frac{3}{8}$ cu. in.

1. $11\frac{1}{8}$ sec. 2. 490 m. 3. (1) 4410 cm.; (2) 196 ft. 4. 500,000.

Page 75. — 5. 300. 6. 11,760. 7. 20° . 8. 95° .

1. 121 sq. in. 2. 90.9 cu. in. 3. 433 ft. 4. 22° . 5. 13.5 sq. ft.

Page 81. — 9. 10. 11. 7.4. 12. 4.1. 13. — 17.1. 14. — $15\frac{1}{4}$.
15. .6. 16. — 1.6. 17. — $3\frac{1}{2}$. 18. 1.7. 19. 7. 20. — $11\frac{6}{4}$.
21. $6\frac{1}{3}$. 22. 0. 23. — 7. 24. — 13. 25. — 8. 26. — 6.
27. — 2. 28. — 44, or 44 B.C. 29. Distances from Juniors'
25-yard line: 3 yd.; 16 yd.; 11 yd.; 22 yd.

Page 82. — 30. 56 yd. from goal nearest start. 31. 139 mi.

Page 83. — 2. — 3. 4. — 4. 6. — 5. 8. — 7. 9. — 1.
10. — 9. 11. — 2. 12. — 11. 13. 4. 14. — 5. 15. 5.
16. — 4. 17. 6. 18. — 3. 19. 7. 20. 1. 21. 11. 22. 2.
23. 12.1. 24. — 23.7. 25. 12. 26. 2. 27. 0. 28. 7.
29. 2.7. 30. 8. 31. 27.1. 32. — 12.4. 33. — 8. 34. $20\frac{3}{4}$.
35. 16.1. 36. — 13. 37. 48. 38. — 15. 39. — 25.

Page 84. — 40. — 13. 41. 18. 42. 3. 43. — 3. 44. 7;
11; 9; 6; 13; 2; — 4. 45. 3 is 8 greater; — 2 is 3 greater; 6 is
1 greater; — 2 — (— 8) is 16 greater. 46. Duluth, 8° ; Helena, 8° ;
Montreal, 30° ; New Orleans, 12° ; New York, 22° . 47. 44° .
48. 30° , 28° ; 32° , 24° ; 76° , 80° . 49. 127 ft.; — 127 ft.

Page 86. — 1. By + 2: 6, 10, — 12, 20, — 16, — 18, — 24; by
— 2: — 6, — 10, 12, — 20, 16, 18, 24. 2. (a) — 49.2; (b) 27;
(c) — 34.02; (d) — $33\frac{3}{4}$; (e) — 20. 3. (a) 35; (b) — 16;
(c) — 70; (d) 132; (e) — 32. 4. (a) — 50; (b) — $11\frac{1}{4}$; (c) 79.54;
(d) $14\frac{2}{3}$; (e) $3\frac{3}{25}$.

Page 87. — 1. — 1. 2. 16. 3. 36. 4. — 90. 5. 36.
6. 112. 7. — 72. 8. 98.
1. 3; — 3; 5; — 5; 7; — 6; — 9; 11; — 4. 2. — 3; 3;
— 5; 5; — 7; 6; 9; — 11; 4.

Page 88. — 3. — 2. 4. — 1. 5. — 3. 6. 9. 7. — 9.
8. — 5. 9. — 11. 10. — 7. 11. — 4. 12. — 82. 13. 4.
14. 1. 15. — 9. 16. — 7. 17. 9.1. 18. — 9. 19. — 20.
20. — 4. 21. 7. 22. 9.2.
7. 0. 8. — 14. 9. 8.8. 10. — .3. 11. — 11.7. 12. 10.

Page 89. — 13. 3.2. 14. — 21.1. 15. — $\frac{5}{8}$. 16. $72\frac{3}{4}$. 17. 39.4.
18. — 7.27. 19. — $24\frac{1}{4}$. 20. $10\frac{3}{8}$. 21. — 28.4. 22. 54.4.
23. $1\frac{1}{5}$. 24. $52\frac{11}{12}$. 25. — 25.5. 26. — 56.8. 27. 15. 28. — $1\frac{1}{8}$.
29. 18° ; 17° ; 8° . 30. 160 yr.; 700 yr. 31. Gain, \$200. 32. 27.
33. 56. 34. — 216. 35. 54. 36. — 24. 37. 6. 38. — 12.
39. — 25. 40. 16. 41. 9.

Page 91. — 23. — $1.2ab^2$. 24. $4(a-b)^2$. 25. — $2(2a+b)^2$.
26. — $1.8a^2bc^3$. 27. 16 a. 28. 11 y. 29. 5 m. 30. ab.
31. — $4x^2y^2$. 32. $3(x+y)$. 33. $18(a+b)^2$. 35. $2a+x$.
36. $c-5m$. 37. $3w+3z$. 38. $3r-t$. 39. — $\frac{1}{2}q$.
40. — $.1b+d+.6$. 41. — $2x^2-2xy+5y^2$. 42. — $2r^2+6rs+9s^2$.
43. a^2b+3b^3 .

- Page 92. — 2. $-3a + 5b$. 3. $6r + 4s$. 4. $2x^2 + xy + y^2$.
 5. $2a^2b - 13ab^2 + 6ac^2$. 6. $4a + 6b + c$. 7. $9w - y$.
 8. $-3l - m + 14n$.

- Page 93. — 9. $11r - 5s - 12t$. 10. $13x - 7y - 15z$.
 11. $a + 4b + 3c$. 12. $5x^2 - 12y^2$. 13. $-2x^2y^2 + 2y^2$.
 14. $3r + 2s$. 15. $5x + y + 3z$. 16. $2an - a^2n^2 - 4$.
 17. $a - b + 7c + 3d$. 18. $x + z$. 19. $8a + b - c$. 20. $3a + \frac{1}{2}b$.
 21. $-2(2a + b)v + 2(a + c)y$. 22. $a - b$. 23. $2a^2 - 2ab + 7b^2$.
 24. $x^3 - 2x + 2$. 25. $-a + b + 9x - 2y$. 26. $2x^3 + .6x^2 + 2.8$.

- Page 94. — 27. $8\sqrt{x} + \sqrt{xy} - 2\sqrt{y}$. 28. $2(c - d)$
 29. $2x + y - \frac{1}{4}z$. 30. $2\sqrt{a} + \sqrt{b}$. 31. $2x + 3y - 2(a + 1)$
 32. $(c + 2)x + (a + d)y$. 33. $x^3 - 2x + 2$. 34. $c + 3m - n$.
 35. $3a^4 - a^2 + 2a$. 36. $6x^6 - x^5 + 4x^4 - 8x^2 + 2x - 1$.
 37. $2(a + b) - (b + c)$. 38. $2(a + b)x + (c - d)y$.

- Page 95. — 19. n . 20. 0 . 21. $5s$. 22. $8b$. 23. $12x$.
 24. $5r$. 25. $16y$. 26. $6z$. 27. $3v$. 28. $6n$. 29. $13y$.

- Page 96. — 2. $-t$. 3. $7x - 4y$. 4. $3r - 3s$. 5. $6n^2 - 3n^3$.
 6. $r - 4s$. 7. $3a^2 - ab - 5b^2$. 8. $3x - 2y - 7z$.
 9. $7a^3 - a^2b - b^3$. 10. $5a^2 - 4b^2 - 14c^2$. 11. $12x - 11y + 11z$.
 12. $3a^2 - 2ab + b^2$. 13. $-m^2 + 2mn - n^2$. 14. $2x^2 + 8xy - y^2$.
 15. $-ab$. 16. $7xy - 7y^2$. 17. $2ab$. 18. $2z$. 19. $m^3 - 3m^2n + m$.

- Page 97. — 20. $3a^2 + b^2$. 21. $a^2 - 4ab + 3b^2$. 22. $a^2 - b^2$.
 23. $-a^2 + 4ab - 3b^2$.

- Page 99. — 1. $n = 2$. 2. $r = 5$. 3. $x = 1$. 4. $s = 8$.
 5. $x = 2$. 6. $x = 3$. 7. $n = 11$. 8. $p = 1$. 9. $x = 9$.
 11. $x = 4$. 12. $x = 2$. 13. $x = 9$. 14. $x = 5$. 15. $x = 2$.
 16. $x = 3$. 17. $x = 11$. 18. $x = 3$. 19. $x = 13$.

- Page 100. — 1. 12,000 freight elevators. 2. 162 ft. 3. Un-
 named, 1473 islands; named, 1668 islands. 5. Katherine, \$40;
 James, \$150. 6. $751\frac{1}{3}$ ft. 7. 30 games.

- Page 101. — 8. Length, 14 ft.; width, 9 ft. 9. Length, 19 rd.;
 width, 12 rd. 10. 35 in. 11. 20 yr. 12. Adrian, 5 yr.;
 sister, 13 yr. 13. Dark skin, \$600; brown skin, \$200. 14. 1200 lb.;
 3000 lb. 15. \$2.53. 16. 450 ft.; 20 ft. 17. 3300 ft. 18. 30
 19. George, 25 yr.; James, 10 yr. 20. John, \$42.40; Frank, \$30.20.

- Page 102. — 21. 20 ft. 22. 10. 23. 21. 24. Construc-
 tion, \$4,000,000; furnishings, \$9,000,000. 25. \$1200.

- Page 103. — 21. $a - 2m + n$. 22. $4a$. 23. 0 .
 24. $-x^2 + xy + 4y^2$. 25. $5m - 4n$. 26. $-x - 4y$.
 27. $x^2 - 6xy + 4y^2$. 28. $3a - 8c$.

- Page 104. — 30. $b - 4a - 2x + 7$. 31. $8y - 4a + 3b - 5$.
 32. a . 33. $6a - 6y + 1$. 34. $-2m - 7n + 5p - 3$.
 35. $4.4m^2 - 10$. 36. $-.4a^2 - 12$.

- Page 105. — 1. $(ax^2 + 2x^2) + (ab + 2b)$. 2. $(a^3 + b^3) + (3a^2b + 3ab^2)$.
 3. $a^2 - (b^2 + 2bc + c^2)$. 4. $a^2 - (b^2 - 2bc + c^2)$.
 5. $xy - xz - (y^2 - 2yz + z^2)$. 6. $ac + bc - (a^2 + 2ab + b^2)$.
 7. $a^2 + 2ab + b^2 - (c^2 - 2cd + d^2)$. 8. $a^2 - 2ab + b^2 - (c^2 + 2cd + d^2)$.
 2. $(a + n)x$. 4. $(c - d)y$. 6. $-(m + n)p$. 8. $(3a + b + c)x$.
 9. $(m - n)x$. 10. $(a^2 - b^2)x$. 11. $(d - e)y + (a + b)z$.
 12. $(m - n)x + (m - n)y$. 13. $(a - 2)x - (b - c)y$. 14. $(r - 3s)y$.

- Page 106. — 16. $(a - b + d)x - (b + c + e)y$. 17. $(5a - 2d - 5)x + (3a + n - 1)y$.
 18. $(3a - 2b + c - l)x + (7a - t)y$.
 19. $-(2a - b + c)x + (b + c - d)y$. 20. $(a + 1)x + (a - 1)y$.
 21. $(1 - a)x - (1 + a)y$. 22. $bx - (2a - b + c)y$.
 23. $(r - s)x - (a - 2c)y$. 24. $(a + e)x^3 + (b - d)x^2 - (c + f)x$.
 25. $(1 - 3a)x^3 + (3 - a)x^2 + (3 + b)x$. 26. $-(1 + mn)x^3 + (1 - b)x^2 - (ab + c - d)x$.
 27. $(a - 1)x^4 - (ab - 1)x^3 - (a - 1)x^2 + (a - 1)x$.
 1. 13. 2. - 53. 3. 13. 4. 3. 5. - 13. 6. - 25.
 7. - 258. 8. - 268.

- Page 108. — 17. $-4a^2$. 18. $6a^{m+3}$. 19. $15y^{n-1}$.
 20. $-5x^{n+1}$. 21. a^{m+n} . 22. $-30m^7n^5$. 23. $-14a^{4r}b^{6z}$.
 24. $a^{n+2}b^nx^3y^n$. 25. $-x^byz^{n+5}$. 26. $12a^{2r-2}b^{2s}$. 27. $-ap^4q^4$.
 28. $-x^{n+1}y^{n+1}$. 29. x . 30. $ab^{2n+2}c^4d^m$. 31. y . 32. $-16a^2b^5m^{10}n^{10}$.
 33. $-8x^{2n}$. 34. $a^{2n}b^nc^{n+2}$. 35. 2^2z^a , or $4z^a$. 36. x^ny^{n-1} .

- Page 109. — 9. $a^5b + 4a^4b + 6a^3b + 4a^2b + ab$. 10. $1.25a^3 + 1.5a^2 + .5a$.
 11. $\frac{4}{3}x$. 12. $3.8a^2b^2 - 5.2a^2b - 7.5ab^2$. 13. 16.
 2. $x^2 + 10x + 24$. 3. $x^2 - x - 2$.

- Page 110. — 4. $8x^2 + 10x - 3$. 5. $x^3 + 1$. 6. $6a^2 + 5ab - 3ac + b^2 - bc$.
 7. $x^2 + 2xy + y^2$. 8. $x^2 + 5x - 36$.
 9. $6x^2 - 7x - 5$. 10. $20y^2 - 7yz - 3z^2$. 11. $2x^2 + 7x + 6$.
 12. $12x^2 - 13x - 4$. 13. $20n^2 - 21n - 5$. 14. $3h^2 - 5hk - 2k^2$.
 15. $6l^2 + 8lt - 30t^2$. 16. $8y^2 - 8by - 6b^2$. 17. $10b^2 + 21bc - 10c^2$.
 18. $a^2x^2 - b^2y^2$. 19. $c^4 - d^4$. 20. $9a^2 + 6ab + b^2$.
 21. $2n^4 + 3ln^2 - 2l^2$. 22. $6x^2 - 1.3xy + .05y^2$. 23. $6a^{2n} + 11a^nb + 3b^2$.
 24. $6b^2 + 31bc + 40c^2$. 25. $\frac{1}{4}a^2 + \frac{1}{3}ab + \frac{1}{5}b^2$. 26. $\frac{1}{2}x^2 + \frac{7}{8}xy - \frac{1}{8}y^2$.
 27. $8x^2y^2 + 2xy^2 - 21y^2$. 28. $16a^2x^2 + 24abxy + 9b^2y^2$. 29. $x^4 - 1$.
 30. $x^4 - 6x^3 + x^2 + 24x - 20$. 31. $14 + 5x - 15x^2 - 5x^3 + x^4$.
 32. $a^4 + 3a^3 - 15a^2 - 19a + 30$. 33. $a^5 - 19a - 6$. 34. $x^4 - 2x^2y^2 + y^4$.
 35. $2b^4 + 7b^3 - 27b^2 - 8b + 16$.

- Page 111. — 36. $a^6 + 2a^5 + 5a^4 + 3a^3 + a$. 37. $8y^4 - 8y^3 - 6y^2 + 40y - 50$.
 38. $93x^4 - 50x^3 - 3$. 39. $120 - 240x - 190x^2 + 160x^3 - 10x^4$.
 40. $40x - 18x^2 - 29x^3 + 36x^4 - 20x^5$.
 41. $6a^4 - 11a^3b - 15a^2b^2 + 17ab^3 + 15b^4$. 42. $a^2 + 2ad - b^2 - 2bc - c^2 + d^2$.
 43. $20x^4 + 9x^3y + 7x^2y^2 - 36y^4$. 44. $3m^5 + 4m^4n - 2m^3n^2 + m^2n^3 - 6mn^4$.
 45. $a^3 - 3abc + b^3 + c^3$. 46. $m^{10} + n^{15}$.
 47. $x^{5a} - 5x^{4a} + 10x^{3a} - 10x^{2a} + 5x^a - 1$. 48. $a^2x^{4n} - a^2y^{4n}$.
 49. $9a^4 + 30a^2b^3 + 25b^6$. 50. $16c^2 - d^2$. 51. $\frac{1}{4}x^2 + xy + y^2$.
 52. $9x^6 - 3x^3y + \frac{1}{4}y^2$. 53. $\frac{1}{4}a^2 - \frac{1}{5}b^2$. 54. $\frac{1}{2}x^2 + \frac{1}{12}xy + \frac{1}{12}y^2$.
 55. $8x^3 + 24x^2y + 24xy^2 + 8y^3$. 56. $a^2 + 2ab + 2ac + b^2 + 2bc + c^2$.
 57. $a^4 - b^4$. 58. $a^{4m} - b^{4n}$. 59. $3a^2x^{2n-2} + 5ax^{n-1}y^{n-1} + 2y^{2n-2}$.

60. $x^{m+1} + x^m y - nx^n y - \frac{1}{2} nx^{n-1} y^2 + \frac{1}{2} nx^{n-2} y^3$. 61. $x^{4n} + x^{2n} y^{2n} + y^{4n}$.

62. $a^{8n} - b^{8c}$.

1. $5x$; x^5 . 2. $x \times y$; $x \cdot y$; xy .

Page 112. — 3. $2 + 3$; $x - y - z$. 4. $25 - 10$, or 15 ; $25 - n$.
5. $4 + 9$, or 13 ; $a + b$. 6. $20 - (10 + 4)$, or 6 ; $z - (10 + y)$.
7. $10 - 6$, or 4 ; $4 - m$. 8. $a - 10$. 9. $(10 - ad)$ dollars;
 $100(10 - ad)$ cents. 10. Albert, $(x - m)$ dollars; Charles, $(y + m)$
dollars. 11. $40 - x$. 12. $2n + 2$, $2n + 4$; $2n - 1$, $2n + 1$.

13. $2n + 3$, $2n + 5$. 14. $\frac{n}{d}$; $\frac{n-2}{d+3}$. 15. $(100d - cp)$ cents.

16. $10t + u$. 17. $100h + 10t + u$. 18. $24t + 17u$; 257 .

Page 113. — 19. $xy(x + y - d)$. 21. $12(a - b)$ dollars.
22. $m(x + y)(x + z)$. 23. $x = d(y - c)$. 24. $60(60c + d)$ sec.
25. $12xyz$ cents. 26. ab mi.; $b(a - c)$ mi. 27. $4b^2$ sq. rd.;
 $(x - y)^2$ sq. rd. 28. $(a + b)(a - b)$ sq. rd. 29. b^2 sq. rd.;
 $b(a - b)$ sq. rd. 30. $(a - b)(10t + u)$.

Page 115. — 2. $c - 5d$. 3. $z + 3 - xz$. 4. $-x + 2xy - 3y$.
5. $b^2 - 3ab + 4a^2$. 6. $-3a^2 - 4ab - 5b^2$. 7. $2a^3b^2 + 7a$.
8. $-7yz^3 - 9x^2yz$. 9. $-3a - 5bc$. 10. $4r + 5 - 7s$.
11. $-x - 4yz^2 + 5z^4$. 12. $2a^3 + 7a^2b - 4ab^2 - b^3$.
13. $z + 5x^2yz^2 + 2xz^2 - 6x^3y^2$.

Page 116. — 14. $a - 2y$; $-a + 2y$. 15. $3x + 5$; $-3x - 5$.
16. $z^2 + 3 - xz$; $-z^2 - 3 + xz$. 17. $x - 2x^3 + 3x^5 - 4x^7$;
 $-x + 2x^3 - 3x^5 + 4x^7$. 18. $2r^2s^3 + rs^2 - 3s^4 + 5$; $-2r^2s^3 - rs^2 + 3s^4 - 5$.
19. $-t^4 - t^3v + u^3v - t^2u^2v^2 + uv^3$; $t^4 + t^3v - u^3v + t^2u^2v^2 - uv^3$.

Page 117. — 1. $x + 2$. 2. $x - 2$. 3. $x - 3 + \frac{-8}{x+4}$. 4. $x + 4$.
5. $x + 3$. 6. $l^2 - 8 + \frac{32}{l^2 + 2}$. 7. $3 + x$. 8. $x + 5 + \frac{3}{x+3}$.
9. $m + 3$. 10. $x + y$.

Page 118. — 11. $3a^2 - 2a + 3$. 12. $5x^2 + x + 2$.
13. $5x^3 + 4x^2 + 3x + 2$. 14. $a^7 - a^6 + 2a^5 - 2a^4 + 3a^3 - 3a^2$
 $+ 3a + \frac{-1}{a+1}$. 15. $b^6 + 2b^3 + b + 1 + \frac{4}{b^3 + 4}$. 16. $z^4 + z^2 + 2$.
17. $x^2 - xy + y^2$. 18. $4m^2 + 2mn + n^2$. 19. $2a^2 - 3ab + b^2$.
20. $20 - 2a - a^2 + a^3$. 21. $2y^2 - 3y + 1$. 22. $x^2 + 5x + 7$.
23. $2a + 3b$. 24. $am^3 - 3m^2$. 25. $6a^2 + 6a - 12$.
26. $4a^2 - a + 4$. 27. $x^2 - ax - b$. 28. $5x^2 + 2xy - 3y^2$.
29. $a^2 - 2ax + x^2$.

Page 119. — 31. $x^4 - 2x^3 + 4x^2 - 8x + 16$. 32. $x^4 - x^2y^2 + y^4$
 $+ \frac{-2y^6}{x^2 + y^2}$. 33. $m^4 - m^3n + m^2n^2 - mn^3 + n^4 + \frac{-2n^5}{m+n}$.
34. $m^4 - m^3n + m^2n^2 - mn^3 + n^4$. 35. $x^6 + x^5 - x^4 - x^3 + x^2 + x - 1$.
36. $2x^2 - 3x - 4$. 37. $7x^2 - 2x - 1$. 39. $a^{2n} - 2a^n + 4$.
40. $x^{3a} + 2x^{2a} + 7x^a + 20$. 41. $a^{3n} - 4a^{2n} + 11a^n - 24$.
42. $5x^{3n} + 4x^{2n} + 3x^n + 2$. 43. $a^{7n} - a^{6n} + 2a^{5n} - 2a^{4n} + 3a^{3n}$
 $- 3a^{2n} + 4a^n + 1$. 44. $2y^{n+2} + 3y^{n+1} - y$.

Page 120. — 1. 500 cents; $\frac{5}{100}$ dollars; 100 m cents; $\frac{m}{100}$ dollars.

2. $x = 8$. 3. $\frac{300}{25}$, or 12; $\frac{x}{10}$; $\frac{s}{r}$. 4. $\frac{a^3 - x^5}{m + n^2}$. 5. $\frac{lmn}{lm}$. 6. $\frac{5}{100}x$;
 $\frac{y}{100}z$. 7. ac da.; $\frac{ac}{2}$ da.; $\frac{ac}{x}$ da. 8. $\frac{D}{d} = q + \frac{r}{d}$. 9. $\frac{s}{r}$ mi.;
 $\frac{m}{h}$ mi.; $\frac{s + m}{r + h}$ mi. 10. $\frac{r + 60}{2}$ cents. 11. $\frac{4}{5}D$ dollars. 12. $(3f + 5)$ in.
 13. 88 m ft.; $\frac{22}{15}m$ ft. 14. $\frac{3m}{h}$ mi.; $\frac{am}{h}$ mi.

Page 121. — 15. $\frac{ce}{b}$ lb. 16. $\frac{i}{36}$ yd.; $\frac{f}{3}$ yd. 17. $(y - 6)$ yr.

18. $(d + \frac{c}{100})$ dollars. 19. Area, bc sq. in.; perimeter, $(2b + 2c)$ in.
 2. Length, 80 ft.; width, 40 ft. 3. Width, 45 ft.; depth, 90 ft.

Page 122. — 4. 15. 5. 16, 17. 6. 70 seniors.

7. Wrapping paper, 150 tons; newspapers, 200 tons. 8. 32 ft.
 9. 4,200,000 ft. 10. Stanley, 6 fish; Edward, 8 fish.
 11. 40,000 Chinese. 12. 25 words.

Page 123. — 13. 22 words. 14. 8 min. 15. Charles, 12 yr.; Alice, 8 yr. 16. 54 times. 17. John, 50¢; father, 90¢.

1. $ax - 12bx + 5by$.

Page 124. — 7. $4a + 11b - 2c$.

8. $2x + y - 4w$.
 9. $2r^2 + 8rs + 2s^2$. 10. $18x - 5y - 3z + 2a$.
 11. $3a + 7b - 4c$. 12. 6. 13. 20. 14. -4 . 15. 210.
 16. $(ax^2 - 2ax^2 + ax^2) + (ax - ax) - (cy - cy) + (2cy^2 - cy^2)$.
 17. $(7a - 22a^2 - 17b - 4)x - (3b - 5a - c - 13)y$.
 18. $b = -2\frac{1}{3}$. 19. $a = -1$. 20. $x = 4$. 21. $x = 3$. 22. $x = \frac{5}{7}$.
 23. $x = -4$. 24. $x = -9\frac{1}{3}$. 25. $x = 2$.

Page 126. — 2. $5x^2(x - 1)$. 3. $2x^2(4 + x^2)$. 4. $3x^2(x - 2y)$.

5. $2a(2a - 3b)$. 6. $m(5m - 3n)$. 7. $3x^2y^2(x - y)$.
 8. $2a^2b(2a - 3b)$. 9. $5m^3n(m - 2n)$. 10. $3x^2(x^2 - 3x - 2)$.
 11. $a^2(3a^2 - 2ab + 1)$. 12. $x^9(x^3 + x^2 + x - 1)$.
 13. $c(a - b - y - ab)$. 14. $3xy(x^2y^2 - xy + 4)$.
 15. $3m(m^4 - 4m^2n^2 + 2n^4)$.

Page 127. — 18. $(x - y)(a + 5)$.

19. $(a + b)(4 - x)$.
 20. $(y + z)^2(c^2 - d)$. 21. $(x - y)(5 + 2a)$. 22. $(3x - 5y)(2a - b)$.
 3. $(a + x)(m - n)$. 4. $(b + x)(c - d)$. 5. $(p - r)(q - x)$.
 6. $(a - b)(y - b)$. 7. $(x - 5)(x - y)$. 8. $(x - a)(x + y)$.
 9. $(c + a)(c - 4)$. 10. $(1 + 2x)(2x - y)$. 11. $(1 - m)(1 + n)$.
 12. $(1 + 3p)(2p + q)$.

Page 128. — 13. $(a + b)(b - c)$.

14. $(a - s)(r - b)$.
 15. $(x^2 + 1)(x + 1)$. 16. $(y^2 - 3)(y + 1)$. 17. $(x^3 + y)(x^2 + 1)$.
 18. $(2 - n^2)(1 - n)$. 19. $(x + a)(x - 1)$. 20. $(3x - 2y)(x^2 - 5)$.
 21. $a(4 - a)(3a^2 - 2b)$. 22. $(a + 3mn)(m - 3n)$.
 23. $b(5a - 3c)(3b - 7)$. 24. $2(2a - b)(4x + 3y)$.

25. $(ax - ay + 1)(x - 1)$.
 27. $(a - b - c)(x - 1)$.
 29. $(b + y)(x^2 - x - 1)$.
 2. $a^2 + 8a + 16$. 3. $m^2 + 6m + 9$. 4. $x^2 - 10x + 25$.
 5. $x^2 - 16x + 64$. 6. $y^2 - 4y + 4$. 7. $r^2 - r + \frac{1}{4}$.
 8. $a^2 + 2ab + b^2$. 9. $a^2 + 6ab + 9b^2$.

- Page 129. — 11. $a - b$; $a^2 - 2ab + b^2$. 14. $y^2 + 6y + 9$.
 15. $a^2 - 4ab + 4b^2$. 16. $a^2 - 8at + 16t^2$. 17. $k^2 + 20k + 100$.
 18. $k^2 - 20k + 100$. 19. $a^2 + 2ab + b^2$. 20. $4y^2 - 20y + 25$.
 21. $9a^2 + 12abc + 4b^2c^2$. 22. $64x^2 - 80axy + 25a^2y^2$.
 23. $x^2 + 2mx + m^2$. 24. $m^2 - 2mn + n^2$. 25. $x^2 - 12x + 36$.
 26. $p^2 - 16p + 64$. 27. $r^2 + 14r + 49$. 28. $x^4 + 18x^2 + 81$.
 29. $y^4 - 12y^2 + 36$. 30. $4a^2 - 24ax + 36x^2$.
 31. $9m^2 - 48mn + 64n^2$. 32. $16x^2 + 8xy + y^2$.
 33. $p^2 + 6pq + 9q^2$. 34. $16a^2 - 24a + 9$. 35. $25x^2 + 40x + 16$.
 36. $a^2b^2 - 6ab + 9$. 37. $4a^2 + 12ab + 9b^2$.
 38. $4x^2 - 28xy + 49y^2$. 39. $64x^2 - 80xy + 25y^2$.
 40. $9x^2 - 12xy + 4y^2$. 41. $25m^2 - 30mn + 9n^2$.
 42. $9p^2 - 30pq + 25q^2$. 43. $16x^2y^2 - 56xy + 49$.
 44. $x^2 + x + \frac{1}{4}$. 45. $y^2 - \frac{2}{3}y + \frac{1}{9}$. 46. $a^2 + \frac{5}{3}a + \frac{16}{9}$.
 47. $m^2 - \frac{4}{7}m + \frac{4}{49}$. 48. $25x^2 + 10xz + z^2$. 49. $4a^2 + 4ax + x^2$.
 50. $a^2b^2 + 2abcd + c^2d^2$. 51. $r^2 + \frac{5}{9}r + \frac{16}{81}$. 52. $s^2 - \frac{6}{5}s + \frac{9}{25}$.
 53. $x^2 - \frac{8}{11}x + \frac{16}{121}$. 54. $z^2 - \frac{3}{4}z + \frac{9}{16}$. 55. $a^6 + 2a^3b^3 + b^6$.
 56. $a^{10} + 2a^5b^5 + b^{10}$. 57. $a^{2n} + 2a^nb^n + b^{2n}$.
 58. $4x^2 - 2x + \frac{1}{4}$. 59. $9x^2 + 2x + \frac{1}{9}$. 60. $4x^2 - \frac{4}{5}x + \frac{1}{25}$.
 61. $9x^2 - \frac{6}{5}x + \frac{1}{25}$. 62. $25x^2 + 20xy + 4y^2$.
 63. $49z^2 + 42cz + 9c^2$. 64. $9b^2 + 30bx + 100x^2$.

- Page 130. — 65. $4a^2 - 4ax + x^2$. 66. $9m^2 - 6mn + n^2$.
 67. $16x^2 - 8xy + y^2$. 68. $m^2 - 8mn + 16n^2$.
 69. $p^2 - 6pq + 9q^2$. 70. $a^2 - 14ab + 49b^2$. 71. $16a^2 - 24a + 9$.
 72. $25x^2 - 40x + 16$. 73. $a^2b^2 - 6ab + 9$.
 74. $4z^2 - 28yz + 49y^2$. 75. $49x^2 - 70xy + 25y^2$.
 76. $81w^2 - 36vw + 4v^2$. 77. $9x^2 - 12x + 4$.
 78. $4x^2 - 20xy + 25y^2$. 79. $25m^2 - 30mn + 9n^2$.
 80. $16p^2 - 40pq + 25q^2$. 81. $a^{2n} - 2a^nb^n + b^{2n}$.
 82. $x^{2m} - 2x^my^n + y^{2n}$. 83. $a^4 - 4a^2b^2 + 4b^4$.
 84. $y^6 - 12xy^3 + 36x^2$. 85. $a^2b^2 - 4abc^2 + 4c^4$.
 86. $x^{2t+2r} - 2x^{t+r}y^{2r} + y^{4r}$. 87. $m^2x^{2m} - 2mnx^my^n + n^2y^{2n}$.
 88. $x^{2m-2} - 2x^{m-1}y^{n-1} + y^{2n-2}$. 89. $.16x^2 - .12xy^2 + .0225y^4$.
 90. $4a^2 + 1.2ab + .09b^2$. 91. $2.25a^4 + 6.3a^2 + 4.41$.
 92. $.49x^{2a} - 1.54x^ay^b + 1.21y^{2b}$. 93. $1.69x^{2m} - 1.04x^my^n + .16y^{2n}$.
 94. $4a^2b^2 + 3.2abc^2 + .64c^4$. 95. $1 + .06 + .0009$, or 1.0609 .
 96. $1 + .08 + .0016$, or 1.0816 . 97. 1.1236 . 98. 1.1025 .
 99. 4.1209 . 100. 4.2025 . 101. $30\frac{1}{4}$. 102. $72\frac{1}{4}$. 103. 1681 .
 104. 1764 . 105. 2809 . 106. $156\frac{1}{4}$. 107. 341 . 108. 1521 .
 109. $13,225$. 110. 2025 . 111. 56.25 . 112. 90.25 .

- Page 131. — 1. $x^2 + 2xy + y^2$. 2. $a^2 - 2ab + b^2$.
 3. $y^2 + 2yz + z^2$. 4. $c^2 - 2cd + d^2$. 5. $x^2 + 4xy + 4y^2$.
 6. $r^2 - 8rs + 16s^2$. 7. $a^2 + 14ab + 49b^2$. 8. $p^2 - 8pq + 16q^2$.
 9. $x^2 + 6xy + 9y^2$. 10. $(x + y)(x + y)$. 11. $(p - q)(p - q)$

Page 131 (continued). — 12. $(c + d)(c + d)$. 13. $(x - 1)(x - 1)$.
 14. $(x + 2)(x + 2)$. 15. $(x + 3)(x + 3)$. 16. $5(x + 3)(x + 3)$.
 17. $(4p - 3)(4p - 3)$. 18. $(3x - 7)(3x - 7)$. 19. $(3 + 7b^3)(3 + 7b^3)$.
 20. $(6n^4 - 1)(6n^4 - 1)$. 21. $16(x^2 - 3y)(x^2 - 3y)$.

Page 132. — 22. $(2 - a)(2 - a)$. 23. $a(2 - a)(2 - a)$.
 24. $(m - 4)(m - 4)$. 25. $(a - 8)(a - 8)$. 26. $(1 + 4b)(1 + 4b)$.
 27. $3(x + y)(x + y)$. 28. $2(m - 2n)(m - 2n)$.
 29. $(1 - 3a^3)(1 - 3a^3)$. 30. $(4x - y)(4x - y)$.
 31. $(2 + 3m^2)(2 + 3m^2)$. 32. $(3 - 2ab)(3 - 2ab)$.
 33. $(s - 6r)(s - 6r)$. 34. $(2x + ay)(2x + ay)$.
 35. $(3a + 2.5b)(3a + 2.5b)$. 36. $3(5x - 7y)(5x - 7y)$.
 38. $(2a + 3b^{2n})(2a + 3b^{2n})$. 39. $.01(3a^{3n} + 5b)(3a^{3n} + 5b)$.
 40. $(2xy - 5)(2xy - 5)$. 41. $(2x + 3yz)(2x + 3yz)$.
 42. $2x(1 + 5a^2)(1 + 5a^2)$. 43. $2b(2a + 5b)(2a + 5b)$.
 44. $4(x^a + y^b)(x^a + y^b)$. 45. $(x^n - y^{nz^n})(x^n - y^{nz^n})$.
 46. $4(x^m + 2y^n)(x^m + 2y^n)$. 47. $(1 + 5mn)(1 + 5mn)$.
 48. $a(x + 3y^2)(x + 3y^2)$. 49. $(ab - cd)(ab - cd)$.
 50. $(3xy + 2rs)(3xy + 2rs)$. 51. $(4x - 7y)(4x - 7y)$.
 52. $.09(5a^2b - 1)(5a^2b - 1)$. 53. $b^2(7a - 1.3)(7a - 1.3)$.
 54. $5(3x^3 - 4y)(3x^3 - 4y)$. 56. $(2x - y)(2x - y)$.
 57. $(2 - t)(2 - t)$. 58. $(4c - 3a)(4c - 3a)$.
 59. $(r + s - 2)(r + s - 2)$. 60. $(4 + 3l - 3t)(4 + 3l - 3t)$.
 61. $(x - y + 7)(x - y + 7)$. 62. $(a - c)(a - c)$.
 63. $(a - 2b + x)(a - 2b + x)$. 64. $16(a + b)(a + b)$.

Page 133. — 1. $(x + 3)(x + 3)$. 2. $(m + 2n)(m + 2n)$.
 3. $(rs - 4)(rs - 4)$. 4. $(4p - 3)(4p - 3)$. 5. $(2x^2 + y^2)(2x^2 + y^2)$.
 6. $(3 + 7b^3)(3 + 7b^3)$. 7. $(3a - 10b)(3a - 10b)$.
 8. $(1 - 3m^4)(1 - 3m^4)$. 9. $(2xy - 5)(2xy - 5)$.
 10. $(2x + 3yz)(2x + 3yz)$. 11. $(7x + 5)(7x + 5)$.
 12. $(x - 10y)(x - 10y)$. 13. $(7a^2 + 8)(7a^2 + 8)$.
 14. $4(2b - c)(2b - c)$. 15. $(5x + 7y)(5x + 7y)$.
 16. $(x - \frac{1}{3})(x - \frac{1}{3})$. 17. $(a - \frac{1}{5})(a - \frac{1}{5})$. 18. $(2a + \frac{1}{4})(2a + \frac{1}{4})$.
 19. $(3b - \frac{1}{5})(3b - \frac{1}{5})$. 20. $(4m - \frac{1}{3}n)(4m - \frac{1}{3}n)$.
 2. $a^2 - b^2$. 3. $m^2 - 100$. 4. $r^2 - 16$. 5. $x^2 - y^2$.
 6. $x^2 - 4y^2$. 7. $x^2 - 16$. 8. $c^2 - 25$. 9. $25m^2 - 4b^2$.
 10. $9t^2 - 625$.

Page 134. — 1. $x^2 - y^2$. 2. $a^2 - c^2$. 3. $p^2 - q^2$. 4. $p^2 - 25$.
 5. $x^2 - 1$. 6. $x^4 - 1$. 7. $a^2b^2 - c^4$. 8. $16y^2 - z^4$.
 9. $r^2s^2 - 25m^2$. 10. $9m^2 - 16n^2$. 11. $l^2x^2 - 25m^2$.
 12. $9m^2 - 16n^2$. 13. $16x^6y^6 - 25$. 14. $144 - x^2y^2$.
 15. $9m^4 - b^2$. 16. $4x^6 - 25y^4$. 17. $4a^4 - 9b^4$. 18. $25n^2 - b^2$.
 19. $\frac{1}{4}x^2 - 9$. 20. $\frac{4}{9}a^2 - 16b^2$. 21. $x^2 - 4y^2$.
 22. $9x^{2m} - 49y^{2n}$. 23. $m^2x^{2a} - 4y^{2b}$. 24. $a^{2n}b^{2m} - a^{2m}b^{2n}$.
 25. $x^{2m-2} - y^{2n+2}$. 26. $25a^6b^4 - 4x^{2z}$. 28. 399. 29. 899.
 30. 1584. 31. 3584. 32. 8099. 33. 2475. 34. 884. 35. 836.
 36. 3591. 37. 3596. 38. 1591. 39. 9975. 40. 4875. 41. 891.
 42. 8091. 43. 3564.

Page 135. — 1. $(x + 2)(x - 2)$. 2. $(a + 3)(a - 3)$.
 3. $(x + y)(x - y)$. 4. $(z + 2)(z - 2)$. 5. $(c + d)(c - d)$.
 6. $(a + 4)(a - 4)$. 7. $(x + m)(x - m)$. 8. $(5 + c)(5 - c)$.

9. $(x^2 + 7)(x^2 - 7)$. 10. $(a^2 + 9)(a + 3)(a - 3)$.
 11. $(5x + 1)(5x - 1)$. 12. $(bc + 4)(bc - 4)$. 13. $(ax + 2c)(ax - 2c)$.
 14. $(5 + 3x)(5 - 3x)$. 15. $(3b + cd)(3b - cd)$.
 16. $(a^3 + b^4)(a^4 + b^2)(a^2 + b)(a^2 - b)$. 17. $(2x + 5y)(2x - 5y)$.
 18. $(3a + 7b)(3a - 7b)$. 19. $(m^2 + 4n^2)(m + 2n)(m - 2n)$.
 20. $(4a^2 + 9b^2)(2a + 3b)(2a - 3b)$. 21. $x(x + y)(x - y)$.
 22. $2(2s + t)(2s - t)$. 23. $(13 + ab)(13 - ab)$.
 24. $2(x^2 + y^2)(x + y)(x - y)$. 25. $(11b + ac)(11b - ac)$.
 26. $4(10x + 3y)(10x - 3y)$. 27. $16(3m + n)(3m - n)$.
 28. $2(3c + 5)(3c - 5)$. 29. $3(x^2 + y^3)(x^2 - y^3)$.
 30. $9(2a^2 + 5)(2a^2 - 5)$. 31. $(x^2y^2 + 16)(xy + 4)(xy - 4)$.
 32. $3m(m^2 + 1)(m + 1)(m - 1)$. 33. $4(a^2 + b^2)(a + b)(a - b)$.
 34. $5(x^4 + 1)(x^2 + 1)(x + 1)(x - 1)$. 35. $2(x^4 + y^4)(x^2 + y^2)(x + y)(x - y)$.
 36. $(a + \frac{1}{3})(a - \frac{1}{3})$. 37. $(x + .1)(x - .1)$.
 38. $(x^2 + \frac{1}{4})(x + \frac{1}{2})(x - \frac{1}{2})$. 40. $(x^{2a} + y^{2b})(x^a + y^b)(x^a - y^b)$.
 41. $(2a + .5)(2a - .5)$. 42. $(2a^n + b^{2m})(2a^n - b^{2m})$. 44. 65.25.
 45. 256. 46. 70.56. 47. 7000. 48. 50.6. 49. 82.4.

- Page 136. — 2. $(a + b + c)(a - b - c)$. 3. $-b(2a + b)$.
 4. $-4a(a + b)$. 5. $(b + 3c)(c - b)$. 6. $(3b + a - x)(3b - a + x)$.
 7. $5(a - 1)(a + 5)$. 8. $4(2x^2 - y)(y - x^2)$. 9. $8(3a - b)(a + 2b)$.

- Page 137. — 11. $(3a + 4b)(a + 2b)$. 12. $5x(10 - x)$.
 15. $(a + x - 3y)(a - x + 3y)$. 16. $(3a + b - 2c)(3a - b + 2c)$.
 17. $(2x - y + a)(2x - y - a)$. 18. $(6a + 7b)(4a + 5b)$.
 19. $4(3a - 2b)(2a - b)$. 20. $4(3 - x)(1 + 4x)$.
 21. $(2a - 2b - 5)(5 - 2b)$. 22. $(2x + z)(2x - 6y - z)$.
 23. $(a - x + n)(a - x - n)$. 24. $(b + y + n)(b + y - n)$.
 25. $(1 - 2q + a)(1 - 2q - a)$. 26. $(r - x + 4t)(r - x - 4t)$.
 27. $(c + a + b)(c - a - b)$. 28. $(r + s - 5)(r - s + 5)$.
 29. $(x + 3 + 4y)(x + 3 - 4y)$. 30. $(x + 2 + y)(x + 2 - y)$.
 31. $(3a + 5b - 6c)(5b - 3a - 2c)$. 33. $(13x + 3y)(5x + 9y)$.
 32. $3(2x - y - a)(2x - 3y + 3a)$. 35. $4a(b + c)$.
 34. $(x^2 + 2)(x + 1)(x^2 - 2)(x + 1)$. 37. $(2c + x + y)(2c - x - y)$.
 36. $(b + x - y)(b - x + y)$. 39. $b(c + 3a + b)(c - 3a - b)$.
 38. $(3c + x - y)(3c - x + y)$. 40. $a(b + 2a + 3c)(b - 2a - 3c)$.
 41. $(a - b + c - d)(a - b - c + d)$.
 42. $(x - y + m - 5)(x - y - m + 5)$.
 43. $(x + y + a + b)(x + y - a - b)$.
 44. $(2x - 3 + m - 5n)(2x - 3 - m + 5n)$.

- Page 138. — 3. $x^2 + 11x + 30$. 4. $x^2 + 15x + 56$.
 5. $x^2 + x - 56$. 6. $x^2 - x - 56$. 7. $x^2 - 9x + 20$.
 8. $x^2 - 5x + 6$. 9. $x^2 - 6x + 5$. 10. $x^2 + 13x + 40$.
 11. $p^2 - 3p - 4$. 12. $r^2 - 4r + 3$. 13. $n^2 + 9n - 90$.
 14. $x^4 + 2x^2 - 15$. 15. $x^6 - x^3 - 42$.

- Page 139. — 17. $x^{2n} - 3ax^n + 2a^2$. 18. $y^2 + by - 6b^2$.
 19. $z^2 - az - 12a^2$. 20. $4x^2 - 4x - 35$. 21. $9y^2 + 3y - 2$.
 22. $16x^4 - 24x^2 - 7$. 23. $a^2b^2 - 2ab - 24$. 24. $x^4y^4 + ax^2y^2 - 2a^2$.
 25. $y^4 + 2xy^3 - 3x^2y^2$. 26. $b^4c^4 + 2b^2c^2ef + e^2f^2$.
 27. $.25a^2b^2 + 2abc^2 + 4c^4$. 28. $.09x^4 - 4y^4$.

- Page 140. — 3. $(x + 4)(x + 3)$. 4. $(y - 4)(y - 3)$.
 5. $(p - 6)(p - 2)$. 6. $(r + 6)(r + 2)$. 7. $(3 + a)(5 - a)$.
 8. $(b + 4)(b - 3)$. 9. $(6 - r)(5 + r)$. 10. $(c - 9)(c + 8)$.
 11. $(c - 7)(c + 2)$. 12. $(x - 11)(x + 10)$. 13. $(x - 2)(x - 1)$.
 14. $(a + 7)(a + 2)$. 15. $(y - 3)(y - 2)$. 16. $(b + 5)(b - 3)$.
 17. $(x + 5b)(x - 4b)$. 18. $2(x - 3y)(x + 2y)$. 19. $(d - 4r)(d + 3r)$.
 20. $(7y - x)(3y + x)$. 21. $(x + 3a)(x + 2a)$.
 22. $(x - 5a)(x - a)$. 23. $(y - 6b)(y + 2b)$. 24. $(y - 7n)(y + 4n)$.
 25. $(z - 2an)(z + an)$. 26. $(20 - x)(x - 5)$, or $(x - 20)(5 - x)$.
 27. $(x^2 + 10c)(x^2 + 9c)$. 28. $(x^3 + 10a)(x^3 + 2a)$.
 29. $(x^4 - 8b^2)(x^4 - 3b^2)$. 30. $(nx - 6y)(nx - 5y)$.
 1. $(x + 4)(x + 2)$. 2. $(a + 5)(a + 3)$. 3. $(b - 8)(b - 4)$.
 4. $(r - 8)(r - 6)$. 5. $(r - 10)(r - 9)$. 6. $(m - 10)(m + 3)$.
 7. $(x - 10)(x + 2)$. 8. $(y + 11)(y - 3)$.

- Page 141. — 9. $(k + 9)(k - 4)$. 10. $(a + 8)(a - 1)$.
 11. $(x - 8y)(x + 2y)$. 12. $(rs - 8t)(rs - 3t)$.
 13. $(a + 13b)(a + 3b)$. 14. $(c - 12d)(c - 4d)$.
 15. $(x - 27a)(x + 5a)$. 16. $(c - 44x)(c + 10x)$.
 17. $(x - 35)(x + 20)$. 18. $(m - 5)(m - 2)$. 19. $(z + 1)(z + 5)$.
 20. $(x + \frac{1}{2})(x + \frac{1}{4})$. 21. $(x + \frac{3}{2})(x + \frac{3}{2})$. 22. $(y - 2)(y + 1)$.

- Page 142. — 2. $14y^2 + 52y + 30$. 3. $6y^2 + 17y + 12$.
 4. $56x^2 + 11x - 15$. 5. $6x^2 + 5x - 6$. 6. $6r^2 + 14r + 4$.
 7. $6r^2 + 19r + 15$. 8. $4t^2 - 8t - 5$. 9. $6a^2 + 14a - 12$.
 10. $6a^2 + 22a + 12$. 11. $56y^2 + 59y + 15$. 12. $56y^2 + 11y - 15$.
 13. $56y^2 - 11y - 15$. 14. $56y^2 - 59y + 15$. 15. $35x^2 + 6x - 8$.
 16. 12. 17. 6. 18. 6. 19. $12y^2$. 20. 15. 21. $30a^2$. 22. 35.
 23. 24. 24. $6y^2$. 25. $6b^2$.

- Page 143. — 26. $15x^2$. 27. $-21a^2$. 28. $-6y^2$. 29. $18b^2$.
 30. $-4x^2$. 31. $-10x^2$. 32. $22x$. 33. $14xy$. 34. $22x$.
 35. $49xy$. 36. $18a$. 37. $12x^2 + 23x + 10$. 38. $12x^2 + 17x + 6$.
 39. $2x^2 - 17xy + 21y^2$. 40. $6x^2 - 14x - 40$. 41. $12x^2 - 7x - 12$.
 42. $x^2 + 15bx + 54b^2$. 43. $6x^2 + 11cx - 10c^2$.
 44. $6x^2 + 2xy - 20y^2$. 45. $15a^2 - 8ab - 12b^2$.
 46. $6x^2 + 23x + 20$. 47. $6x^2 - 13x + 6$. 48. $12a^2 - 7a - 12$.
 49. $3x^2 - 10xy + 3y^2$. 50. $21z^2 + 11az - 2a^2$.
 51. $10a^2 + 29ab + 10b^2$. 52. $14n^4 - 53n^2p + 14p^2$.
 53. $8m^2 - 24mn + 18n^2$. 54. $15x^4 + 2x^2y - y^2$.
 55. $21y^4 + y^3 - 2y^2$.

- Page 145. — 1. $(2y + 1)(y + 1)$. 2. $(5x - 1)(x + 2)$.
 3. $(3x + 2)(x - 3)$. 4. $(2r + 3)(2r + 1)$. 5. $(3x - 2)(2x - 1)$.
 6. $(2x + 3)(x - 4)$. 7. $(5t + 3)(2t - 1)$. 8. $(3n - 2)(2n - 3)$.
 9. $(2x - 5)(x + 3)$. 10. $(5x + 3)(x + 2)$. 11. $(3x - 2)(x - 5)$.
 12. $(3x + 5)(2x - 7)$. 13. $(5x - 1)(3x + 4)$.
 14. $(5x + 2)(3x - 4)$. 15. $(2x - y)(x + 2y)$.
 16. $(3x - y)(x - 3y)$.

- Page 146. — 1. $x^{2n} + 2x^ny^n + y^{2n}$. 2. $9 - 12y^2 + 4y^4$.
 3. $a^{2n} + 2a^nb^m + b^{2m}$. 4. $1 - 6xy + 9x^2y^2$. 5. $4x^8 + 2x^4 + .25$.
 6. $r^6 - .4r^3s + .04s^2$. 7. $y^{2m} - 2y^{m+n} + y^{2n}$.
 8. $16a^2 + 2.4ab + .09b^2$. 9. $1 + 2r^as^b + r^{2a}s^{2b}$.

10. $.04x^6 - 2x^3 + 25$.
 12. $\frac{1}{8}m^4 - \frac{1}{2}m^2n + \frac{1}{8}n^2$.
 14. $\frac{1}{8}x^2 - xy + \frac{1}{8}y^2$.
 16. $4x^2y^2 + 4xyz^3 + z^6$.
 19. $\frac{1}{4}a^2 - 3a + 8$.
 22. $a^{4n} - b^{2m}$.
 25. $2.25x^2 - 9xy^4 + .09y^8$.
 27. $\frac{1}{16}r^2s^2 - \frac{1}{4}rst^{a+1} + \frac{1}{4}t^{2a+2}$.
 29. $m^{2a+4} - 2m^{a+b+2}n^{c+1} + m^{2b}n^{2c+2}$.
 31. $4a^2 + 12ab + 9b^2 + 16a + 24b + 16$.
 32. $x^{2m-2n} - 2x^{m-n}y^{m+n} + y^{2m+2n}$.
 34. $20.25x^4 - 31.5x^2 + 12.25$.
 36. $.42x^2 + 1.9x - 35$.
 38. $25r^{18} + 15r^{a+18} + 2.25r^{2a-2}$.
 40. $x^2 - 25b^2$.
 43. $4a^2 + 2ax + 2ay + xy$.
 45. $y^4z^4 - 3y^2z^2 - 40$.
 47. $9x^{2n} + 3m^nr^n + 3n^mx^n + m^nn^m$.
 49. $25c^2d^2x^2 - 25cdx - 6$.
 51. $a^2d^4x^6 - 13ad^2x^3 + 30$.
 53. $t^2 - u^2 - 2uv - v^2$.
 56. $49a^4x^6 - 36z^{2n}$.
 58. $8a^2z^6 + 4ayz^3 - 6ay^2z^3 - 3y^3$.
 60. $a^2 + 2ab + b^2 - c^2 - 2cd - d^2$.
11. $\frac{1}{4}x^4 + a^2x^2 + a^4$.
 13. $a^2b^2c^2 - abcd + \frac{1}{4}d^2$.
 15. $9a^2b^2 - 6abc + c^2$.
 17. $x^2 - x - 12$.
 18. $4x^2 + 6x + 2$.
 20. $a^2x^2 + 2ax - 3$.
 21. $x^2y^2 - 16$.
 23. $6r^2 + 34r + 20$.
 24. $25x^2 - .0144$.
 26. $9x^4y^4 + 12a^nx^2y^2z^2 + 4a^{2n}z^4$.
 28. $b^2d^2 - 2bcde + c^2e^2$.
 30. $6.25 + 60p^2q^2r^3 + 144p^4q^4r^6$.
 33. $a^4b^4c^4 + 2a^2b^2c^2d^2e^2 + d^4e^4$.
 35. $\frac{4}{9}m^8n^6 + m^4n^3p^2q^2 + \frac{1}{9}p^4q^4$.
 37. $16d^2e^2f^2 - 2abcdef + \frac{1}{16}a^2b^2c^2$.
 39. $.04a^2 - .5a - 1.5$.
 42. $6a^2 - ab - 2b^2$.
 44. $6y^2 - yz - 2z^2$.
 46. $10r^4 - 21r^2s - 10s^2$.
 48. $r^2 - s^2 - 2st - t^2$.
 50. $21b^2x^6 + 58bx^3 + 21$.
 52. $x^{2a+2b} + x^{a+b}y - 6y^2$.
 55. $2x^{2a-2} + x^{a-1}y - 3y^2$.
 57. $a^{2m-2} + 2a^{m-1}b^{n-1} + b^{2n-2}$.
 59. $x^{2a}y^{2b} - x^{2b}y^{2a}$.

- Page 148. — 1. $8(x+4)(x+3)$.
 3. $a(b-4)(b+3)$.
 6. $3(5-a)(3+a)$.
 9. $2(a^4+b^4)(a^2+b^2)(a+b)(a-b)$.
 11. $n(2m^2+4mn+n^2)$.
 13. $3(x+2)(x-2)$.
 16. $11(a^2+b^2)(a+b)(a-b)$.
 18. $3(x^5-8b^2)(x^5-3b^2)$.
 20. $(y+1)(y-1)(y+1)(y-1)$.
 22. $120(x^2+y^2)(x+y)(x-y)$.
 24. $7(y-6b)(y+2b)$.
 26. $9(x+3a)(x+2a)$.
 28. $3(m^2+2)(m^2-2)$.
 30. $5(a^2b^2+c^2d^2)(ab+cd)(ab-cd)$.
2. $15(y+3)(y-3)$.
 5. $12(x+y)(x-y)$.
 8. $2(x^2+y^2)(x+y)(x-y)$.
 10. $(m^2+1)(m+1)(m-1)$.
 12. $(5x+7)(5x+7)$.
 15. $2(5+x)(5-x)$.
 17. $(nx-6y)(nx-5y)$.
 19. $5(x-11)(x+2)$.
 21. $2(a+1)(a-1)(a+1)(a-1)$.
 23. $13(a^2+b^2)(a+b)(a-b)$.
 25. $8(x-5a)(x-a)$.
 27. $10(b+4)(b-3)$.
 29. $2(m^2+8)(m^2-8)$.

Page 149. — 1. $25(17+8)$, or 625; $15(18-13)$, or 75; $7^2(12-9)$, or 147. 2. 2100; 8600; 600. 3. $A = 16.5$; $A = 111$; $A = 502\frac{1}{2}$; $A = 15.1$.

- Page 150. — 4. $2x^2(4x-1)$.
 6. $4(x^2y^4+z^4)(xy^2+z^2)(xy^2-z^2)$.
 8. $3(x^a+2)(x-2)$.
 10. $(y^a+1)(y^a-1)$.
 13. $y(y^n+1)(y^n-1)$.
 15. $(a+.2b)(a-.2b)$.
 17. $(2a^n+2.5b^{2n+3})(2a^n-2.5b^{2n+3})$.
 19. $2(x+2)(x-4)$.
 5. $(b^4+9)(b^2+3)(b^2-3)$.
 7. $3x(x^2+1)(x+1)(x-1)$.
 9. $y(x^2+y^2)(x+y)(x-y)$.
 11. $a^2b^2(5b+4a)(5b-4a)$.
 14. $(\frac{1}{5}x^{2n}+\frac{1}{4}y^4)(\frac{1}{3}x^n+\frac{1}{2}y^2)(\frac{1}{3}x^n-\frac{1}{2}y^2)$.
 16. $d(d+3)(d-3)$.
 18. $4x(x+1)(x-1)$.
 21. $(x+1)(x+2)$.

Page 150 (continued).—22. $(a + b + c)(a - b - c)$.

23. $3.5(x + 2)(x - 3)$. 24. $(2a - 3b)(a + 2b)$.
 25. $(7x + y)(x - 3y)$. 26. $(x - 3)(x + 1)$. 27. $5a^2b^2(1 + a^2b^2)$.
 28. $(x + .3)(x + .2)$. 29. $(6x - 6y + z)(6x - 6y - z)$.
 30. $3(2a^{n-1} + 3.5b^2)(2a^{n-1} - 3.5b^2)$. 31. $(x^a + y^{a+1})(x^a - y^{a+1})$.
 32. $(a^{2n} + b)(a^{2n} - b)$. 33. $(x^{n+1} + 1)(x^{n+1} - 1)$. 34. $2x^3y^3(4xy + 5)$.
 35. $3(y - \frac{1}{2})(y - \frac{1}{2})$. 36. $(x + 2y)(x + y)$. 37. $(x + 3)(x + 1)$.
 38. $(b + 1.4)(b - 1.4)$. 39. $(a^2 + 16)(a + 4)(a - 4)$.
 40. $2(15 + a)(15 - a)$. 41. $(1 - 3m^a)(1 - 3m^a)$.
 42. $(x + 15)(x - 6)$. 43. $(3x + 4)(x - 2)$. 44. $3(5 - 3x)(1 + x)$.
 45. $(17 + a)(1 - a)$. 46. $(ab + 8)(ab - 7)$. 47. $x(6 - x)(1 + x)$.
 48. $(ax - 3)(ax - 1)$. 49. $(x - 9a)(x + 8a)$.
 50. $(n - 10a)(n + 9a)$. 51. $(x - 12y)(x + 11y)$.
 52. $3(x + 9)(x + 1)$. 53. $(3x - 5)(2x - 3)$. 54. $a(4 + x)(1 - x)$.
 55. $(x^n + y^p)(x^n + y^p)$. 56. $c(b^2 - 7)(b^2 - 6)$.
 57. $2x(x + 11)(x + 3)$. 58. $(y + 18a)(y - 2a)$.
 59. $a(x + 13)(x - 3)$. 60. $7(x - 12y)(x + y)$.

Page 151.—61. $(y - 17x)(y - 8x)$.

63. $(4x^2 + 7)(2x^2 - 5)$. 62. $a^2(z^2 + 1)(z^2 + 1)$.
 65. $5(x^4 + 3)(x^2 + 1)(x + 1)(x - 1)$. 64. $(11 - 10x)(7 + 3x)$.
 67. $(8a + 3b)(a - 3b)$. 66. $(5x - y)(x - 5y)$.
 69. $(10a + 3x)(6a - x)$. 68. $(2m + 3n)(2m + 3n)$.
 71. $(5x + 6y)(5x + 6y)$. 70. $c(5a - 1)(2a + 7)$.
 73. $(5x + 3y + 4z)(5x - 3y - 4z)$. 72. $a(3x - 2y)(2x + 3y)$.
 75. $(6 + y^2)(1 - y)$. 74. $(3 + x^2)(7 + x)$.
 76. $(y + x)(y - z)$. 77. $(3a + 5b)(3a + 5b)$.
 78. $(a + 1)(b + 1)$. 79. $(a + 2)(a - 2)(3 + b)(3 - b)$.
 80. $(m - 3n + 4xy)(m - 3n - 4xy)$.
 81. $(x^2 + y^2 + 2ab)(x^2 + y^2 - 2ab)$.
 82. $(4a^2 + 3a - b)(4a^2 - 3a + b)$. 83. $(m + 7)(m + 1)$.
 84. $(y + 2)(y + 2)(y - 2)(y - 2)$. 85. $(a - b + c - d)(a - b - c + d)$.
 86. $(x + 10y)(x + 10y)$. 87. $(x^n + 5y)(y - 5x^n)$.
 88. $(2a + b)(2a + 3n)$. 89. $a(b + 2a + 3c)(b - 2a - 3c)$.
 90. $(3b - c + 3)(3b - c - 3)$. 91. $(2.5y^2 + 1 + 2a)(2.5y^2 - 1 - 2a)$.
 92. $(z + 1)(z - 9)$. 93. $(3 + 2d)(3 + 2d)$. 94. $(z + 4b)(z - 3b)$.
 95. $(x^2 + 2y^2 - z^2)(x^2 + z^2)$. 96. $(c - 9)(c - 8)$.
 97. $(2a + 13)(2a + 3)$. 98. $(a + 6d)(a - 12d)$.
 99. $(4a + 5b)(4a - 3b)$. 100. $(3x^2 + 2y^2)(3x^2 + 2y^2)$.
 101. $3(a^2 - 5b^2)(a^2 - 5b^2)$. 102. $(3 - 2x)(6x - 5)$, or
 $(2x - 3)(5 - 6x)$. 103. $(m + 2b)(m - 4a - 2b)$.
 104. $(4a + 3d)(3a - 5d)$. 105. $(a + b + c)(a - b + c)$.
 106. $5(100 + 10)(100 - 10)$, or 49,500.
 107. $16a^2(a^x + 2b^{2y})(a^x - 2b^{2y})$.
 108. $3(x - \frac{3}{4})(x - \frac{1}{4})$. 109. $(2a + 3b - 2c)(2a + 3b - 2c)$.
 110. $(a + b)(a - b)(2a - 3b)$. 111. $b(5a - 7c)(3x + 2y)$.
 112. $23(15.5 + 11.5 - 2)$, or 575. 113. $17(28.2 - 26 + 17.8)$,
 or 340.

Page 153.—1. $\frac{3}{4}$. 2. $-\frac{2}{5}$. 3. $-\frac{a+x}{2x}$. 4. $\frac{4c}{b+y}$. 5. $\frac{a+b}{c+d}$.

6. $-\frac{2}{a+y}$. 7. $\frac{2+m}{2+n}$. 8. $\frac{3(x+y)}{4(a+b)}$. 9. $\frac{4(a+b)}{5(x+y)}$.

Page 154. — 15. $\frac{1}{(a-b)(b-c)}$.

16. $\frac{(n-m)(m+n)}{(a-c)(a-b)}$.

17. $\frac{(a-b)(a-b-c)}{(x-y)(y-z)(x-z)}$.

18. $\frac{2a}{(a+3)(a-3)}$.

19. $\frac{2a-3}{a+5}$.

Page 157. — 12. $\frac{x}{x+y}$.

13. $\frac{1}{a+b}$.

14. $\frac{b^3}{b+1}$.

15. $\frac{c+d}{x+y}$.

16. $\frac{1}{a}$. 17. $\frac{a+2b}{c+3d}$. 18. $\frac{a-3b}{2(a-2b)}$. 19. $\frac{5}{7}$. 20. $\frac{1}{x-y}$.

21. $\frac{1}{d-e}$. 22. $\frac{1}{a+b}$. 23. $\frac{1}{b+c}$. 24. $\frac{x-y}{x+y}$. 25. $\frac{1}{3y-1}$.

26. $\frac{1}{5a+6b}$. 27. $\frac{1}{4a-7}$. 28. $\frac{1}{a-b}$. 29. $\frac{1}{x+y}$. 30. $\frac{1}{c+2}$.

31. $\frac{1}{s-3}$. 32. $\frac{b^2}{b-3}$. 33. $\frac{x+y}{x-y}$. 34. $\frac{b-2}{b+1}$. 35. $\frac{a+2}{a-1}$.

Page 158. — 36. $\frac{1+3m}{1+4m}$.

37. $\frac{a-3}{a-4}$.

38. $\frac{y-9}{y-2}$.

39. $\frac{a-8}{a+2}$.

40. $\frac{x+1}{x-4}$. 41. $\frac{x-3}{3x-11}$. 42. $\frac{x-1}{x+7}$. 43. $\frac{3-x}{x+6}$. 44. $\frac{5a+b}{7a+5b}$.

45. $\frac{7a^n+3b}{2a^n-5b}$. 46. $\frac{2x+5y}{5x-7y}$. 47. $\frac{x}{(x+1)(x-1)}$. 48. $\frac{5(x-y)}{2}$.

49. $x+1$. 50. $\frac{b+4}{3(b+6)}$. 51. $\frac{x^2+1}{a+b-1}$. 52. $\frac{a(a+2b)^2}{(a-2b)^2}$.

53. $\frac{x(x+6)}{2(x+1)}$. 54. $\frac{1}{(a-b)^2}$. 55. $\frac{ac}{5x-y}$. 56. $\frac{2x+5y}{5x-y}$.

57. $\frac{3.5x-4y}{3x-y}$. 58. $\frac{x-2y}{x+3y}$. 59. $\frac{x(2x^n+y^m)}{x^n-y^m}$.

Page 159. — 2. $\frac{35a}{42}$.

3. $\frac{15x}{55b}$.

4. $\frac{18ay}{84xy}$.

5. $\frac{16a^2y^2}{20y^3}$.

6. $\frac{x^2-4x+3}{(x-1)^2}$. 7. $\frac{6x^2-19x+10}{(2x+5)(3x-2)}$. 8. $\frac{3x-4}{3x^2-10x+8}$.

9. $-\frac{2ab^2+5ab}{2b^2-b-15}$. 10. $\frac{2x^2-5x-25}{2x+5}$. 11. $\frac{14l^2+11lt-15t^2}{7l-5t}$.

Page 160. — 2. $3a$. 3. 2 . 4. $3xy^2z$. 5. $4a+1$. 6. $a+\frac{c^m}{a}$.

7. $2x-1$. 9. $2x^2-4x+1-\frac{1}{2x}$. 10. $a-e\frac{cd-d^2}{b}$.

11. $ax^2-x-\frac{x+1}{ax}$. 12. $x+3$.

Page 161. — 13. $b+\frac{d}{a}$.

14. $x+\frac{m}{y}$.

15. $2x+\frac{1}{2}$.

16. $x+2+\frac{1}{x}$. 17. $b+1+\frac{1}{b}$. 18. $1+\frac{x-12}{x^2}$. 19. $x-5$.

20. $a+3b+\frac{b^2}{a+b}$. 21. $x^2+xy+y^2-2y-\frac{2y^2}{x-y}$.

Page 161 (*continued*). — 22. $ax^2 - x - \frac{x+1}{ax}$.

23. $x^2 - 4x + 6 + \frac{3}{x-2}$.

24. $x^2 + 3x - 3$.

25. $x - 7 - \frac{x-19}{x^2-3}$.

26. $a^2 + 2b^2 - \frac{b^4}{a^2+b^2}$.

27. $2x + 7 - \frac{7}{2x+4}$. 28. $-3 + \frac{x^2+4y^2}{2xy}$. 29. $x - 2y - \frac{y^2+z}{x+y}$.

30. $x^3 + 3x^2 + 6x + 23 + \frac{68}{x-3}$.

31. $a + 3b - \frac{ab^2+3b^2}{a^2+b}$.

32. $x^3 + 3x^2y + 3xy^2 + y^3 - \frac{y^4}{x+y}$.

33. $m^2 - n - \frac{4n^2+2n}{m^2-n}$.

Page 162. — 2. $\frac{2m(m-n)}{m-n}, \frac{m+n}{m-n}$.

3. $\frac{5x}{10}, \frac{6y}{10}$.

4. $\frac{2a}{5b}, \frac{15bx}{5b}$.

5. $\frac{a^2bd}{c^2d^2}, \frac{ab^2c}{c^2d^2}$.

6. $\frac{2ax}{4ay}, \frac{3}{4ay}$.

7. $\frac{9by}{3bc}, \frac{2ac}{3bc}$.

8. $\frac{3x^2}{x^4y^4}, \frac{-6xy}{x^4y^4}, \frac{3y^2}{x^4y^4}$.

Page 163. — 9. $\frac{3b^3c}{8ab^2c^2}, \frac{14a^3c}{8ab^2c^2}, \frac{40ab}{8ab^2c^2}$.

10. $\frac{m^2-n^2}{a(m+n)}, \frac{2(x^2-y^2)}{12}$.

$\frac{2a(m+n)}{a(m+n)}, \frac{a^2}{a(m+n)}$.

11. $\frac{6(x+y)}{12}, \frac{3(x-y)}{12}$.

12. $\frac{x^2}{x^2-1}, \frac{x(x-1)}{x^2-1}, \frac{x(x+1)}{x^2-1}$.

13. $\frac{a^3}{a^4-16}, \frac{a(a^2-4)}{a^4-16}$.

$-2a(a^2+4)$.

14. $\frac{-4a(a+b)}{a^2-b^2}, \frac{3b(a-b)}{a^2-b^2}, \frac{1}{a^2-b^2}$.

15. $\frac{a(1+ax)}{1-a^2x^2}, \frac{x(1-ax)}{1-a^2x^2}, \frac{ax(1+ax)}{1-a^2x^2}$.

16. $\frac{x-1}{(x+5)(x+2)(x-1)}, \frac{x+5}{(x+5)(x+2)(x-1)}$.

17. $\frac{3x(x-3)}{(x-1)(x-2)(x-3)}, \frac{x+3}{(x+3)(x-2)}$.

$\frac{(x+5)(x+2)(x-1)}{(x-1)(x-1)}$.

$\frac{(x-1)(x-2)(x-3)}{(a+5)(a-5)}$.

$\frac{(x-1)(x-2)(x-3)}{(a-2)(a-1)}$.

18. $\frac{(a-1)(a-3)(a-5)}{(a+1)(a-3)}$.

$\frac{(a-1)(a-3)(a-5)}{(a-1)(a-3)(a-5)}$.

$\frac{(a-1)(a-3)(a-5)}{(a-1)(a-3)(a-5)}$.

Page 165. — 1. $\frac{19x}{10}$.

2. $\frac{20a+18b}{15}$.

3. $\frac{13a}{6b}$.

4. $-\frac{29}{21x}$.

5. $\frac{m}{2}$.

6. $-\frac{17x}{18}$.

7. $\frac{9+y^2}{3y}$.

8. $\frac{a-5b}{6}$.

9. $\frac{3x+34}{12}$.

10. $\frac{19-31x}{36}$.

11. $\frac{3-2x}{54}$.

12. $\frac{9x+2}{30}$.

13. $\frac{15-x}{50}$.

$$14. \frac{5a^2 - 6a - 7}{40} \quad 15. \frac{3 + 13x + 3x^2}{12}$$

$$\text{Page 166. — } 17. \frac{2a + b}{2} \quad 18. \frac{2x - y}{2} \quad 19. \frac{a^2 + 4c^2}{c}$$

$$20. \frac{1 - 13x}{3} \quad 21. \frac{a^2 - 2x^2}{a - x} \quad 22. \frac{a^2 - 2b^2}{a + b} \quad 23. \frac{a - c}{ac}$$

$$24. \frac{4ab}{a^2 - b^2} \quad 25. \frac{2y^2}{y - x} \quad 26. \frac{6x - 4}{x^2 - 4} \quad 27. \frac{2x^2 - 4}{x - 1} \quad 28. \frac{2}{a - b}$$

$$29. \frac{2a^3}{a^4 + a^2 + 1} \quad 30. \frac{c^2 + b^2}{a^3 - ab^2} \quad 31. \frac{4a}{1 - a^2} \quad 32. \frac{2a^2 + 2a - 11}{a^2 - 3a - 10}$$

$$33. \frac{5a^2 + 8a}{a^2 + 3a + 2} \quad 34. \frac{7 - 9a - a^2}{a^2 + a - 6} \quad 35. \frac{11a^2 - 49a + 34}{4a^2 - 16}$$

$$36. \frac{ax^2 - 2ax + 2x^2}{a^2 - x^2} \quad 37. \frac{12}{x - 2} \quad 38. \frac{2a + 3}{a^2 - 1} \quad 39. \frac{2n^2}{n - m}$$

$$40. \frac{1 + x^2}{x + x^2} \quad 41. \frac{a^2}{3a + 2x} \quad 42. \frac{2}{a + b} \quad 43. \frac{2a + 2x}{a - x}$$

$$44. \frac{6a - 7}{a^2 - 4} \quad 45. 1. \quad 46. \frac{10x + 12}{x^2 - 4} \quad 47. \frac{4ax}{a - x}$$

$$\text{Page 167. — } 48. \frac{4x^2 + 5xy + 2y^2}{x^2 + 2xy + y^2} \quad 49. \frac{10b - 12 - b^2}{6b^2 - 12b}$$

$$50. \frac{a - 3b}{2a + 2b} \quad 51. \frac{5a + 45}{a^2 - 9} \quad 52. \frac{x^2 + 3xy - y^2}{x^2y - y^3}$$

$$\text{Page 168. — } 3. \frac{5}{8} \quad 4. \frac{1}{8} \quad 5. \frac{b}{2ax} \quad 6. \frac{3x^2}{4cy} \quad 7. \frac{6b^2}{5a^2c}$$

$$8. -\frac{5bn}{4my} \quad 9. -\frac{5ab}{3xy} \quad 10. \frac{ab}{a^2 - b^2} \quad 11. \frac{5y}{4x} \quad 12. \frac{5x - 1}{x - 1}$$

$$13. \frac{3}{a - c} \quad 14. a + b \quad 15. 2x + 2y \quad 16. \frac{1}{b - 1} \quad 17. 2a + 2b$$

$$18. 5x - 10 \quad 19. \frac{x + y}{x + 3} \quad 20. 4 \quad 21. \frac{9}{20a} \quad 22. \frac{4xy + 4y}{x - 1}$$

$$23. a \quad \text{Page 169. — } 24. \frac{4}{3a + 3b} \quad 25. \frac{3}{4x - 12} \quad 26. \frac{a - 3}{a + 3}$$

$$27. \frac{xy + y^2}{x} \quad 28. \frac{x - 1}{x + 7} \quad 29. 1. \quad 30. \frac{b}{a} \quad 31. \frac{3x}{4y^2}$$

$$32. \frac{2x - y}{2} \quad 33. p.$$

$$1. \frac{b}{a} \quad 2. \frac{p}{3m} \quad 3. \frac{1}{rs} \quad 4. \frac{bc}{2a^2} \quad 5. \frac{1}{3tu} \quad 6. 3m.$$

$$7. \frac{ab}{4} \quad 8. \frac{b - y}{a - x}$$

$$\text{Page 170. — } 2. \frac{ax}{4bm} \quad 3. \frac{3m}{7x} \quad 4. \frac{m^2 - 2my + y^2}{my + y^2} \quad 5. \frac{ab - b^2}{a^2 + ab}$$

$$6. 10a \quad 7. a^2 + ab \quad 8. \frac{1}{xy} \quad 9. \frac{1}{1 - x} \quad 10. 1.$$

Page 171. — 2. $\frac{ad}{bc}$. 3. $\frac{4m}{49}$. 4. $\frac{18}{p^2}$. 5. $\frac{x}{y}$. 6. $\frac{3b}{2a}$. 7. $\frac{y}{2}$.
 8. $\frac{2t}{q}$. 9. $\frac{3c}{2b}$. 10. $\frac{4a+1}{4b-2}$. 11. $\frac{bdp+ad}{bdq-bc}$.
 12. $\frac{(x+2y)(w+4z)}{(x-2y)(w-4z)}$. 13. $\frac{bd(wz+xy)}{xz(ad-bc)}$.

Page 173. — 1. $x = 5$. 2. $x = 12$. 3. $x = 5$. 4. $x = 21$.
 5. $x = 15$. 6. $x = 3$.

Page 174. — 7. $x = 4$. 8. $x = 1$. 9. $x = 40$. 10. $x = 2$.
 11. $x = 2$. 12. $n = 9$. 13. $x = 1.13$. 14. $x = 1.64$. 15. $x = 2$.
 16. $m = 17$. 17. $x = 2$. 18. $x = 8$. 19. $x = 6$.

Page 175. — 20. $p = 4$. 21. $x = \frac{1}{4}$. 22. $n = \frac{1}{5}$. 23. $x = -\frac{2}{3}$.
 24. $n = 4$. 25. $n = -4$. 26. $n = -2\frac{3}{5}$. 27. $x = \frac{1}{10}$.
 28. $x = 10$. 29. $x = -2$. 30. $x = -1\frac{4}{5}$. 31. $x = 1$.
 32. $x = -1\frac{1}{2}$. 33. $x = 4\frac{1}{2}$. 34. $x = 5$.

1. $\frac{p}{m-n}$. 2. $n\left(\frac{l+m}{2}\right)$. 3. $\frac{s(r-1)}{(t+v)^2}$.

Page 176. — 4. $\left(m - \frac{m}{a}\right)$ marbles. 5. $\frac{z}{x}$. 6. $\frac{a+b-c}{(a+b)^2}$.
 7. $\frac{t}{s}$ hr. 8. $\frac{v+w}{x}$ dollars. 9. $\left(\frac{s}{x-y} - \frac{s}{x}\right)$ hr. 10. $\frac{a+b}{c-b}$.
 11. $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$. 12. $\left[x - \left(\frac{x}{m} + \frac{x}{n} + \frac{x}{s} + \frac{x}{r}\right)\right]$ dollars.
 1. $\frac{3}{8}$.

Page 177. — 2. $\frac{5}{8}$. 3. $\frac{4}{3}$. 4. Monoplane, 6 hr.; biplane, 5 hr.
 5. 12 columns. 6. South America, 19,000 tons; Africa, 11,400 tons.
 7. 20 oz. 8. 32, 24. 9. 6064 steamships; 5335 sailing vessels.
 10. 12,000 tons.

Page 178. — 11. 64, 8. 12. 60. 13. 29, 25. 14. $\frac{mr+n}{2r}$,
 $\frac{mr-n}{2r}$. 15. 6 5-cent pieces; 4 dimes; 3 quarters. 16. \$.21.
 17. George, 2 a yr.; father, 4 a yr. 18. $15\frac{3}{4}$ mi. 19. 28 mi.
 20. $\frac{rst}{s+t}$ mi. 21. $\frac{100mn}{6n-1}$ dollars.

Page 179. — 6. $\frac{2a-3b}{x(2a+3b)}$. 7. $\frac{3ab-6b^2}{a^2-4ab+4b^2}$.
 8. $\frac{3ab-6b^2}{a^2-4ab+4b^2}$. 9. $(b)\frac{y}{x}, \frac{x-y}{a+b}, \frac{1}{x}$. 10. $\frac{x^2-y^2-2y}{x+y}$. 11. $-\frac{2mn}{m+n}$. 12. $\frac{x^2}{x-3}$.
 13. $-\frac{9}{2x+3}$.

Page 180. — 15. $\frac{s^2+2st}{2rls+2rtl}$. 16. 0. 17. 0. 18. $\frac{2c}{(c-a)(b-c)}$.
 19. $\frac{ayz}{c}$. 20. $\frac{1}{3acy}$. 21. 4. 22. $3ax-3ay$. 23. $\frac{1}{1-x}$.

24. $x + y$. 25. a . 26. $\frac{4}{3a + 3b}$. 27. $\frac{3}{4x - 12}$. 28. $r - 2s$.
 29. $\frac{1}{x - 3y}$. 30. $\frac{1}{a + b}$. 31. 1. 32. $\frac{a - 1}{2a}$. 33. $\frac{x - 5}{3x}$.
 34. $\frac{x^2 - 3x}{2x - 2}$.

Page 181. — 37. $x = 8$. 38. $x = 3$. 39. $x = 1$. 40. $x = 27$.
 41. $x = -2\frac{3}{5}$. 42. $x = 3$. 43. $x = 20$. 44. $x = 4$. 45. $y = 7$.
 46. $v = 41$. 47. $z = 4$. 48. $u = 10$. 49. $t = 7$. 50. $x = 5$.
 51. $y = 3$.

Page 182. — 1. $2; \frac{1}{2}$. 2. $\frac{2}{3}; \frac{3x}{4y}; \frac{a}{b}; \frac{2a}{c}; \frac{1}{m - n}$. 3. $\frac{3}{4}; \frac{4}{9}; \frac{2}{3}$.
 4. $2; \frac{3}{2}; \frac{8}{9}$. 5. $\frac{1}{2}; 3; \frac{1}{320}$. 6. $\frac{2}{x}; \frac{3b}{2c}; \frac{1}{3}; \frac{7}{8}; \frac{x + y}{x - y}$.

Page 183. — 7. 1. 8. $\frac{8}{9}; \frac{9}{8}$. 9. 40 mi. an hour.
 10. 21 billion cu. ft.

Page 184. — 1. $x = 33$. 2. $a = 2\frac{1}{2}$.

Page 185. — 3. $x = 6$. 4. $x = 3\frac{3}{4}$. 5. $x = \pm 3$. 6. $x = 3$.
 7. $x = 5$. 8. $x = 4$. 9. $m : p = q : n; p : m = n : q; p : n = m : q$;
 8 : 24 = 3 : 9; 24 : 8 = 9 : 3; 24 : 9 = 8 : 3. 10. ± 8 . 11. ± 15 .
 1. Copper, 900 lb.; zinc, 100 lb. 2. Copper, 48 oz.; zinc, 24 oz.

Page 186. — 3. 40%. 4. 270 million steel pens. 5. 150 ft.
 6. 1000 lb. 7. Old machine, 60 pins; new machine, 900 pins.
 8. $\frac{1}{2}$. 9. Lake Erie, 250 mi.; Lake Michigan, 300 mi. 10. Lisbon,
 60,000; Messina, 115,000. 11. 345,000 lb. 12. 3 caps.

Page 187. — 1. $a = 6$ in. 2. $p' = 315$ rd. 3. $b = 2.5$ in.
 4. $A' = 36$ sq. ft.

Page 188. — 5. $A = 12\frac{1}{2}$ sq. in. 6. $\frac{1}{9}$. 7. $c' = 186\frac{5}{7}$ in. 8. $\frac{2}{3}$.
 9. $A = 154$ sq. in. 10. $\frac{4}{3}$. 11. $\frac{25}{16}$. 12. $\frac{r^2}{R^2}$. 13. $8\frac{1}{8}$ in.;
 26 sq. in.; $40\frac{5}{8}$ sq. in.

Page 189. — 14. $\frac{2}{3}; \frac{4}{9}$.

1. $A = 81$ sq. ft. 2. $\frac{1}{8}$. 3. $\frac{8}{27}$. 4. $V' = 1000$ cu. in.

Page 190. — 5. $V = 161$ cu. in. 6. $S = 160\pi$ sq. in. 7. $\frac{4}{9}$.
 8. $\frac{1}{4}$. 9. $\frac{1}{2}$.

Page 191. — 11. $\frac{27}{125}$. 12. $\frac{1000}{343}$. 14. $\frac{S}{S'} = \frac{r^2}{r'^2} = \frac{d^2}{d'^2}$;
 $\frac{V}{V'} = \frac{r^3}{r'^3} = \frac{d^3}{d'^3}$. 15. $\frac{27}{8}$. 16. $\frac{9}{4}; \frac{27}{8}$.

Page 192. — 17. 893.2 cu. in. 18. 400 sq. in.

1. \$25.50, \$34. 2. 578.88 ft.; 176.88 ft. 3. $\frac{am}{m + n}$ in.;
 $\frac{an}{m + n}$ in. 4. \$1250, \$5000. 5. 12, 8. 6. 19, 6. 7. 42 mi.,
 56 mi. 8. \$18,750, \$31,250. 9. 550 ft., 300 ft.

Page 193. — 10. \$960, \$1200. 11. 4. 12. 7. 13. 12 da.; 20 da. 14. \$14,495,000; \$2,230,000. 15. 48 lb. 16. $1\frac{1}{15}$ gal. 17. 5 oz. of mixture 10% pure; 3 oz. of mixture 50% pure. 18. $\frac{4}{7}$ gal.

Page 194. — 19. 20 sq. in. 20. $p:W = D:d$, or $W:p = d:D$. 21. $W = \frac{pd}{D}$; 160 lb. 22. 1 ft. 23. $b = 6.4$. 24. $\$37\frac{1}{2}$, $\$37\frac{1}{2}$, $\$48\frac{3}{4}$, $\$56\frac{1}{4}$. 25. $66\frac{2}{3}^\circ$, $266\frac{2}{3}^\circ$, $13\frac{1}{3}^\circ$, $13\frac{1}{3}^\circ$. 26. $11\frac{1}{4}$ horse power. 27. Mr. Ball, \$1200; Mr. Keats, \$1800; Mr. Hunt, \$2250.

Page 196. — 8. $x = \frac{2}{7}$. 9. $x = \frac{1}{3}$. 10. $x = 3$. 11. $x = 4$. 12. $x = 2\frac{1}{2}$. 13. $x = 2$. 14. $x = 10$. 15. $r = 10$. 16. $n = 5$. 17. $t = 4$. 18. $x = 3$. 19. $w = 1\frac{3}{4}$. 20. $x = 3$. 21. $x = 1$. 22. $x = 3$. 23. $x = 13\frac{1}{3}$. 24. $y = 8$. 25. $x = -2$. 26. $x = -5$. 27. $x = 1$.

Page 197. — 28. $x = 5$. 29. $x = 2$. 30. $x = -640$. 31. $s = 4\frac{1}{2}$. 32. $x = 4$. 33. $x = 8$. 34. $x = 5$. 35. $x = 8$. 36. $x = 10$. 37. $x = 3.64$. 38. $x = 1\frac{2}{3}$. 39. $x = 2$. 40. $x = 7$. 41. $r = 4\frac{1}{2}$. 42. $x = \frac{1}{2}$. 43. $t = 7$. 44. $r = 20$. 45. $r = 7$. 46. $x = 2$. 47. $x = 5$.

Page 199. — 16. $x = \frac{8-a}{7}$. 17. $x = a + b$. 18. $x = \frac{bc}{b+c}$. 19. $y = 2$. 20. $x = \frac{c+d}{2}$. 21. $x = \frac{a+b}{2}$. 22. $x = 2a$. 23. $y = 2b^2$. 24. $x = a + c$. 25. $x = r - s$. 26. $x = -(a^2 + b^2)$. 27. $x = b + bc$. 28. $y = b^2 + 2bc + c^2$. 29. $x = \frac{2bc}{b^2 + c^2}$. 30. $y = \frac{a}{2}$. 31. $y = 3a$. 32. $x = \frac{b-a}{b}$. 33. $x = 0$. 34. $x = b$. 35. $x = \frac{2ab}{a^2 + b^2}$. 36. $x = 7b + c$. 37. $x = ab + 7$.

Page 200. — 1. $h = \frac{2A}{b}$; $h = 16.9$. 2. (1) $h = \frac{2A}{b_1 + b_2}$; (2) $b_1 = \frac{2A}{h} - b_2$; (3) $b_2 = \frac{2A}{h} - b_1$; $b_1 = 10.2$. 3. $h = \frac{3V}{b}$; $h = 8.1$. 4. $t = \frac{a-p}{pr}$; (1) $p = \frac{a}{1+rt}$, (2) $r = \frac{a-p}{pt}$; $t = 3.5$. 5. $F = \frac{2}{5}C + 32^\circ$; $F = 42^\circ$; $F = -4^\circ$. 6. $g = \frac{2s}{t^2}$; $g = 32.16$. 7. $r = \frac{E}{C} - R$; $r = 2.5$. 8. $P_2 = \frac{V_1P_1}{V_2}$; $P_2 = 125$. 9. $f_1 = \frac{ff_2}{f+f_2}$; $f_1 = 10.2$. 10. $e = \frac{Er}{R+r}$; $e = 1.4$.

Page 201. — 1. John, 8 fish; father, 4 fish. 2. 12 to 6. 3. Shell, 1000 lb.; powder, 265 lb. 4. 435 tons. 5. \$10.50; \$3.50. 6. 4 20-candle-power lamps; 8 25-candle-power lamps.

Page 202. — 7. Small pump, 400 gal.; large pump, 1600 gal. 8. 12 shots. 9. Width, 151 ft.; length, 252 ft. 10. 108 votes. 11. 7 bbl. @ \$10.50; 55 bbl. @ \$8.50. 12. 9 boxes @ \$9; 16 boxes @ \$6.50. 13. 24 @ \$6; 16 @ \$7; 32 @ \$5; 24 @ \$8. 14. *United States*, 650 mi.; *Lotus II*, 375 mi. 15. \$47 $\frac{1}{2}$; \$52 $\frac{1}{2}$.

Page 203. — 16. Clear, 121 da.; cloudy, 127 da.; partly cloudy, 117 da. 17. $\frac{2}{3}\%$. 18. $\frac{1}{8}$. 19. $\frac{9}{17}$. 20. 25 bu. 21. 48 cakes @ 3 for \$.25; 24 cakes @ \$.10. 22. Former, 7.42¢; latter, 9.32¢.

Page 204. — 24. $3\frac{9}{37}$ da. 25. $3\frac{3}{13}$ da. 26. $1\frac{1}{3}$ da. 27. 5 da.; Arthur, 30 da.; Louis, 15 da.; John, 10 da. 28. Albert, $3\frac{1}{3}$ hr.; Benjamin, $3\frac{1}{3}$ hr.; Charles, $7\frac{7}{9}$ hr. 29. Indians, 38%; whites, 19%; mixed races, 43%.

Page 205. — 31. 93. 32. 48. 33. \$500 at 6%; \$300 at 5%. 34. \$240,000. 35. Mr. Bailey, \$6000; Mr. Day, \$4800. 36. New York Central train, 50.3 mi. per hour; other train, 53.3 mi. an hour. 37. 24 mi. 38. 1.8 mi.

Page 206. — 39. 40 ft. 40. \$8. 41. $\frac{1200n}{a}\%$. 42. \$12,600. 43. One, 45 mi. an hour; other, 15 mi. an hour. 44. \$4000. 45. Aluminum, 15.6%; copper, 84.4%. 46. 45 cabinets. 47. Tin, 16 lb.; lead, 24 lb. 48. $11\frac{1}{3}$ hr.

Page 215. — 1. $x = 5, y = 4$. 2. $x = 3.5, y = 2$. 3. $x = 4, y = 3$. 4. $x = -1, y = -4$. 5. Inconsistent. 6. $x = 5.5, y = 1.2$. 7. $x = 1.6, y = 1.6$. 8. $x = 1.1, y = .4$. 9. $x = -6, y = 0$. 10. Indeterminate. 11. $x = 2, y = 1$. 12. Inconsistent. 13. Indeterminate. 14. $x = -6, y = 3$. 15. $x = 4.6, y = 1.3$. 16. $x = 4.9, y = 3.1$. 17. Inconsistent. 18. Indeterminate. 19. $x = 1.2, y = 2.3$. 20. $x = 5, y = 2$. 21. $x = -1, y = -1.5$. 22. $x = 1.5, y = .5$. 23. $x = .3; y = 1.0$. 24. $x = -2, y = 3$.

Page 217. — 2. $x = 11, y = 5$. 3. $x = 7, y = 1$. 4. $x = 5, y = 2$. 5. $x = 10, y = \frac{1}{2}$. 6. $a = 4, b = 3$. 7. $d = 7, y = 1$. 8. $p = 4, q = 2$. 9. $a = 7, z = 3$. 10. $x = 8, y = 3$. 11. $m = 4, n = 5$. 2. $x = 10, y = 6$. 3. $x = 8, y = 2$. 4. $x = 1, y = 4$. 5. $x = 3, y = 8$.

Page 218. — 6. $x = 6, y = -2$. 7. $x = -2, y = 3$. 8. $x = 1, z = 3$. 9. $x = 4, z = 5$. 10. $x = 2, y = \frac{1}{3}$. 11. $x = -11, y = 4$. 1. $x = .9, z = .4$. 2. $x = 3, y = 1$. 3. $x = 2, y = -2$. 4. $x = 6, y = .2$. 5. $x = -2, y = 4$. 6. $x = 3, y = 3$. 7. $x = -4, y = 8$. 8. $x = 5, y = 3$. 9. $x = 9, y = 6$. 10. $x = 12, y = 15$. 11. $x = 4, y = 6$. 12. $x = 18, y = 6$. 13. $x = 4, y = 8$.

Page 219. — 14. $x = 11, y = 7$. 15. $x = 60, y = 40$. 16. $x = 1, y = 4$. 17. $x = 24, y = 3$. 18. $x = 9, y = 6$. 19. $x = 5, y = 2$.

Page 220. — 21. $x = 5, y = 1$. 22. $x = 7, y = 4$. 23. $x = \frac{1}{2}, y = \frac{1}{3}$. 24. $x = 2\frac{2}{5}, y = 2\frac{2}{3}$. 25. $x = 2, y = 3$. 26. $x = 2, y = 5$. 27. $x = 3, y = 2$. 28. $x = 1, y = -1$. 29. $x = 4, y = 24$. 30. $x = 1, y = 3$. 31. $x = \frac{1}{5}, y = \frac{1}{8}$. 32. $x = 12, y = 10$. 33. $x = 2, y = 3$. 34. $x = 8, y = 6$.

Page 221. — 2. $x = \frac{am + bc}{a^2 + b^2}$, $y = \frac{bm - ac}{a^2 + b^2}$. 3. $x = \frac{dm - br}{ad - bc}$,
 $y = \frac{cm - ar}{ad - bc}$. 4. $x = \frac{ab^2}{a + b}$, $y = \frac{a^2b}{a + b}$. 5. $x = \frac{an + 2am}{2mn}$,
 $y = \frac{an - 2am}{2mn}$. 6. $x = \frac{an - 5c}{ab - ac}$, $y = \frac{an - 5b}{ab - ac}$. 7. $x = \frac{a^2 + b^2}{2a}$,
 $y = \frac{a^2 + b^2}{2b}$. 8. $x = -a$, $y = b$. 9. $x = \frac{a^2 + ab - 2b^2}{a + b}$,
 $y = \frac{ab - b^2}{a + b}$.

Page 222. — 10. $x = a$, $y = b$. 11. $x = \frac{2ab}{a + b}$, $y = \frac{2ab}{b - a}$.
 12. $x = -(a + b)$, $y = a + b$. 13. $x = a$, $y = b$. 14. $x = a + b$,
 $y = a - b$. 15. $x = a + 1$, $y = a - 1$. 16. $x = \frac{abcd - ab^2c}{ac - b^2}$,
 $y = \frac{abc^2 - b^2cd}{ac - b^2}$. 17. $x = m + n$, $y = m - n$. 18. $x = a + b$,
 $y = a - b$.
 1. $x = 5$, $y = 4$. 2. $x = 3.5$, $y = 2$. 3. $x = 4$, $y = 3$.
 4. $x = -1$, $y = -4$. 5. Inconsistent. 6. $x = 5.5$, $y = 1.2$.
 7. $x = 1.6$, $y = 1.6$. 8. $x = 1.1$, $y = .4$. 9. $x = -6$, $y = 0$.
 10. Indeterminate.

Page 223. — 1. Length, 100 yd.; width, 80 yd. 2. 8 hats @ \$5;
 10 hats @ \$.50. 3. Raspberries, \$.24; cherries, \$.20. 4. 350
 glasses @ \$.10; 500 glasses @ \$.15. 5. 20 pineapples @ \$.15; 16 pine-
 apples @ \$.20. 6. 200 tickets @ \$.50; 100 tickets @ \$.25. 7. 75
 wreaths; 225 wreaths. 8. Height, 265 ft.; span, 325 ft.

Page 224. — 9. Detroit chimney, 100 ft.; Chicago chimney, 150 ft.
 10. 26 1-dollar bills; 12 2-dollar bills. 11. 7 boxes each containing
 120 oranges; 3 boxes each containing 150 oranges. 12. 8 shots @
 \$400; 12 shots @ \$70. 13. First team won 107 games and lost
 45 games; second team won 91 games and lost 63 games. 14. 8 in.;
 12 in. 15. 840 boys; 160 girls. 16. 18 in.; 12 in.

Page 225. — 17. 650 motor trucks; 725 pleasure cars. 18. 1700
 Americans; 1094 British. 19. Eastern university, 9000 students;
 western university, 6000 students. 20. \$98.55. 21. \$1500 at 5%;
 \$2500 at 4%. 22. \$97,500; \$95,000. 23. 6 silver fox skins;
 4 black fox skins.

Page 226. — 24. Peanuts, 225 million lb.; imported nuts, 90 mil-
 lion lb. 25. 1200 admission tickets; 400 grand-stand tickets.
 26. 48 tons; 160 tons. 27. $6\frac{1}{2}$ lb.; 100 lb. 28. $\frac{5}{8}$. 29. $\frac{1}{2}$.
 30. 63.

Page 227. — 31. Steel, 2200 tons; iron, 3400 tons. 32. 23 1-dollar
 bills; 17 5-dollar bills. 33. One, 75 lb.; other, 90 lb. 34. Larger,
 34 oz.; smaller, 30 oz. 35. Alvah, \$16; Malcolm, \$10. 36. 5, 2.
 37. First day, 305 knots; second day, 457 knots. 38. Length,
 600 ft.; width, 80 ft. 39. 4 mi. an hour; 2 mi. an hour. 40. $8\frac{1}{3}$ mi.
 an hour; $16\frac{2}{3}$ mi. an hour.

Page 228. — 41. $\frac{cm + dm}{2cd}$ mi. an hour; $\frac{dm - cm}{2cd}$ mi. an hour.
 42. 8, $\frac{1}{4}$. 43. 66, 16. 44. 60, 8. 45. $\frac{1}{12}$. 46. 10 da.; 15 da.
 47. 9000. 48. 45 ft. a second. 49. Length, 80 ft.; width, 50 ft.

Page 229. — 1. $x = 2, y = 5, z = 7$. 2. $x = 24, y = 6, z = 23$.

Page 230. — 3. $x = 39, y = 21, z = 12$. 4. $x = 11, y = 13, z = 15$.
 5. $x = 8, y = 9, z = 12$. 6. $x = 2, y = 3, z = 4$.
 7. $x = 60, y = 60, z = 20$. 8. $x = 3\frac{1}{2}, y = 5\frac{1}{2}, z = 1\frac{1}{2}$.
 9. $x = \frac{1}{4}, y = \frac{1}{2}, z = -1$. 10. $x = a + b, y = a + c, z = -a$.
 11. $x = \frac{1}{3}, y = \frac{1}{2}, z = \frac{1}{4}$. 12. $x = \frac{1}{2}, y = \frac{1}{4}, z = \frac{1}{6}$. 13. $x = 9, y = 6, z = 8$.
 14. $x = \frac{1}{3}, y = -\frac{1}{2}, z = -1$. 15. $x = a + b, y = a - b, z = 2a$.
 16. $x = \frac{1}{a}, y = \frac{1}{b}, z = \frac{1}{c}$. 17. $x = \frac{1}{2}, y = \frac{1}{10}, z = \frac{1}{14}$.

Page 231. — 1. 6, 12, 20. 2. 100, 200, 500. 3. John, $14\frac{2}{7}$ da.; Paul, $34\frac{2}{7}$ da.; Guy, $18\frac{6}{7}$ da. 4. Fred, $\frac{2rst}{rt + st - rs}$ da.; Hugh, $\frac{2rst}{rs - rt + st}$ da.; David, $\frac{2rst}{rs + rt - st}$ da. 5. 239. 6. Ann, \$500; Eva, \$300; Dora, \$700. 7. 14 gal., 10 gal., 8 gal. 8. Scow a , 450 cu. yd.; scow b , 360 cu. yd.; scow c , 400 cu. yd.

Page 233. — 18. $x = 2, y = -2$. 20. Inconsistent.

Page 234. — 23. $x = -2, y = -3$. 24. $56\frac{1}{4}^\circ, 33\frac{3}{4}^\circ$. 25. John, \$.81; Paul, \$.90. 26. $90^\circ, 67\frac{1}{2}^\circ, 22\frac{1}{2}^\circ$. 27. $118\frac{1}{8}$ mi.; $11\frac{1}{4}$ hr.
 28. First pile: 100 5-dollar bills, 200 10-dollar bills; second pile: 200 5-dollar bills, 100 10-dollar bills. 29. 8 in., 7 in., 5 in.

30. $\frac{ac}{a+b}$ in., or $4\frac{1}{2}$ in.; $\frac{bc}{a+b}$ in., or $2\frac{1}{2}$ in.

Page 238. — 1. $a^2b^4c^6$. 2. $a^{12}b^8c^4$. 3. $8a^6c^3$. 4. $49a^4m^{10}$.
 5. $-64c^6y^{15}$. 6. $16a^{12}n^{20}$. 7. $a^mb^mc^mx^m$. 8. $64e^{12}x^{6c}$. 9. -1 .
 10. 1. 11. $3^nb^{nc^n}$. 12. $2^na^{2n}x^{3n}$.
 1. $\frac{1}{16b^2}$. 2. $\frac{4x^2}{49y^2}$. 3. $\frac{9x^4}{100y^6}$. 4. $\frac{25}{a^2b^2}$. 5. $\frac{16}{81x^4}$. 6. $-\frac{27x^3}{8y^3}$.
 7. $\frac{64a^6}{b^6}$. 8. $\frac{b^6c^4}{a^4x^2}$. 9. $\frac{a^{2n}b^{3n}}{x^ny^{4n}}$. 10. (1) $5.29a^2b^2$; (2) $14.44x^4$;
 (3) $.49a^2$.

Page 239. — 1. $9x^2 + 24xy^3 + 16y^6$. 2. $25m^{2a} - 40m^{an}b + 16n^{2b}$. 3. $25m^6 - 60ax^m + 36a^{2x}$. 4. $16r^4 + s^4 + t^2 + 8r^2s^2 + 8r^2t + 2s^2t$. 5. $4a^2 + b^2 + c^2 + 4ab + 4ac + 2bc$.
 6. $9a^2 + 4b^2 + 16c^2 + 12ab - 24ac - 16bc$. 7. $25a^2 + 4c^2 + 9e^2 - 20ac - 30ae + 12ce$. 8. $9r^4 + 4s^2 + t^3 + 12r^2s + 6r^2t^4 + 4st^4$.
 9. $4x^4 + 9y^2 + z^6 - 12x^2y + 4x^2z^3 - 6yz^3$.

Page 240. — 10. $x^3 + 3x^2y + 3xy^2 + y^3$. 11. $x^3 - 3x^2y + 3xy^2 - y^3$. 12. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$. 13. $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$. 14. $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.
 15. $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$.

- Page 240 (continued). — 17. (a) $a^3 + 3a^2b + 3ab^2 + b^3$;
 (b) $a^3 - 3a^2b + 3ab^2 - b^3$. 18. $a^3 + 6a^2b + 12ab^2 + 8b^3$.
 19. $8a^3 - 12a^2b + 6ab^2 - b^3$. 21. $27a^3 + 135a^2b + 225ab^2 + 125b^3$.
 22. $8x^3 - 84x^2y + 294xy^2 - 343y^3$. 23. $\frac{1}{8}x^3 + \frac{1}{4}x^2y + \frac{1}{6}xy^2 + \frac{1}{7}y^3$.
 24. $64a^3 + 144a^2b + 108ab^2 + 27b^3$.
 25. $49x^2 + 4y^2 + 9z^2 - 28xy + 42xz - 12yz$.
 26. $25a^2 + 9b^2 + c^2 + 30ab - 10ac - 6bc$. 27. $9a^2 + 4b^2 + 49c^2 - 12ab - 42ac + 28bc$.
 28. $25a^2 + 16b^2 + 9c^2 - 40ab + 30ac - 24bc$. 29. $16x^4 - 24x^3 + 25x^2 - 12x + 4$. 30. $9 - 12a - 2a^2 + 4a^3 + a^4$.
 31. $.04x^2 + .09y^2 + z^2 + .12xy - .4xz - .6yz$.
 32. $\frac{a^2}{4} + \frac{b^2}{9} + \frac{c^2}{16} + \frac{ab}{3} - \frac{ac}{4} - \frac{bc}{6}$.

- Page 242. — 1. 25. 2. 9. 3. 7. 4. 21. 5. 22. 6. 8.
 7. 24. 8. 6. 9. 17. 10. 10. 11. 3. 12. 1.

- Page 243. — 3. ab^2c^3 . 4. $\pm a^3b^2c^4$. 5. a^2xy^3 . 6. $\pm a^n b^2 c^3$.
 7. $\pm x^{2n} y^4 z^m$. 8. $-2a^2b^5$. 9. $-2x^2y^2$. 10. $\pm 4x^2y$. 11. $-ab^2x$.
 12. $-3y^n$. 13. $-mb^3$. 14. $-a^6b^3$. 15. $-a^2bc^3$. 16. $3p^3r$.
 17. $2a^2c^3d^3$. 19. $\pm \frac{8a^2b^3}{9mn^4}$. 20. $\frac{(x-y)^2}{2x^2}$. 21. $-\frac{5x}{y^3}$. 22. $-\frac{2ax^2}{3y^3}$.
 23. $\frac{9a^2}{16b^4}$. 24. $\pm \frac{2a^2}{3b}$. 25. $\frac{6a^{m+1}}{c^{2n}}$. 26. $\pm \frac{4(a-b)}{3(a+3b)}$.
 27. $\frac{2a-3b}{3}$. 28. $\pm \frac{25a^{n+1}}{11b^{2n-1}}$. 29. $\pm \frac{17(2a-b)^2}{13a^{3n}}$. 30. $\pm \frac{21}{2a^{2n+1}}$.
 31. $-\frac{2(3a^2-b)}{3a}$. 32. $-\frac{3a^2b^3}{2x}$. 33. $\pm \frac{2(a-b)}{3xy^2}$.

- Page 245. — 3. $\pm(2x + 3y^2)$. 4. $\pm(4x - 7y)$.
 5. $\pm\left(\frac{2a}{b} + \frac{2b}{a}\right)$. 6. $\pm\left(\frac{a}{3} + 5\right)$. 7. $\pm\left(\frac{5}{2n} + 3n\right)$.
 8. $\pm\left(\frac{5d}{4r} - 2\right)$. 9. $\pm(1 + 3x - 2x^2)$. 10. $\pm(2a^2 - a + 1)$.
 11. $\pm\left(\frac{a^2}{3} + 2a - 3\right)$. 12. $\pm\left(\frac{x^2}{8} + \frac{x}{2} - 1\right)$. 13. $\pm\left(x + 1 - \frac{1}{x}\right)$.
 14. $\pm\left(x^2 + \frac{x}{2} + \frac{1}{5}\right)$. 15. $\pm(a^2 - b + c^2)$. 16. $\pm(2a - 3b + 4c)$.
 17. $\pm(3x - 5y + 3z)$. 18. $\pm(a^2 + 7a + 2)$.
 19. $\pm(5x^2 - 3ax + 4a^2)$. 20. $\pm(2x^3 - 5x - 1)$.
 21. $\pm(1 + 5x - 2x^3)$. 22. $\pm(2c^2 - c + 1)$.
 23. $\pm(.2x^2 + .2x - .5)$. 24. $\pm\left(\frac{a^2}{3} - 2a + \frac{b}{2}\right)$.
 25. $\pm\left(\frac{2m^3}{3} - m^2 + \frac{m}{5} + \frac{3}{4}\right)$.

- Page 248. — 1. 23. 2. 47. 3. 69. 4. 95. 5. 97. 6. 99.
 7. 89. 8. 81. 9. 83. 10. 71. 11. 87. 12. 239. 13. 205.
 14. 49.8. 15. 3.31. 16. .035. 17. $\frac{25}{27}$. 18. $\frac{24}{8}$. 19. $\frac{13}{18}$.
 20. $\frac{14}{34}$. 21. $\frac{17}{18}$. 22. $\frac{19}{20}$. 23. $\frac{24}{25}$. 24. $\frac{17}{31}$. 26. $\frac{10a^2}{3b^{n+1}}$.
 27. $\frac{30x^{2n-4}}{a^3}$. 28. $7y^{3-n}$. 29. $\frac{105v^2}{a^2}$. 30. $2a^{3-n}$. 31. $\frac{2.4x^3}{77z}$.

Page 249. — 32. 36.5. 33. 37.1. 34. 3.14. 35. .047.
 36. .302. 37. .977. 38. .866. 39. .894. 40. .790. 41. .471.
 42. .559. 43. .645. 44. $c = 55$. 45. $a = 24$. 46. $d = \sqrt{a^2 + b^2}$.
 47. $2\frac{1}{2}$ in. 48. 10.5 rd.
 1. 8. 2. 10. 3. 12. 4. 14. 5. 20. 6. 15. 7. 16.
 8. 50. 9. 24. 10. $16a^2b^3$. 11. 2.8. 12. 2.5.

Page 250. — 1. 51.84. 2. 705,600. 3. 941.192. 4. 5.1.
 5. .97. 6. 94.09. 7. 9.9. 8. .83. 9. 7.7. 10. 7.8.
 11. 8.2. 12. .6241. 13. .96. 14. .830584. 15. 5.1.
 16. 9.1. 17. 5.5 in. 18. 10.4 in. 19. $m = 6.4$. 20. $t = 7.5$.
 21. (a) $m = 21.4$; (b) $t = 10.0$.

Page 255. — 1. $\frac{1}{8}$. 2. 1. 3. $-\frac{1}{64}$. 4. 1. 5. 75. 6. $\frac{1}{36}$.
 7. -8 . 8. 1. 9. $\frac{3}{4}$. 10. $-7\frac{1}{4}$; $-24\frac{1}{4}$; -1 .

Page 256. — 11. $(\frac{1}{3})^3$; $(-\frac{1}{3})^{-4}$. 12. 5^{-1} . 13. a^{-2} . 14. 2^{-n} .
 15. ax^{-3} . 16. $b^{-1}mn^{-2}$. 17. $a^{-2}cx^{-3}$. 18. $a^{-2}b^{-3}c^2d$.
 19. $ab^{-1}m^3x^{-n}$. 20. $a^mb^{-n}c^x$. 21. $\frac{3}{y^2}$. 22. $\frac{2a}{x}$. 23. $\frac{x}{yz^4}$.
 24. $\frac{b^2}{ac^3}$. 25. $\frac{a^n}{b^{3n}}$. 26. $\frac{5a^2c^5}{b^4}$. 27. $4x^3 - 2x^2 + 5x - 6 + \frac{3}{x} - \frac{5}{x^2}$.
 28. $\frac{a^3}{b^3} - \frac{a^2}{b^2} + \frac{a}{b} - 1 + \frac{b}{a} - \frac{b^2}{a^2} + \frac{b^3}{a^3}$. 29. $a^{-2}b^{-2}c^2d^2$. 30. $a^2b^{-3}m^3n^{-2}$.
 31. m^3n^{-1} . 32. $3a^2c^{-2}x^{-2}$. 33. $4x^4$. 34. $x^{-1}y^{-1}$. 35. a^3b^{-6} .
 36. $a^{-3}b^3$. 38. 2. 39. 8. 40. 2. 41. 8. 42. $-\frac{1}{2}$.
 43. $\frac{4}{25}$. 44. 27. 45. $\frac{1}{16}$. 46. $64^{-\frac{2}{3}}$; $(\frac{1}{64})^{-\frac{3}{2}}$. 47. 529.

Page 257. — 49. $c^{\frac{1}{2}}d^{\frac{1}{2}}$. 50. $a^{\frac{1}{3}}b^{\frac{1}{3}}$. 51. $x^{\frac{3}{2}}y^{\frac{3}{2}}$. 52. $m^{\frac{4}{3}}n^{\frac{4}{3}}$.
 53. $\frac{x^{\frac{3}{2}}}{y^{\frac{1}{2}}z^{\frac{3}{2}}}$. 54. $\frac{3b^{\frac{x}{3}}}{a^{\frac{2}{3}}c^{\frac{1}{3}}}$. 55. $\sqrt{x}$. 56. $\sqrt{a^3}$. 57. $\sqrt{ab^3}$.
 58. $\sqrt[3]{x^2y}$. 59. $\sqrt[5]{m^3n^4}$. 60. $\sqrt[3]{\frac{a^2}{b^4}}$. 61. $\sqrt[3]{\frac{a^2}{b}}$. 62. $\sqrt[4]{\frac{c}{d^5}}$.
 63. $4x^{\frac{2}{3}} - 4x^{\frac{1}{3}} - 5$. 64. $2x^{\frac{1}{3}} + 4\frac{3}{4} - \frac{1}{x^{\frac{1}{3}}}$.

1. a^{-3} . 2. a^2 . 3. ab . 4. m . 5. $an^{\frac{1}{2}}$. 6. a^{m-p} .
 7. $x + x^{\frac{4}{5}}y^{\frac{1}{5}} + x^{\frac{3}{5}}y^{\frac{2}{5}} + x^{\frac{2}{5}}y^{\frac{3}{5}} + x^{\frac{1}{5}}y^{\frac{4}{5}}$. 8. $x^n + x^{n-1}y + x^{n-2}y^2 + x^{n-3}y^3$.

Page 258. — 10. $a - b$. 11. $x^{\frac{4}{3}} - y^{\frac{4}{3}}$. 12. $x^{\frac{4}{5}} + x^{\frac{2}{5}} - 20$.
 13. $x^{\frac{3}{5}} + y^{-\frac{3}{5}}$. 14. $x^{-3} + x^{-1} + x$. 15. $a^{-2} + 3a^{-1}c^{\frac{1}{2}} - b^{-1} + b^{-\frac{1}{2}}c^{\frac{1}{2}} + 2c$.
 16. n^{-1} . 17. n^2 . 18. n^8 . 19. r^4 . 20. $s^{\frac{3}{2}}$. 21. x^{2-2n} .
 22. $x^2y^{-2} + 1 + x^{-2}y^2$. 23. $a^{-2}b^{-1} - 1 + a^2b$.
 24. $a^{-4} - 2a^{-3}x^{-1} + a^{-1}x^{-3} - x^{-4}$. 26. $a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$.
 27. $a^{\frac{4}{5}} - a^{\frac{3}{5}}b^{\frac{1}{5}} + a^{\frac{2}{5}}b^{\frac{2}{5}} - a^{\frac{1}{5}}b^{\frac{3}{5}} + b^{\frac{4}{5}}$. 28. $x^{\frac{1}{3}} - 1$. 29. $x^{\frac{7}{2}} - x^{-\frac{7}{2}}$.
 30. a . 31. a^{-2} . 32. a^{-8} . 33. $-a^{\frac{3}{2}}$. 34. a^8 . 35. $-a^{-\frac{4}{3}}$. 36. $a^{-\frac{2}{3}}$.
 37. $x^{\frac{2}{3}}y^{-\frac{2}{3}}$. 38. $a^{-\frac{1}{5}}b^{-1}$. 39. $-8a^{-6}$. 40. $216ab^{-\frac{1}{2}}$. 41. $\frac{1}{3}m^{-\frac{1}{2}}n^{-\frac{1}{4}}$.

- Page 259. — 43. $\left(\frac{1}{a} + \frac{1}{b}\right)\left(\frac{1}{a} - \frac{1}{b}\right)$. 44. $\left(3 + \frac{1}{x}\right)\left(3 - \frac{1}{x}\right)$.
 45. $\left(4 + \frac{1}{a^2}\right)\left(2 + \frac{1}{a}\right)\left(2 - \frac{1}{a}\right)$. 46. $2\left(2x + \frac{3}{x^2}\right)\left(2x - \frac{3}{x^2}\right)$.
 47. $3\left(3 + \frac{4}{x^3}\right)\left(3 - \frac{4}{x^3}\right)$. 48. $b\left(5b + \frac{1}{y}\right)\left(5b - \frac{1}{y}\right)$.
 49. $\left(a + \frac{1}{a}\right)\left(a + \frac{1}{a}\right)$. 50. $\left(b + \frac{2}{b}\right)\left(b - \frac{2}{b}\right)\left(b + \frac{2}{b}\right)\left(b - \frac{2}{b}\right)$.
 51. $\left(4 + \frac{1}{x}\right)\left(3 - \frac{1}{x}\right)$. 52. 4. 53. 4. 54. $\frac{1}{48}$. 55. $\frac{27}{x^2y}$.
 56. $3a$. 57. $\frac{a^2}{b^2}$. 58. $\frac{m^{\frac{1}{2}}}{8r^3}$. 59. $m^6n^2x^{\frac{1}{2}}$. 60. $\frac{2}{x^3}$. 61. $9^{\frac{1}{3}}$.
 62. $\frac{2}{a^2}$. 63. x . 64. 128. 65. 1. 66. a . 67. $x + y$. 68. $\frac{4}{9}$.

- Page 262. — 3. $2\sqrt{3}$. 4. $5\sqrt{3}$. 5. $2\sqrt[3]{2}$. 6. $8\sqrt{2}$.
 7. $5\sqrt[3]{2}$. 8. $3\sqrt{6}$.

- Page 263. — 10. $4\sqrt{5}$. 11. $5\sqrt{5}$. 12. $3\sqrt{5}$. 13. $18\sqrt{3}$.
 14. $20\sqrt{7}$. 15. $8\sqrt{10}$. 16. $27\sqrt{2}$. 17. $30\sqrt{2}$. 18. $14(2)^{\frac{1}{2}}$.
 19. $32(2)^{\frac{1}{2}}$. 20. $18(5)^{\frac{1}{2}}$. 21. $2\sqrt[3]{2}$. 22. $4a\sqrt{3ab}$. 23. $5b^{\frac{1}{2}}$.
 24. $7c\sqrt{2c}$. 25. $5(2a)^{\frac{1}{2}}$. 26. $5a^2x^5\sqrt{3}$. 27. $2a^2b(9)^{\frac{1}{2}}$.
 28. $2a^2\sqrt{5b}$. 29. $2(x-4)\sqrt{3}$. 30. $x\sqrt[3]{x^3-1}$. 31. $4a\sqrt{1-3b}$.
 32. $3b\sqrt{a-3b}$. 33. $3xy\sqrt{3x-4y}$. 34. $2a\sqrt{3b^2-7b}$.
 35. $4\sqrt{x^2-1}$. 36. $6a\sqrt{1+2a}$. 37. (1) $c = 2\sqrt{29}$;
 (2) $c = 3\sqrt{5}$; (3) $a = 24$; (4) $b = \sqrt{299}$. 38. $(y+2)\sqrt{b}$.
 39. $(x-y)\sqrt{5}$. 40. $3c\sqrt[3]{c^3-1}$.

- Page 264. — 41. $6\sqrt{3}$; $12\sqrt{3}$; $\frac{15}{2}\sqrt{3}$; $\frac{1}{3}\sqrt{3}$; $\frac{m}{2}\sqrt{3}$; $2m\sqrt{3}$;
 $\frac{a+b}{2}\sqrt{3}$. 42. $t = 2\sqrt{2}$. 43. $y = 2\sqrt{6}$. 44. $v = 2\sqrt{5}$.
 45. $r = 5\sqrt{10}$. 46. $r = 2\sqrt{15}$.
 1. $\frac{1}{2}\sqrt{2}$. 2. $\frac{1}{3}\sqrt{6}$. 3. $\frac{1}{4}\sqrt{6}$. 4. $\frac{1}{8}\sqrt[3]{100}$. 5. $\frac{1}{6}\sqrt[3]{90}$. 6. $\frac{1}{5}\sqrt{10}$.
 7. $\frac{2}{3}\sqrt{6}$. 8. $\frac{2}{3}\sqrt{6}$. 9. $2x\sqrt{2}$. 10. $2a\sqrt{6}$. 11. $8x\sqrt{3}$.
 12. $\frac{20}{9b}\sqrt{3a}$. 13. $\frac{a+b}{a-b}\sqrt{a^2-b^2}$. 14. $\frac{1}{x-2y}\sqrt{2y(x-2y)}$.
 15. $\frac{(a+b)^2}{(a-b)^2}\sqrt{a^2-b^2}$.

- Page 265. — 2. $\sqrt{8}$. 3. $\sqrt{45}$. 4. $\sqrt{50}$. 5. $\sqrt[3]{81}$. 6. $\sqrt{80}$.
 7. $\sqrt{2}$. 8. $\sqrt[3]{\frac{a^3}{4}}$. 9. $\sqrt{\frac{9x^5}{16}}$. 10. $\sqrt[4]{\frac{c}{16b^3}}$. 11. $\sqrt{\frac{14a^2}{5b^2}}$.
 12. $\sqrt{\frac{11a^2}{4}}$. 13. $\sqrt[3]{\frac{1}{3}}$.

2. $\sqrt[3]{4}$, $\sqrt[3]{3}$. 3. $\sqrt[3]{125}$, $\sqrt[3]{36}$. 4. $\sqrt[3]{7}$, $\sqrt[3]{100}$. 5. $\sqrt[3]{10}$,
 $\sqrt[3]{8}$, $\sqrt[3]{25}$. 6. $\sqrt[3]{16}$, $\sqrt[3]{8}$, $\sqrt[3]{729}$. 7. $\sqrt[3]{13}$, $\sqrt[3]{3125}$, $\sqrt[3]{16}$. 8. $\sqrt[3]{729}$,
 $\sqrt[3]{625}$, $\frac{1}{3}\sqrt[3]{27}$. 9. $\sqrt[3]{a^6b^6}$, $\sqrt[3]{a^4b^8}$, $\sqrt[3]{8}$. 10. $\sqrt[3]{a^6}$, $\sqrt[3]{b^4}$, $\sqrt[3]{x^3}$, $\sqrt[3]{y^2}$.
11. $\sqrt{(a+b)^2}$, $\sqrt{(x+y)^3}$.

- Page 266. — 12. $\sqrt{2}$; $\sqrt{3}$. 13. $\sqrt[3]{3}$; $3\sqrt[3]{4}$. 14. $\sqrt[3]{7}$, $\sqrt{2}$, $\sqrt[3]{3}$.
15. $\sqrt{2}$, $\sqrt[3]{5}$, $\sqrt[3]{4}$. 16. $\sqrt[3]{3}$, $\sqrt[3]{30}$, $\sqrt[3]{2}$. 17. $\sqrt{2}$, $\sqrt{2\frac{1}{2}}$, $\sqrt[3]{4}$, $\sqrt[3]{5}$.
18. $\sqrt[3]{4}$, $\sqrt[3]{63}$, $\sqrt[3]{48}$, $\sqrt{7}$. 19. $\sqrt[3]{5}$, $\sqrt[3]{150}$, $\sqrt[3]{13}$, $\sqrt[3]{4}$.
2. $15\sqrt{2}$. 3. $10\sqrt{3}$. 4. $9\sqrt{5}$. 5. $(2a+3b)\sqrt{2ab}$.
6. $(5a^2+4bc)\sqrt{3b}$. 7. $15\sqrt{7}$. 8. $10\sqrt[3]{2}$. 9. $\frac{2}{3}\sqrt[3]{2}$.
10. $12\sqrt[3]{5}$. 11. $6\sqrt[3]{4}$.

- Page 267. — 12. $3a\sqrt{2a}$. 13. $3x\sqrt{2}$. 14. $(3a-6+9x)\sqrt{2}$.
15. $\frac{3a}{25} + 2a^2\sqrt{3}$. 16. $4\sqrt{2a}$. 17. $\frac{9a}{2}\sqrt{2a}$. 18. $4\sqrt{2}$.
19. $8\sqrt{3}$. 20. $9\sqrt{3}$. 21. $3\sqrt[3]{5} + 7\sqrt{7}$.
22. $(2a-21)\sqrt{3b} + 2a\sqrt{6b}$. 23. $\frac{29a-20b+8}{4}\sqrt{2b}$.
24. $\sqrt{5}$. 25. $11\sqrt{3}$. 26. $34\sqrt{2}$. 27. $8\sqrt[3]{2}$. 28. $5\sqrt{7}$.
29. $2b\sqrt{c}$. 30. $4\sqrt[3]{abx}$. 31. $13\sqrt{3a}$. 32. $9\sqrt{2(a+b)}$.
33. $\frac{yz+xz-xy}{xyz}\sqrt{a}$. 34. $\frac{a+b+c}{abc}\sqrt{abc}$. 35. $\frac{8}{9}\sqrt[3]{25}$.
36. $\frac{3x-3y+1}{x-y}\sqrt{x^2-y^2}$.

- Page 268. — 2. 4. 3. $\sqrt[3]{288}$. 4. $\sqrt[3]{30375}$. 5. $30\sqrt{2}$.
6. $12\sqrt{10}$. 7. $18\sqrt[3]{1125}$. 8. $12\sqrt[3]{96}$. 9. 25. 10. $\frac{1}{5}\sqrt{10}$.
12. 2. 13. 5. 14. $26-4\sqrt{30}$. 15. $\sqrt[3]{4} + 6\sqrt[3]{108} + 27$.
16. -7 . 17. $11-2\sqrt{30}$. 18. $4\sqrt{5}$. 19. 111.
20. $-27\sqrt{2}-33$. 21. $6a^2-a\sqrt{5}-5$.

- Page 269. — 22. $x\sqrt{x} + yz\sqrt{yz}$. 23. $x^2 + (x-y)\sqrt{xy} - y^2$.
24. x^2-6x+9 . 25. $\frac{4}{3}\sqrt{3} + 3\sqrt{2} - \sqrt{6}$. 26. $12\frac{3}{5}$.
27. $(a+b)(a^2+ab-2b^2)^{\frac{1}{2}}$. 28. $56+24\sqrt{5}$.
29. $4+2a-6\sqrt{2a}-5$. 30. $9a^2b-6a\sqrt{ab}+a$.
31. $23+4\sqrt{15}$. 32. $4a+4\sqrt{3ab}+2b$. 33. $\frac{1}{6}$.
34. $20a-2a\sqrt{2a}$. 35. $25-13\sqrt{6}+\sqrt{15}+\sqrt{10}$.
36. $11\frac{1}{24} + \frac{1}{3}\sqrt{6}$. 37. $8a^3-8a\sqrt{ab}+2b-4a^2b^2+4b^2\sqrt{ab}-\frac{b^3}{a}$.
38. 6. 39. a^2+ab+c^2 . 40. $(a+4b)(a^2-3ab+2b^2)^{\frac{1}{2}}$.
41. $(2x-3y)(2x^2+13xy+15y^2)^{\frac{1}{2}}$. 42. $(3x+2y)(4x^2-12xy+5y^2)^{\frac{1}{2}}$.

- Page 270. — 3. $2\frac{1}{2}$. 4. $\sqrt{3}$. 5. $\sqrt{2}$. 6. $\sqrt{2}$. 7. $\frac{1}{2}\sqrt{21}$.
8. $\frac{1}{2}\sqrt{2}$. 9. $2\sqrt[3]{5}$. 10. $\frac{1}{2}\sqrt[3]{8}$. 11. $\sqrt[3]{45}$. 12. $2\sqrt[3]{12}$.
13. $\frac{5a}{6b}\sqrt[3]{36b^2}$. 14. $\frac{2}{a}\sqrt[3]{12a}$. 15. $\sqrt[3]{32}$. 16. $1-\frac{2}{3}\sqrt[3]{3}$. 17. $1\frac{1}{3}$.

- Page 270 (*continued*). — 18. $\frac{3b}{a} \sqrt{2b}$. 19. $\frac{5a}{3} \sqrt{3}$. 20. $\frac{2}{3} \sqrt{3}$.
 21. $\frac{2a^n}{3} \sqrt{6}$. 22. $\frac{3}{4} \sqrt{6}$. 23. $\frac{1}{a+b} \sqrt{a^2 - b^2}$. 24. 3.
 25. $\sqrt{6} - 1$. 26. $2\sqrt{3} - \sqrt{6} + 2$. 27. $2 - \sqrt{3}$. 28. $a + x$.
 29. $1\frac{1}{3}$. 30. $\frac{2}{3}(a + b)$. 31. $2\frac{5}{8}$. 32. $\frac{36 + 13\sqrt{6}}{10}$.

- Page 271. — 1. $\frac{2 - 6b}{3b} \sqrt{15ab}$. 2. $(a - 1)\sqrt{10b}$.
 3. $(a + b)\sqrt{a - b}$. 4. $2 - 2x + 2\sqrt{1 - 2x}$.
 5. $4x - 4x\sqrt{2x} + 2x^2$. 6. $3 + 4\sqrt{x - 1} + x$.
 7. $x - 2ax + a^2x$. 8. $\frac{2}{15}\sqrt{3}$. 9. $b\sqrt{a}$. 10. $93 + 20\sqrt{5}$.
 11. $6 + \sqrt{6}$. 12. 93. 13. 14. 14. $(3 - 6a)\sqrt[3]{a} - 7a$.
 15. $2b - a^2 - a\sqrt{b}$. 16. 15. 17. 15,750. 18. $2 - 4\sqrt[3]{3} + 3\sqrt[3]{6}$.
 19. $\frac{7 + 4\sqrt{10}}{8}$.

- Page 272. — 1. .8. 2. .9. 3. 1.2. 4. 2.1. 5. .8. 6. .4.
 7. 2.1. 8. .1. 9. .4. 10. .3. 11. $\frac{\sqrt{ab}}{b}$. 12. $\frac{\sqrt{x}}{x^4}$. 13. $\frac{\sqrt{2ax}}{2a}$.
 14. $\frac{3\sqrt{14x}}{4b}$. 15. $\frac{2\sqrt{aby}}{by}$. 16. $\frac{\sqrt[3]{4}}{2}$. 17. $\frac{\sqrt[3]{x}}{x}$. 18. $\frac{\sqrt{3xy}}{3y}$.
 19. $\frac{\sqrt{r^2 - 1}}{r + 1}$. 20. $\frac{\sqrt{a^2 - b^2}}{a - b}$. 21. $\frac{\sqrt{x^2 - 4}}{x + 2}$. 22. $\frac{\sqrt{a + b}}{a + b}$.

- Page 273. — 2. $\frac{12 - 4\sqrt{2}}{7}$. 3. $\frac{c\sqrt{a} + c\sqrt{b}}{a - b}$. 4. $5 - 2\sqrt{6}$.
 5. $\frac{4 + \sqrt{2}}{7}$. 6. $\frac{a + 2\sqrt{ax} + x}{a - x}$. 7. $\frac{a^2 + 4a\sqrt{b} + 4b}{a^2 - 4b}$.
 8. $\frac{14\sqrt{5} - 7\sqrt{3}}{17}$. 9. $\frac{a\sqrt{6a} - 6a\sqrt{2b}}{2a - 24b}$.

- Page 274. — 10. $\frac{35 - 12\sqrt{6}}{38}$. 11. $2x^2 - 2x\sqrt{x^2 - 1} - 1$.
 12. $\frac{4ab\sqrt{15a} - 16ab\sqrt{3b}}{15a - 48b}$. 13. $\frac{a^2 + a + 2 - 2\sqrt{a^2 + a + 1}}{a^2 + a}$.
 14. $\frac{a - \sqrt{a^2 - b^2}}{b}$. 15. $\frac{\sqrt{x^4 - 4} - x^2}{2}$. 16. 1.1. 17. 10.
 18. 4.6. 19. 9.9. 21. $\frac{\sqrt{10} - \sqrt{35}}{5}$. 22. $\frac{2\sqrt{3} + 3\sqrt{2} - \sqrt{30}}{12}$.
 23. $\frac{29 + 12\sqrt{6} + 14\sqrt{3} + 15\sqrt{2}}{23}$. 24. $\frac{5\sqrt{6} - 2 + 4\sqrt{15} + \sqrt{10}}{4}$.

- Page 275. — 3. $4xy$. 4. $9\sqrt[3]{4a^2}$. 5. $b^3\sqrt[3]{9b}$. 6. $2cy^4\sqrt{d}$.
 7. $16\sqrt{2}$. 8. $27x^2$. 9. $c^2d^2\sqrt[3]{cd}$. 10. $2n\sqrt{n}$. 11. $81x^2y^2$.

12. $-8\sqrt{2}a$. 13. $729a^{3m}x^{2n}$. 14. $-32a^2b\sqrt[6]{a^3b^4}$. 16. $\sqrt[4]{ab}$.
 17. $\sqrt[6]{3c}$. 18. $\sqrt[4]{2a}$. 19. $\sqrt[6]{x^n}$. 20. $\sqrt[5]{a^2b^2}$. 21. $\sqrt[2n]{c^n x^2}$.
 22. $\sqrt[6]{3a}$. 23. $\sqrt[6]{7a^3}$. 24. $\sqrt[9]{ab^3x}$. 25. $-2a$. 26. $-\sqrt[5]{a^n b^n}$.
 27. $-3\sqrt[5]{xz}$. 28. $a\sqrt[3]{x^2}$. 29. $\sqrt[3]{2x^2y^2}$. 30. $\sqrt[n]{ax^2}$. 31. $\sqrt[2n]{xy}$.
 32. $\sqrt{3}$. 33. $-\frac{5}{3}$.

- Page 276. — 2. $3 - \sqrt{2}$. 3. $\sqrt{7} + \sqrt{5}$. 4. $\sqrt{10} - \sqrt{6}$.
 5. $\sqrt{13} + \sqrt{2}$. 6. $\sqrt{11} - \sqrt{5}$. 7. $\sqrt{6} + \sqrt{5}$. 8. $\sqrt{5} - \sqrt{2}$.
 9. $3 - \sqrt{3}$. 10. $3 + 2\sqrt{2}$. 11. $\sqrt{10} + \sqrt{2}$. 12. $\frac{1}{3}\sqrt{6} + \frac{1}{3}\sqrt{3}$.
 13. $\frac{1}{2}\sqrt{6} + \frac{1}{2}\sqrt{2}$. 14. $\frac{1}{2}\sqrt{14} + \frac{1}{2}\sqrt{10}$. 15. $\frac{1}{2}\sqrt{10} - \frac{1}{2}\sqrt{6}$.
 16. $\frac{3}{2} - \frac{1}{2}\sqrt{5}$.
 1. $3\sqrt{3}$.

- Page 277. — 2. $V = 24\pi$. 3. $V = 9\pi$. 4. $r = 2\sqrt{10}$.
 5. $r = 5\sqrt{2}$. 6. $36\sqrt{3}$. 7. $x = \pm 6\sqrt{2}$. 8. $a\sqrt{2}$.

- Page 278. — 2. $n = 9$. 3. $x = 144$. 4. $n = 36$. 5. $d = 72$.
 6. $d = 5$. 7. $n = 164$. 8. $x = c^2 - d^2$. 9. $r = 10$.
 10. $x = a^3 - b^3$. 11. $x = (a - b)^3$. 12. $x = 6$. 13. $x = 4$.
 14. $x = 121$. 15. $x = 6$. 16. $s = 25$. 17. $x = 64$. 18. $y = 32$.
 19. $x = 49$. 20. $m = 5$. 21. $b = 9$. 22. $x = \frac{1}{2}t$. 23. $x = 4$.
 24. $v = 121$. 25. $r = 2$. 26. $n = 11$.

- Page 279. — 2. Impossible; $\sqrt{2r} - \sqrt{2r - 3} = 1$. 3. $s = 3$.
 4. Impossible; $2\sqrt{x} - \sqrt{4x - 11} = 1$. 5. $x = 4$. 6. Impossible;
 $\sqrt{4y + 5} + 2\sqrt{y - 1} = 9$.

- Page 283. — 4. $x = 3$ or 1. 5. $x = 4.4$ or 1.6. 6. $x = 3.4$ or .6.
 7. Roots imaginary. 8. Roots imaginary. 9. $x = 2.7$ or $-.7$.
 10. $x = 3$ or 3. 11. $x = -.6$ or -3.4 . 12. Roots imaginary.
 13. $x = 4.2$ or $-.2$. 14. $x = 2.2$ or $-.2$. 15. $x = -1.3$ or -2.7 .
 16. $x = 4.2$ or 1.8. 17. $x = 2.9$ or .1. 18. Roots imaginary.

- Page 285. — 3. $x = \pm 3$. 4. $x = \pm 4$. 5. $x = \pm \frac{1}{6}$. 6. $x = \pm 6$.
 7. $x = \pm 5$. 8. $x = \pm 7$. 9. $x = \pm 2\sqrt{2}$. 10. $n = \pm 4$.
 11. $x = \pm 2\sqrt{2}$. 12. $u = \pm 3$. 13. $x = \pm \sqrt{6}$. 14. $x = \pm \sqrt{2}$.
 15. $s = \pm \sqrt{5}$. 16. $r = \pm \frac{1}{2}\sqrt{21}$. 17. $x = \pm 2\sqrt{2}$. 18. $x = \pm 5$.
 19. $x = \pm 10$. 20. $x = \pm \frac{1}{2}$. 21. $x = \pm a\sqrt{b - 1}$.
 22. $x = \pm \sqrt{a^2 - b^2}$. 23. $x = \pm 6$. 24. $x = \pm 3$. 25. $n = \pm 6$.
 26. $a = \pm \sqrt{-1}$. 27. $x = \pm 4$. 28. $t = \pm 6$. 29. $x = \pm 5$.
 30. $x = \pm 3a^2$.

- Page 286. — 1. -1 . 2. Width, 20 rd.; length, 80 rd. 3. 80 rd.
 4. Thickness, 4 in.; width, 20 in. 5. Length, 50 ft.; width, 30 ft.
 6. 20 tons @ \$10. 7. Length, 32 rd.; width, 24 rd. 8. 10,000 ties;
 40 bd. ft. 9. \$.40; 20 wk.

$$1. t = \sqrt{\frac{2s}{g}} \quad 2. v = \sqrt{\frac{2E}{M}}$$

Page 287. — 3. $l = \frac{gt^2}{\pi^2}$.

4. $v = \sqrt{\frac{FR}{m}}$.

5. $d = \sqrt{\frac{mm'}{G}}$.

6. 2.5 sec.

7. 12.5 sec.

8. $b = \sqrt{c^2 - a^2}$.

9. $m = \frac{1}{2}\sqrt{2(a^2 + b^2) - c^2}$.

10. $d = \sqrt{\frac{A}{.7854}}$.

11. $r = \sqrt{\frac{3V}{\pi h}}$.

12. $c = 10.0$.

13. $a = 12.0$.

14. $a = 7.1$.

15. $a = \frac{b}{2}\sqrt{3}$.

Page 288. — 16. $m = 8.5$.

17. $d = 35.7$ ft.

18. $r = 7$ cm.

Page 289. — 1. $x = 3$ or 2.

2. $x = -7$ or -3 .

3. $x = 2$ or -14 .

4. $x = 17$ or 3.

5. $x = 8$ or -3 .

6. $x = 4$ or -1 .

7. $x = 3$ or -13 .

8. $x = 12$ or 5.

9. $x = 7$ or -6 .

10. $x = 5$ or -3 .

11. $x = 14$ or -3 .

12. $x = -11$ or -5 .

13. $x = 3$ or $\frac{1}{2}$.

14. $z = \frac{3}{2}$ or -1 .

15. $v = 2$ or $-\frac{4}{3}$.

16. $r = \frac{1}{5}$ or $\frac{5}{2}$.

17. $s = \frac{2}{3}$ or $\frac{3}{2}$.

18. $t = -1$ or -2 .

19. $x = 3$ or $-\frac{2}{3}$.

20. $x = 1$ or 2.

Page 290. — 1. $x = 5$ or -1 .

2. $x = 7$ or -1 .

3. $x = 9$ or -1 .

4. $x = 3$ or -5 .

Page 291. — 5. $x = 6$ or -4 .

6. $x = -3$ or -5 .

7. $x = 7$ or -9 .

8. $x = 11$ or 1.

9. $x = 10$ or -4 .

10. $y = 2$ or -5 .

11. $v = 2$ or -7 .

12. $n = 2$ or -1 .

13. $x = \frac{1}{4}(1 \pm \sqrt{41})$.

14. $x = \frac{5}{3}$ or 1.

15. $x = \frac{1}{6}(-1 \pm \sqrt{43})$.

16. $x = \frac{8}{3}$ or -1 .

17. $x = \frac{5}{2}$ or -1 .

18. $x = 1 \pm \sqrt{2}$.

19. $x = -5$ or -3.5 .

20. $v = \frac{1}{2}(-3 \pm \sqrt{13})$.

21. $x = 1 \pm \sqrt{5}$.

22. $r = -2 \pm \sqrt{11}$.

23. $l = 7$ or 4.

24. $x = \frac{7}{2}$ or $\frac{5}{2}$.

25. $x = \frac{1}{12}(-1 \pm \sqrt{61})$.

26. $x = \frac{4}{3}$ or 1.

27. $x = \frac{1}{4}(3 \pm \sqrt{21})$.

28. $x = 1$ or $-\frac{4}{5}$.

29. $x = 1$ or $-\frac{9}{16}$.

30. $x = \frac{1}{3}(1 \pm \sqrt{3})$.

Page 292. — 1. $x = 4$ or $\frac{3}{2}$.

2. $x = 2$ or $\frac{4}{5}$.

3. $x = 1$ or $-\frac{7}{2}$.

4. $x = -\frac{3}{2}$ or -2 .

5. $x = 2$ or $\frac{1}{3}$.

6. $x = 1$ or $-\frac{3}{4}$.

7. $a = 7$ or $-\frac{5}{2}$.

8. $a = -\frac{4}{3}$ or -3 .

9. $x = \frac{3}{2}$ or -2 .

10. $x = 2$ or $\frac{1}{2}$.

11. $x = \frac{2}{5}$ or $-\frac{1}{3}$.

12. $x = \frac{2}{3}$ or $-\frac{1}{4}$.

13. $x = 1$ or $-\frac{1}{11}$.

14. $x = -\frac{3}{5}$ or -1 .

15. $x = 1$ or $\frac{5}{6}$.

16. $x = -\frac{1}{3}$ or $-\frac{1}{2}$.

17. $x = -\frac{1}{2}$ or -2 .

18. $x = \frac{2}{3}$ or $\frac{1}{5}$.

19. $x = \frac{1}{2}$ or $-\frac{3}{2}$.

20. $x = \frac{3}{2}$ or -3 .

21. $m = -\frac{4}{3}$ or $-\frac{5}{2}$.

22. $p = -\frac{3}{10}$ or $-\frac{5}{3}$.

23. $x = 3 \pm \sqrt{-1}$.

24. $x = -2 \pm 2\sqrt{-2}$.

25. $x = 6$ or 2.

26. $x = \frac{1}{5}(3 \pm 9\sqrt{-1})$.

27. $x = \frac{1}{3}(-2 \pm \sqrt{-2})$.

28. $x = 1 \pm \sqrt{-3}$.

Page 293. — 1. $x = 1$ or 5.

2. $x = 5$ or -2 .

3. $x = \frac{1}{3}$ or -2 .

4. $x = \frac{1}{4}(7 \pm \sqrt{65})$.

5. $x = 14$ or -2 .

6. $x = 4$ or $-\frac{2}{3}$.

7. $x = 15$ or -2 .

8. $x = 2$ or -6 .

9. $x = -\frac{2}{3}$ or -3 .

10. $x = \pm 7$.

11. $x = \frac{1}{8}(3 \pm \sqrt{41})$.

12. $x = \pm 8$.

13. $x = 2$ or $\frac{1}{2}$.

14. $x = 3$ or $\frac{2}{3}$.

15. $x = 4$ or -2 .

16. $x = \frac{1}{5}(3 \pm \sqrt{-91})$.

17. $x = \frac{1}{3}(5 \pm 2\sqrt{-5})$.

18. $x = 2$ or -3 .

19. $x = \frac{1}{4}$ or $-\frac{1}{2}$.

20. $x = 10$ or $\frac{5}{3}$.

21. $x = 1$ or 4.

22. $x = 3$.

23. $x = 2$ or -1 .

24. $x = 3$ or $-\frac{7}{3}$.

25. $x = 1$ or 10.

26. $x = 4$.

27. $x = \pm 4$.

28. $x = \frac{1}{2}$ or $-\frac{2}{3}$.

29. $x = 7$ or $\frac{6}{7}$.

30. $y = -4$ or $-\frac{27}{5}$.

31. $x = 6$ or -18 .

32. $x = 0$ or $-\frac{1}{3}$.

- Page 294. — 33. $x = 4.24$ or $-.24$. 34. $v = -1.59$ or -4.41 .
 35. $x = 2.73$ or $-.73$. 36. $v = -.10$ or -3.90 .
 37. $u = -1.63$ or -3.37 . 38. $t = 10.42$ or 1.58 .
 39. $y = -.59$ or -3.41 . 40. $x = 2.23$ or $.27$.

- Page 295. — 3. 9, 6 or $-11\frac{1}{2}$, $-7\frac{2}{3}$. 4. 12 or -12 . 5. 150.
 6. Length, 29 ft.; width, 14 ft. 7. 120 rd. 8. 16 or -4 .
 9. 3, 7. 10. 5, 11. 11. 12, 8 or -8 , -12 .
 12. 16, 17 or -17 , -16 . 13. 7, 8 or -8 , -7 . 14. $a - 1$,
 $a + 1$ or $-a - 1$, $-a + 1$. 15. Width, 6 ft.; length, 30 ft.
 16. 16 ft. square.

- Page 296. — 17. 10 sheets. 18. 30 pins. 19. Length, $2\frac{1}{2}$ in.;
 width, $2\frac{1}{2}$ in.; height, 5 in.; volume, $31\frac{1}{4}$ cu. in. 20. 16 cm.; 13 cm.
 21. $\frac{3}{2}$ or $\frac{2}{3}$. 22. 2, 3. 23. c or -1 . 24. 112 crates; 60 fowls.
 25. 200 tons. 26. 6 hr. 27. 500 ft.

- Page 297. — 28. 8 boys. 29. Length, 53 ft.; width, 50 ft.
 30. 3 in. 31. Larger, 45 min.; smaller, 55 min. 32. Larger,
 60 min.; smaller, 84 min. 33. John, 7 da.; Arthur, $5\frac{1}{4}$ da.
 34. Larger, 6 hr.; smaller, $7\frac{1}{2}$ hr. 35. $d = 72.45$. 36. \$20.
 37. 3%.

- Page 298. — 38. 18¢. 39. $\frac{1}{2}$ in. 40. Altitude, 12 in.; base,
 22 in. 41. 58. 42. $\frac{2}{3}$. 43. 8 ft. 44. 200 boxes; 10 lb.
 45. 13 in. square. 46. 9 in. 47. Length, 85 ft.; width, 75 ft.

- Page 299. — 1. $r = \frac{1}{2\pi} \sqrt{\pi S}$. 2. $d = \frac{1}{n} \sqrt{2.5 n P}$.
 3. $S = \frac{1}{1.5 W} \sqrt{1.5 F R W}$. 4. $d = \frac{2}{\pi h} \sqrt{3 \pi h V}$.
 5. $r = \frac{ab}{2(b^2 - a^2)} \sqrt{b^2 - a^2}$. 6. $t = \frac{1}{32} (v \pm \sqrt{v^2 + 64 a - 64 S})$.
 7. $w = \frac{2}{z} \sqrt{2 h r - h^2}$. 8. $r = \frac{v}{5} \sqrt{10 \pi v}$; $v = \frac{1}{\pi} \sqrt[3]{2.5 \pi^2 r^2}$.
 9. 6 ft.; 16 ft. 10. Lower base, 14 ft.; upper base, 9 ft.; altitude,
 8 ft. 11. $h = \sqrt{a^2 + b^2}$. 12. $d = s\sqrt{2}$. 13. 11.3 ft.; 4.5 ft.
 14. 127.3 ft.

- Page 300. — 15. $d = \sqrt{a^2 + b^2}$; $a = \sqrt{d^2 - b^2}$. 16. Width, 6 ft.;
 length, 8 ft. 17. $A = a^2$; $A = a^2 - b^2$. 18. 3 in.
 19. $r = \frac{1}{\pi} \sqrt{\pi^2 R^2 - \pi A}$. 20. 104 ft. 21. $V = 10\sqrt{2 P}$.
 22. 60 mi. per hour. 23. $a = 5$; $b = 3$.

- Page 301. — 24. $t = \frac{T}{L} \sqrt{L l}$. 25. Once in 3 sec.
 26. $a:b = b:c$; $b = \sqrt{ac}$. 27. Longer part, 2.8 ft.; shorter part,
 1.7 ft.

1. $x = \pm \sqrt{b}$. 2. $x = \pm 3$. 3. $x = c$ or $-d$. 4. $x = 6b$ or $-2b$.
 5. $x = -b \pm b\sqrt{2}$. 6. $x = 2a$ or $-5a$. 7. $x = 0$ or $a - b - c$.
 8. $x = \pm a$. 9. $x = \frac{b}{a}$ or $-\frac{a}{b}$. 10. $x = 2b \pm b\sqrt{11}$.

- Page 301 (*continued*). — 11. $x = \pm(a - b)$. 12. $x = 2c$ or $\frac{1}{2}c$.
 13. $x = \frac{3}{4}a$ or $\frac{1}{4}a$. 14. $x = c$ or $\frac{1}{c}$. 15. $x = \pm(a - 1)$.
 16. $x = \frac{1}{2}a$ or $-\frac{7}{2}a$. 17. $x = \frac{b}{5}(5 \pm 2\sqrt{15})$. 18. $x = \frac{2}{a}$ or $-\frac{b}{6}$.
 19. $x = a - b$ or c . 20. $x = \frac{a - b}{b - c}$ or 1. 21. $x = \pm a\sqrt{b - 1}$.
 22. $x = \pm\sqrt{a^2 - b^2}$. 23. $x = \frac{b^2}{ac}$ or $\frac{b^{\frac{11}{2}}}{ac}$. 24. $x = \frac{3}{4}b$ or $\frac{1}{2}b$.

- Page 302. — 25. $x = 4a$ or $-\frac{1}{4}a$. 26. $x = \frac{n+1}{n-1}$ or -1 .
 27. $x = 2a - 1$ or $a + 1$. 28. $x = a + b$ or $\frac{a+b}{2}$.
 29. $x = \frac{a+b}{2}$ or $-\frac{3b}{2}$. 30. $x = \pm \frac{a+b}{2}\sqrt{2}$.
 31. $x = a - b$ or $-b$.

- Page 303. — 1. $x = 4$. 2. $x = 0$ or 1. 3. $x = 49$.
 4. $x = 8$. 5. $x = 1$. 6. $x = \pm 4$. 7. $x = -24$. 8. $x = 5$.
 9. $x = a + 2b$ or $-b$. 10. $x = 6$. 11. $x = \pm 2.45$.
 12. $x = \pm 7.07$. 13. $x = \pm 1.15$. 14. $x = 3.85$ or -2.85 .
 15. $x = 119.42$ or 8.58 . 16. $x = 9.90$ or $.10$. 17. $x = 10$.
 18. $x = \pm 2a$. 19. $x = \frac{2a}{3}\sqrt{3}$. 20. $x = 3a^2$. 21. $x = 17$.
 22. $x = 1$. 23. $x = \frac{5}{8}$. 24. $x = \pm \frac{a^3}{a^4 - 4}\sqrt{a^4 - 4}$.

- Page 304. — 3. $x = \pm 2$ or $\pm\sqrt{-21}$. 4. $x = \pm 1$ or $\pm\frac{2}{3}\sqrt{-6}$.
 5. $x = \pm 1$ or $\pm\frac{1}{5}\sqrt{-55}$. 6. $x = 4$ or -3 . 7. $x = \pm 2$ or $\pm\sqrt{-10}$.
 8. $x = 64$ or 1. 9. $x = 256$. 10. $x = 1$.
 11. $x = 216$ or -8 . 12. $x = \pm 1$ or ± 64 .

- Page 305. — 14. $x = 0$ or 25. 15. $x = 0, 1$, or 16.
 16. $x = 0, 1$, or $\frac{1}{8}$. 17. $x = 0, 4$, or 9. 18. $x = 0, 1$, or 8.
 19. $x = 0$ or $\frac{1}{9}$. 21. $x = 15$. 22. $x = 2$. 23. $x = 3$ or -5 .
 24. $x = 5$ or $\frac{1}{3}$. 25. $x = 4$ or -3 . 26. $x = 5$ or $-\frac{1}{3}$.

- Page 311. — 6. $\begin{cases} x = 5.7, -3.9; \\ y = 1.8, -4.6. \end{cases}$ 7. $\begin{cases} x = 5, -4.4; \\ y = 0, 3.8. \end{cases}$
 8. $\begin{cases} x = 4.1, -5.1; \\ y = .7, -2.4. \end{cases}$ 9. $\begin{cases} x = 1, 1; \\ y = -1, -1. \end{cases}$ 10. $\begin{cases} x = 2, 3.3; \\ y = 0, -1.3. \end{cases}$
 11. $\begin{cases} x = 4, 1, -1, -4; \\ y = -\frac{1}{2}, -2, 2, \frac{1}{2}. \end{cases}$ 12. Two imaginary roots;
 also $\begin{cases} x = .2, .2; \\ y = 3.0, -3.0. \end{cases}$ 13. $\begin{cases} x = 4, -4, 4, -4; \\ y = 0, 0, 0, 0. \end{cases}$
 14. Four imaginary roots. 15. $\begin{cases} x = 7.5, 7.5, -7.5, -7.5; \\ y = 2.7, -2.7, 2.7, -2.7. \end{cases}$
 16. $\begin{cases} x = 5, 4; \\ y = 3, 0. \end{cases}$ 17. $\begin{cases} x = 5, 4, -4, -5; \\ y = 4, 5, -5, -4. \end{cases}$

- Page 312. — 2. $\begin{cases} x = 6, -6; \\ y = 2, -2. \end{cases}$ 3. $\begin{cases} x = 4, 2; \\ y = -1, 1. \end{cases}$ 4. $\begin{cases} x = 27, 5; \\ y = 19, -3. \end{cases}$
5. $\begin{cases} x = 4, -4; \\ y = -3, 3. \end{cases}$ 6. $\begin{cases} x = 2, -\frac{1}{3}; \\ y = 4, \frac{5}{3}. \end{cases}$ 7. $\begin{cases} m = 5, -11; \\ n = 2, -6. \end{cases}$
8. $\begin{cases} y = 2, 6; \\ z = -2, -10. \end{cases}$ 9. $\begin{cases} x = 3, 5; \\ y = 5, 3. \end{cases}$ 10. $\begin{cases} x = 3, -2; \\ y = 1, -4. \end{cases}$
11. $\begin{cases} x = 3, \frac{1}{2}; \\ y = 4, \frac{13}{2}. \end{cases}$ 12. $\begin{cases} x = 2, \frac{18}{5}; \\ y = 1, -\frac{11}{5}. \end{cases}$ 13. $\begin{cases} x = 2, -\frac{52}{15}; \\ y = 2, -\frac{22}{15}. \end{cases}$
14. $\begin{cases} x = \frac{1}{3}(3 + \sqrt{3}), \frac{1}{3}(3 - \sqrt{3}); \\ y = \frac{1}{3}(-5 + \sqrt{3}), \frac{1}{3}(-5 - \sqrt{3}). \end{cases}$ 15. $\begin{cases} x = -2, -\frac{1}{17}; \\ y = -3, -\frac{22}{17}. \end{cases}$

- Page 313. — 16. $\begin{cases} x = 2, \frac{7}{3}; \\ y = -1, -\frac{5}{7}. \end{cases}$ 17. $\begin{cases} x = 1, -\frac{5}{4}; \\ y = 4, -\frac{11}{4}. \end{cases}$
18. $\begin{cases} x = 3, -1; \\ y = 2, -6. \end{cases}$ 19. $\begin{cases} x = 4, -4; \\ y = 4, 8. \end{cases}$ 20. $\begin{cases} x = 0, 4; \\ y = 5, 3. \end{cases}$
21. $\begin{cases} x = 2, 5; \\ y = 5, 2. \end{cases}$ 22. $\begin{cases} x = 3 + \sqrt{-71}, 3 - \sqrt{-71}; \\ y = -3 + 3\sqrt{-71}, -3 - 3\sqrt{-71}. \end{cases}$
23. $\begin{cases} x = 5, -\frac{37}{13}; \\ y = 1, -\frac{55}{13}. \end{cases}$ 24. $\begin{cases} x = 5, -71; \\ y = 6, 82. \end{cases}$ 25. $\begin{cases} x = 4, -\frac{76}{5}; \\ y = 2, \frac{74}{5}. \end{cases}$
1. 5, 3. 2. 5, 4. 3. 12, 4 or -12, -4. 4. Length, 30 rd.; width, 15 rd. 5. Length, 24 rd.; width, 10 rd. 6. 12, 4 or 10, 2, 5, 3. 7. 30, 15 or -30, -15.

- Page 314. — 2. $\begin{cases} x = 6, 5; \\ y = 5, 6. \end{cases}$ 3. $\begin{cases} x = 6, 1; \\ y = 1, 6. \end{cases}$ 4. $\begin{cases} x = 12, 3; \\ y = 3, 12. \end{cases}$
5. $\begin{cases} x = 4, -1, 1, -4; \\ y = 1, -4, 4, -1. \end{cases}$ 6. $\begin{cases} x = 3, 1; \\ y = 1, 3. \end{cases}$ 7. $\begin{cases} x = 5, 3; \\ y = 3, 5. \end{cases}$
8. $\begin{cases} x = 7, -1, 1, -7; \\ y = 1, -7, 7, -1. \end{cases}$ 9. $\begin{cases} x = 6, 1; \\ y = 1, 6. \end{cases}$ 10. $\begin{cases} x = 3, 2; \\ y = 2, 3. \end{cases}$
11. $\begin{cases} x = 4, 3; \\ y = 3, 4. \end{cases}$ 12. $\begin{cases} x = 2, 1; \\ y = 1, 2. \end{cases}$ 13. $\begin{cases} x = 5, -1, 1, -5; \\ y = 1, -5, 5, -1. \end{cases}$
14. $\begin{cases} x = 2, 2, -2, -2; \\ y = 2, 2, -2, -2. \end{cases}$

- Page 315. — 2. $\begin{cases} x = 3, 1, \frac{1}{2}(-5 + \sqrt{13}), \frac{1}{2}(-5 - \sqrt{13}); \\ y = 1, 3, \frac{1}{2}(-5 - \sqrt{13}), \frac{1}{2}(-5 + \sqrt{13}). \end{cases}$
3. $\begin{cases} x = 7, -2; \\ y = 2, -7. \end{cases}$

- Page 316. — 4. $\begin{cases} x = \frac{1}{7}, -\frac{1}{5}; \\ y = \frac{1}{5}, -\frac{1}{7}. \end{cases}$ 5. $\begin{cases} x = 3, 3; \\ y = -3, -3. \end{cases}$
6. $\begin{cases} x = 3, -3, -4\sqrt{2}, 4\sqrt{2}; \\ y = 1, -1, \frac{5}{2}\sqrt{2}, -\frac{5}{2}\sqrt{2}. \end{cases}$

- Page 317. — 7. $\begin{cases} x = 3, -3, \frac{7}{3}\sqrt{3}, -\frac{7}{3}\sqrt{3}; \\ y = 2, -2, \frac{1}{3}\sqrt{3}, -\frac{1}{3}\sqrt{3}. \end{cases}$
8. $\begin{cases} x = 5, -5, -\sqrt{3}, \sqrt{3}; \\ y = 1, -1, 2\sqrt{3}, -2\sqrt{3}. \end{cases}$ 9. $\begin{cases} x = 7, -7, -5\sqrt{-1}, 5\sqrt{-1}; \\ y = 5, -5, 7\sqrt{-1}, -7\sqrt{-1}. \end{cases}$

Page 317 (*continued*).—10. $\begin{cases} x = .4, -.2; \\ y = .2, -.4. \end{cases}$

11. $\begin{cases} x = 3, 2; \\ y = 2, 3. \end{cases}$

12. $\begin{cases} x = 4 + \frac{1}{8}\sqrt{3}, 4 - \frac{1}{8}\sqrt{3}; \\ y = -4 + \frac{1}{8}\sqrt{3}, -4 - \frac{1}{8}\sqrt{3}. \end{cases}$

13. $\begin{cases} x = 5, -5; \\ y = 1, -1. \end{cases}$

14. $\begin{cases} p = 5, -3, \frac{1}{2}(-3 + \sqrt{69}), \frac{1}{2}(-3 - \sqrt{69}); \\ q = -3, 5, \frac{1}{2}(-3 - \sqrt{69}), \frac{1}{2}(-3 + \sqrt{69}). \end{cases}$

15. $\begin{cases} y = 5, -4; \\ z = 4, -5. \end{cases}$

16. $\begin{cases} x = 5, 1; \\ y = 1, 5. \end{cases}$

17. $\begin{cases} x = \frac{1}{3}, -\frac{1}{3}; \\ y = \frac{1}{2}, -\frac{1}{2}. \end{cases}$

18. $\begin{cases} r = \frac{1}{2}, \frac{3}{2}; \\ s = \frac{3}{2}, \frac{1}{2}. \end{cases}$

19. $\begin{cases} c = 1, 1, -1, -1; \\ d = -1, -1, 1, 1. \end{cases}$

20. $\begin{cases} x = 1, -2; \\ y = 2, -1. \end{cases}$

Page 318.—1. $\begin{cases} x = 2, 2; \\ y = -2, -2. \end{cases}$

2. $\begin{cases} x = 4, -4; \\ y = 6, -6. \end{cases}$

3. $\begin{cases} x = 5, -6; \\ y = 6, -5. \end{cases}$

4. $\begin{cases} x = \frac{5}{3}, \frac{1}{3}; \\ y = -\frac{1}{3}, -1. \end{cases}$

5. $\begin{cases} x = 3, 96; \\ y = 2, -29. \end{cases}$

6. $\begin{cases} x = 2, \frac{8}{5}; \\ y = 3, \frac{1}{5}. \end{cases}$

7. $\begin{cases} x = 5, -2; \\ y = -2, 5. \end{cases}$

8. $\begin{cases} x = 6, 5; \\ y = 5, 6. \end{cases}$

9. $\begin{cases} x = 4, \frac{9}{3}; \\ y = 3, -\frac{1}{3}. \end{cases}$

10. $\begin{cases} x = 3, -3; \\ y = 2, -2. \end{cases}$

11. $\begin{cases} x = \frac{5}{2}, -1; \\ y = -\frac{4}{5}, 2. \end{cases}$

12. $\begin{cases} x = \frac{6}{5}, -24; \\ y = \frac{4}{5}, 12. \end{cases}$

13. $\begin{cases} x = 3, 3, -3, -3; \\ y = 2, -2, 2, -2. \end{cases}$

14. $\begin{cases} x = 7, -7, \frac{11}{2}\sqrt{2}, -\frac{11}{2}\sqrt{2}; \\ y = 4, -4, \frac{3}{2}\sqrt{2}, -\frac{3}{2}\sqrt{2}. \end{cases}$

15. $\begin{cases} x = 8, -8; \\ y = 2, -2. \end{cases}$

16. $\begin{cases} a = 4, 7; \\ b = -8, -5. \end{cases}$

17. $\begin{cases} x = 4, -3, -1 + \sqrt{13}, -1 - \sqrt{13}; \\ y = -3, 4, -1 - \sqrt{13}, -1 + \sqrt{13}. \end{cases}$

18. $\begin{cases} x = 3, 3, -3, -3; \\ y = 8, -5, 5, -8. \end{cases}$

19. $\begin{cases} m = 3, 3, -3, -3; \\ n = 2, -2, 2, -2. \end{cases}$

20. $\begin{cases} p = 3, -7; \\ q = 2, 7. \end{cases}$

21. $m = 10, n = 6.$

Page 319.—22. $\begin{cases} r = 4, 3, -3, -4; \\ s = 3, 4, -4, -3. \end{cases}$

23. $\begin{cases} t = 4, -3, -1 + \sqrt{13}, -1 - \sqrt{13}; \\ u = -3, 4, -1 - \sqrt{13}, -1 + \sqrt{13}. \end{cases}$

24. $\begin{cases} x = 7, -6; \\ y = 6, -7. \end{cases}$

25. $\begin{cases} x = -2, 2; \\ y = 5, -5. \end{cases}$

26. $\begin{cases} x = 4, 4, -4, -4; \\ y = 2, -2, 2, -2. \end{cases}$

27. $\begin{cases} x = 6, 2, -2, -6; \\ y = 2, 6, -6, -2. \end{cases}$

28. $\begin{cases} p = 0, 0, \frac{3}{2}; \\ q = 1, -1, -\frac{1}{2}. \end{cases}$

29. $\begin{cases} x = 9, 5, -5, -9; \\ y = 5, 9, -9, -5. \end{cases}$

30. $\begin{cases} x = 3\sqrt{21}, 3\sqrt{21}, -3\sqrt{21}, -3\sqrt{21}; \\ y = 6\sqrt{5}, -6\sqrt{5}, 6\sqrt{5}, -6\sqrt{5}. \end{cases}$

31. $\begin{cases} x = \frac{1}{2}(\sqrt{123} + \sqrt{-33}), \frac{1}{2}(-\sqrt{123} + \sqrt{-33}), \\ \quad \frac{1}{2}(\sqrt{123} - \sqrt{-33}), \frac{1}{2}(-\sqrt{123} - \sqrt{-33}); \\ y = \frac{1}{2}(\sqrt{123} - \sqrt{-33}), \frac{1}{2}(-\sqrt{123} - \sqrt{-33}), \\ \quad \frac{1}{2}(\sqrt{123} + \sqrt{-33}), \frac{1}{2}(-\sqrt{123} + \sqrt{-33}). \end{cases}$

$$32. \begin{cases} x = 16, 9; \\ y = 9, 16. \end{cases} \quad 33. \begin{cases} m = 3, 6; \\ n = 6, 3. \end{cases} \quad 34. \begin{cases} x = \frac{a^2 + b^2}{a - b}, -\frac{a^2 + b^2}{a - b}; \\ y = \frac{2ab}{a - b}, -\frac{2ab}{a - b}. \end{cases}$$

$$35. \begin{cases} x = 2a, a + 3b; \\ y = -b, \frac{b - a}{2}. \end{cases} \quad 36. \begin{cases} x = 2, -2; \\ y = -\frac{1}{2}, \frac{1}{2}. \end{cases}$$

$$37. \begin{cases} r = 4, 4, -4, -4; \\ s = 3, -3, 3, -3. \end{cases} \quad 38. a = \frac{v^2}{2s}; t = \frac{2s}{v}.$$

$$39. \begin{cases} v = -6 + \sqrt{36 + 2as}, -6 - \sqrt{36 + 2as}; \\ t = \frac{-6 + \sqrt{36 + 2as}}{a}, \frac{-6 - \sqrt{36 + 2as}}{a}. \end{cases}$$

1. 12, 4. 2. 11, 7 or -7, -11. 3. 12, 9 or -12, -9. 4. 6, 2.

Page 320. — 5. 81, 1. 6. Length, 6 in.; width, 4 in. 7. 3 s, $\frac{s}{3}$

or -3 s, $-\frac{s}{3}$. 8. Length, 12 ft.; width, 10 ft. 9. 15, 45 or -15, -45. 10. $6\frac{1}{8}$; $5\frac{7}{8}$. 11. Length, 24 rd.; width, 20 rd. 12. Length, 7 ft.; width, 5 ft. 13. 9, 8 or 2, 1. 14. 12. 15. 4 ft. 16. 33. 17. Altitude, 16 rd. or 30 rd.; base, 30 rd. or 16 rd.

Page 321. — 18. Base, 16 in.; altitude, 9 in. 19. $a + \sqrt{a^2 + b}$, $-a + \sqrt{a^2 + b}$ or $a - \sqrt{a^2 + b}$, $-a - \sqrt{a^2 + b}$. 20. Length, 36 in.; width, $24\frac{1}{2}$ in. 21. 3 yd. @ 25¢. 22. 25 yd. @ \$.24. 23. Length, 137 ft.; width, 92 ft. 24. 6 ft.; 9 ft. 25. \$10. 26. 8 hr. 27. 49 yd.; \$3.

Page 322. — 28. 8 hr. 29. Side of square, 4; rectangle, 7 by 1. 30. 8 in. from length; 4 in. from width. 31. $1\frac{1}{2}$ ft. 32. Length, 40 rd.; width, 28 rd. 33. \$3800 at 5%. 34. \$2000 at 3%. 35. \$10,500 at 6%. 36. 5, 3.

Page 323. — 2. (b) 4. 7. $a^{2n}x^{mn}$; $-a^8$; a^{12} ; a^{12} ; $-a^{10}$.

Page 324. — 15. (b) $x = \pm 2$. 16. (b) $x = 2$ or $-\frac{11}{3}$; $x = \frac{1}{10}(7 \pm \sqrt{209})$. 17. (b) $x = 4$ or $-\frac{3}{2}$. 19. $x = 3$ or $-\frac{3}{2}$; $x = \frac{1}{8}(5 \pm \sqrt{133})$.

Page 325. — 29. $\pm 4a$. 30. $18:3 = 12:2$; $3:18 = 2:12$.

1. (a) $A = 156.2$; (b) $A = 26.0$; (c) $b = \frac{A}{h}$; (d) $b = 8.3$. 2. (a) $A = 240.3$; (b) $A = 818.0$. 3. (a) $A = 42.5$; (b) $A = 83.1$. 4. (a) $A = 66.6$; (b) $A = 159.8$; (c) $h = \frac{2A}{b + b'}$; (d) $h = 9.5$. 5. (a) $A = 220.9$; (b) $A = 291.2$.

Page 326. — 6. (a) $c = 10.4$; (b) $c = 11.5$; (c) $a = \sqrt{c^2 - b^2}$; (d) $a = 17.2$. 7. (a) (1) $c = 77$; (2) $c = 49.0$; (b) (1) $c = 22.6$, (2) $c = 34.7$. 8. (a) (1) $A = 39.6$, (2) $A = 169.8$, (3) $A = 38.5$; (b) (1) $A = 30.7$, (2) $A = 57.6$, (3) $A = 58.7$.

- Page 326 (continued). — 9. (a) $V = 206.0$; (b) $V = 235.6$;
 (c) $l = \frac{V}{wh}$; (d) $l = 4$. 10. (a) $V = 669.9$; (b) $V = 10.9$.
 11. (a) $S = 185.5$; (b) $S = 108.4$. 12. (a) $V = 125.1$; (b) $V = 20.1$;
 (c) $B = \frac{3V}{h}$; (d) $B = 21$.

- Page 327. — 13. (a) $S = 180.3$; (b) $S = 183.2$; (c) $p = \frac{2S}{l}$;
 (d) $p = 73.2$. 14. (a) $V = 1125.2$; (b) $V = 303.4$; (c) $r = \sqrt{\frac{V}{\pi h}}$;
 (d) $r = 3.5$. 15. (a) $S = 774.3$; (b) $S = 297.7$. 16. (a) $V = 158.0$;
 (b) $V = 338.5$; (c) $h = \frac{3V}{\pi r^2}$; (d) $h = 6$. 17. (a) $S = 838.9$;
 (b) $S = 80.0$. 18. (a) $S = 679.1$; (b) $S = 380.3$; (c) $r = \frac{1}{2} \sqrt{\frac{S}{\pi}}$;
 (d) $r = 4.0$. 19. (a) $V = 321.7$; (b) $V = 951.2$.

- Page 328. — 20. (a) $s = 61.1$; (b) $s = 49.4$. 21. (a) $s = 361.0$;
 (b) $s = 228.1$; (c) $t = \sqrt{\frac{2s}{a}}$; (d) $t = 3.5$. 22. (a) $h = 10.6$;
 (b) $a = 6.1$. 23. (a) $M = 10,250$; (b) $M = 56,313$.
 24. (a) $f = 57.8$; (b) $f = 20$; (c) $m = \frac{ft}{v}$; (d) $m = 26.2$.
 25. (a) $w = 98,000$; (b) $w = 6125$. 26. (a) $C = -22.5^\circ$;
 $C = 51.8^\circ$; (b) $F = \frac{2}{3}C + 32^\circ$; (c) $F = 249.4^\circ$, $F = 24.8^\circ$.
 27. (a) $t = \frac{a-p}{pr}$; (b) $t = 4.0$ yr.

- Page 332. — 1. $5b + d$. 2. $\frac{7}{4}x - \frac{1}{4}y - \frac{1}{4}z$.
 3. $c + 3m - n$. 4. $9am$. 5. $8\sqrt{x} + 5\sqrt{xy} + 3\sqrt{y}$. 6. $4x - 9y$.
 7. $4c + 10d$. 8. $-9p + 5q$. 9. $8x - 10y$. 10. $19a + 7b$.
 11. $-26c - 14d$. 12. $2s - 2v$. 13. $-2x + 2y - 2z$.
 14. $-4y$. 15. $-2c - 5d$.

- Page 333. — 16. $-2p - 2q + 6r$. 17. $-2a - 2b$.
 18. $-2xy$. 19. $4ab$. 20. $3x^m + 4y^n$. 21. $7s^2 - 2r^2$.
 22. $2x + y$. 23. $4m^2 + 3n^2$. 24. $1 + 2x - 2x^2 + x^3$.
 25. $-a^2 + ab + ac - b^2 + c^2$. 26. $2x^2 + 3x - 8$.
 27. $-\frac{3}{2}a^2 + 10ab + b^2$. 28. $-16a^2b^2 - 44abcd + 12c^2d^2$.
 29. $233x^2 + 270xy - 30y^2$. 30. $4x^6 - 40x^3 + 112$.
 31. $-3x^{2a} - 14x^ay^b - 8y^{2b}$. 32. $2y^2 - 34$. 33. $2x^4 + 9x^2 - 81$.
 34. $-a^2 + 19ab - 50b^2$. 35. $8a^2 - 18ab + 20b^2$.
 36. $-.05x^2 - 2.02xy - 4.56y^2$. 37. $x^{3n} + x^{2n}y^m - x^ny^{2m} - y^{3m}$.
 38. $120 - 240x - 190x^2 + 160x^3 - 10x^4$.
 39. $40x - 18x^2 - 29x^3 + 36x^4 - 20x^5$.
 40. $6a^4 - 11a^3b - 15a^2b^2 + 17ab^3 + 15b^4$.

- Page 334. — 41. $a^{4n} - 2a^{2n}b^{2n} + b^{4n}$. 42. $x^{2c-2}y^{2a+2} + 2x^{c-1}y^{a+1}$
 $- x^{2a+2}y^{2c-2} + 1$. 43. $z^{4a} + z^{4a-2} + z^{4a-4}$. 44. $3x - 5$.
 45. $a^n + 4b$. 46. $1.6x^2 + x - .3$.

47. $a^2 - 5a - 1 - \frac{1}{2a^2 + 5a - 1}$. 48. $3x - 2 + \frac{15x - 9}{x^2 + x - 1}$.
49. $x^{a+1} + x^a + x^{a-1}$. 50. $a^5 - a^4b + a^3b^2 - a^2b^3 + ab^4$.
1. $m(m + 2n)$. 2. $(4a - b)(4a - b)$. 3. $(3 - 2xy)(3 - 2xy)$.
4. $4(a^n + 2b^m)(a^n + 2b^m)$. 5. $a(x^2 + 2y)(x^2 - 2y)$.
6. $3(3ab + 1)(3ab - 1)$. 7. $2(2 + 7cd)(2 - 7cd)$.
8. $6(2b + ad)(2b - ad)$. 9. $y(x - 2)(x - 1)$.
10. $5(a + 7)(a + 2)$. 11. $x(y - 3)(y - 2)$. 12. $3a(4x + 1)(x - 3)$.
13. $7(3 - 2r)(2 + 3r)$. 14. $2a(4y + 1)(3y + 2)$.
15. $2(3x^n - 2)(x^n + 7)$. 16. $(ab - 4c)(ab + c)$.
17. $(5d + 3x)(2d + 3x)$. 18. $(4cd + 5ab)(cd + ab)$.
19. $(2m + n)(m - 2n)$. 20. $(3c + 4d)(2c - 5d)$.
21. $(3y - x)(x - 2y)$ or $(2y - x)(x - 3y)$. 22. $3x^2(3 - xy)$.
23. $4ab(a + 3b)$. 24. $(x - 2y)(ab - 4)$.
25. $a(m + 3y^2)(m + 3y^2)$. 26. $3(5b + 3c)(5b - 3c)$.
27. $(7a^n + 6)(7a^n - 6)$. 28. $2a(rs + 4y)(rs - 4y)$.
29. $3(2b + ad)(2b - ad)$. 30. $4(2 + 7cd)(2 - 7cd)$.
31. $5(x + 4ay)(x - ay)$. 32. $7(z + 3ab)(z - 6ab)$.
33. $2(b + 8ac)(b - 2ac)$. 34. $5(4 - x)(3 + x)$.

Page 335. — 35. $3(y - 2a)(y + a)$.

36. $7(2 - r)(1 + 2r)$.
37. $2a(2a + b)(a - 2b)$. 38. $3(3a + 4c)(2a - 5c)$.
39. $21a(b + 2ac + ab^2)$. 40. $4x^2(3x^2 - 4)$.
41. $45(2a^2 + 3b^2)(2a^2 - 3b^2)$. 42. $.01(2a + 5b)(2a - 5b)$.
43. $.03(6b + 7c)(6b - 7c)$. 44. $.01(a + 3b)(a + 3b)$.
45. $(t + 3)(r - s)$. 46. $(ax - b)(cx + d)$.
47. $x^{m+1}(x^m - 1)(x^{m-1} + 1)$. 48. $(3x + a)(2x - 3y)$.
1. $\frac{52x + 27}{8x^2}$. 2. $\frac{16x^2 + 28x - 39}{24x^2}$. 3. $\frac{3a^2 + x^2}{2ax}$.

4. $\frac{4x^2 - 38}{(x - 2)(x + 1)(x + 3)}$. 5. $\frac{4a^3b}{(a^2 + b^2)(a^2 - b^2)}$.
6. $\frac{16 - 4x}{(2x + 7)(x - 3)(x - 5)}$. 7. $-\frac{12x^2}{(x^2 - 1)(x^2 - 4)}$.
8. $\frac{(x^2 - 1)(x^2 - 4)}{x^2 - x + 1}$. 9. $\frac{a^2 - 34}{(a + 5)(a - 4)(a - 5)}$.
10. $\frac{x^2 - x + 1}{(x - 2)(x - 3)}$.

Page 336. — 11. $\frac{12y - a}{4}$.

12. $\frac{abx + 1}{x}$. 13. $\frac{a^2 - 4nz^2}{z^2}$.
14. $\frac{5r - s}{2}$. 15. $\frac{3x + 5}{x + 1}$. 16. $\frac{2b}{a - b}$. 17. $\frac{4x^2 - 6xy}{x - y}$.
18. $\frac{2a^2 - b^2}{a + b}$. 19. $\frac{2xy}{x + y}$. 20. $\frac{1}{3ay}$. 21. 4. 22. $\frac{9}{20a}$.
23. $x + y$. 24. $\frac{4xy + 4y}{x - 1}$. 25. $\frac{1}{2x}$. 26. $\frac{3a^2b^2}{5}$. 27. $2s - r$.
28. a . 29. $\frac{4}{3(a + b)}$. 30. $\frac{3}{4(x - 3)}$. 31. $\frac{1}{x - 3y}$. 32. 1.
33. $\frac{5}{x^2}$. 34. $(a + 1)^2$. 35. $\frac{x^4}{x^2 + y^2}$.

- Page 337. — 36. $\frac{1}{2x^2 - 1}$. 37. $\frac{1}{x^2 - 1}$. 38. $\frac{a^2 + a + 1}{2a + 1}$.
39. $\frac{x^2 + y^2}{(x - y)^2}$. 40. 1. 41. $\frac{5}{16}$. 42. $\frac{2 + a - x}{a + x}$.
1. $x = 7$. 2. $r = 2$. 3. $m = .1$. 4. $r = 1.36$. 5. $t = -17$.
6. $p = \frac{1}{4}$. 7. $m = 2$. 8. $y = -1\frac{10}{11}$. 9. $x = 6$. 10. $x = 9$.
11. $m = 17$. 12. $P = 1\frac{1}{2}$. 13. $m = 3$. 14. $x = 8\frac{2}{3}$.

- Page 338. — 1. $(26d - s)$ dollars. 2. $\left(3y + 6 + \frac{i}{12}\right)$ ft.
3. $h(a + b)$ mi. 4. $(12ds + c)$ cents. 5. (1) $2h(l + w)$ sq. yd.;
(2) lw sq. yd. 6. $(12b - 12c - d)$ in. 7. $c + \frac{cr}{100}$. 8. $xy + z$.
9. $\frac{ac}{10}$ apples. 10. $I = d + \frac{cnr}{100}$. 11. $100a + 10b + c$;
 $100c + 10b + a$. 12. $36(e - s)$ dollars. 13. $4(y - 3) = 2(y + 5)$.

- Page 339. — 14. Area, xy sq. yd.; perimeter, $2(x + y)$ yd.
15. $\left(p + \frac{pr}{20}\right)$ dollars. 16. $2n + 2$, $2n + 4$; $2n + 3$, $2n + 5$.
17. $S = L - \frac{dL}{100}$. 18. (1) $(s + c)$ mi. an hour; (2) $(s - c)$ mi. an hour.
1. $\frac{1}{8}$. 2. 7. 3. $2\frac{4}{5}$ da. 4. 6 da. 5. $2\frac{2}{17}$ hr.

- Page 340. — 6. 36 min. 7. $\frac{9}{10}$ gal.; $\frac{3}{8}$ gal. 8. 2 lb.
9. 12 nickels; 6 dimes. 10. 18 mi.
1. $x = 2$, $y = 6$. 2. $x = -\frac{3}{17}$; $y = 3\frac{4}{17}$. 3. $x = 1\frac{1}{3}$, $y = 1\frac{1}{3}$.
4. $x = 2.3$, $y = 4.1$. 5. $x = 5$, $y = 2$. 6. $x = -162$, $y = 42$.

- Page 341. — 7. 84. 8. 253. 9. A, 40 da.; B, 120 da.;
C, 60 da. 10. A, \$525; B, \$600; C, \$160. 11. State, \$22,500;
county, \$7500; town, \$5250. 12. Rate of rowing, 7 mi. per hour;
rate of current, 1 mi. per hour.
1. 12, 8. 2. 12, 4. 3. $15\frac{2}{3}$, $9\frac{1}{3}$. 4. 1, 3.

- Page 342. — 5. $\frac{1}{2}$. 6. 40° , 60° , 80° . 7. 13 boys. 8. 7.
9. 4, 5, $1\frac{1}{3}$, $1\frac{2}{3}$.
1. (a) $\frac{1}{16}$; (b) $\frac{1}{4}$. 2. $\frac{4}{5}$. 3. $\frac{1}{25}$; $\frac{1}{5}$. 4. $31\frac{1}{2}$ in. 5. $\frac{4}{9}$; $\frac{2}{3}$.

- Page 343. — 6. $7\frac{1}{2}$ in. 7. 4 ft. 8. 6 ft. 9. 432 cg.
10. 1024 cg. 11. $a = \sqrt{c^2 - b^2}$; $a = 21.2$. 12. 18.6 in.
13. 19 in. 14. (12) 200 sq. in.; (13) 156 sq. in. 15. 5.2.
16. $T = \frac{1}{2}\sqrt{14} = 9.0$.

- Page 344. — 1. $\sqrt{2b}$. 2. $(a + b)(a^2 + 2ab - 3b^2)^{\frac{1}{3}}$.
3. $8\sqrt{2} - 2\sqrt{30}$. 4. $a - a^{\frac{3}{4}} - 4a^{\frac{1}{2}} + 8a^{\frac{1}{4}} - 6 + \frac{4}{a^{\frac{1}{4}}} - \frac{3}{a^{\frac{1}{2}}} + \frac{1}{a^{\frac{3}{4}}}$.
5. The given values satisfy the first equation, but not the second equation.
6. $\frac{12a^2 + 6ab - 1}{6a}(6)^{\frac{1}{2}}$. 7. $(4a + 3b)\sqrt{a - 3b}$.
8. $\sqrt{15} - 24$. 9. $2a - 2\sqrt{a^2 - x^2}$. 10. $\frac{1}{4}\sqrt{10}$. 11. $8\sqrt{2}$.

12. $\frac{17}{8}(53\sqrt{5} - 113)$. 13. $x + 28 - 10\sqrt{x + 3}$. 14. $\frac{31a}{12}\sqrt{2}$.
 15. $17 + 6\sqrt{3} + \sqrt{6}$. 16. $-\frac{a + \sqrt{a^2 - x^2}}{x}$.

- Page 345. — 1. $\pm(a^2 + 6a - 18)$. 2. $\pm(x^2 - 3x + 2)$.
 3. $\pm\left(x^2 - \frac{x}{2} + \frac{2}{x}\right)$. 4. $\pm\left(\frac{a}{2} + b - \frac{c}{2}\right)$. 5. $\pm\left(y + \frac{a}{3} - \frac{b}{2}\right)$.
 6. $\pm\left(\frac{a}{5} - \frac{3b}{5} + c\right)$.
 1. $y = -.4$ or -3.5 . 2. $x = 7$ or -2 . 3. $x = .05$ or $-.3$.
 4. $x = 2$ or -1.6 . 5. $x = \frac{1}{20}(5 \pm \sqrt{1685})$. 6. $x = \frac{1}{4}(9 \pm \sqrt{57})$.
 7. $x = 3.5$ or -1.9 . 8. $x = 4.3$ or $.7$. 9. $x = 0$ or $.1$.
 10. $x = 4.9$ or -2.8 . 11. $x = .1$ or $-.4$. 12. $y = -.1$ or -3.9 .
 13. $x = -.9$ or -2.9 . 14. $x = \frac{1}{4}(3 \pm \sqrt{101})$. 15. $x = \frac{1}{4}(-1 \pm \sqrt{5})$.
 16. $y = 11\frac{1}{3}$ or 1 . 17. $r = \frac{1}{2}(3 \pm \sqrt{-7})$. 18. $x = 7$ or $1\frac{1}{4}$.
 19. $x = 4$ or $\frac{3}{8}$.

- Page 346. — 20. 8, 9 or -9 , -8 . 21. 5. 22. 6, 8 or -6 , -8 .
 23. 15.5, 4.5. 24. 16 rows. 25. 10 ft., 24 ft., 26 ft. 26. 30 mi.
 an hour. 27. 40 bbl. 28. Length, 60 ft.; width, 20 ft.
 29. 35 rows. 30. Rate going, 24 mi. an hour; rate returning,
 30 mi. an hour.

- Page 347. — 31. 21 mi. an hour. 32. First man, 45 da.; second
 man, 36 da. 33. 15 in., 12 in., 9 in. 34. Length, 24 rd.; width,
 10 rd. 35. 15.45 in., 9.55 in.

1. $\begin{cases} x = 3b, b; \\ y = b, 3b. \end{cases}$ 2. $\begin{cases} x = 5, -1; \\ y = -2, 1. \end{cases}$ 3. $\begin{cases} x = 4, 3; \\ y = 5, 7. \end{cases}$ 4. $\begin{cases} x = 3, -1; \\ y = 1, -3. \end{cases}$
 5. $\begin{cases} x = 2, -\frac{274}{7}; \\ y = -3, -\frac{213}{7}. \end{cases}$ 6. $\begin{cases} x = 3 + \sqrt{-71}, 3 - \sqrt{-71}; \\ y = -3 + 3\sqrt{-71}, -3 - 3\sqrt{-71}. \end{cases}$
 7. $\begin{cases} x = \frac{5}{4}\sqrt{2}, -\frac{5}{4}\sqrt{2}; \\ y = \frac{3}{4}\sqrt{2}, -\frac{3}{4}\sqrt{2}. \end{cases}$ 8. $\begin{cases} x = 3, \frac{2}{5}; \\ y = 2, -\frac{3}{5}. \end{cases}$ 9. $\begin{cases} x = 4, 6; \\ y = 15, 10. \end{cases}$
 10. $\begin{cases} x = 5, \frac{3}{4}; \\ y = 3, -\frac{5}{4}. \end{cases}$ 11. $\begin{cases} x = 3, 2; \\ y = 2, 3. \end{cases}$ 12. $\begin{cases} x = \frac{1}{8}, \frac{1}{2}; \\ y = 1, \frac{1}{3}. \end{cases}$

- Page 348. — 13. 4, 8. 14. 64, 27 or -27 , -64 . 15. Length,
 24 in.; width, 7 in. 16. Length, 88 yd.; width, 55 yd. 17. 9, 5
 or -9 , -5 or $7\sqrt{2}$, $2\sqrt{2}$ or $-7\sqrt{2}$, $-2\sqrt{2}$. 18. $\frac{2}{3}$ or $-\frac{2}{6}$.

1. $3x^2 - x - 5$. 2. (a) $3m(r + 11m)(r - 11m)$;
 (b) $(3y + 2)(y - 3)$; (c) $(M - 4N + 2R)(M - 4N - 2R)$;
 (d) $(3x^{2a} + 5)(3x^{2a} + 5)$; (e) $(5a + b)(3a - 2b)$;
 (f) $(x^a - y^a)(x^a - y^a)$. 3. $-\frac{1}{x + 2}$. 4. $x = 7$.

- Page 349. — 5. $x = 2a$, $y = 3b$. 6. (a) $6\sqrt{2}$; (b) $\frac{4m + 4x + 2}{x}$;
 (c) $\sqrt{2}$. 7. (a) $y = 1$ or $\frac{1}{4}$; (b) $x = 2$ or 4 ; (c) $x = 5$.

Page 349 (continued). — 8. $n = \frac{CR}{e - Cr}$.

9. $l = \frac{2s - an}{n}$.

10. $s = 78.2$. 11. $w = \frac{p - 2l}{2}$. 12. $[m - h(a + b)]$ mi.

Page 350. — 13. $x = 7.4$ or -1.4 .

14. 24 coats.

15. (a) $\left\{ \begin{array}{l} x = \frac{2}{3}, -3; \\ y = \frac{3}{2}, 7; \end{array} \right.$ (b) $\left\{ \begin{array}{l} x = 3, -5; \\ y = 2, -6. \end{array} \right.$ 16. 8.4. 17. 36, 60.

18. $\frac{3m}{250}$ dollars.

19. $\frac{1}{4}n + 1 = n - 2$.

20. $\frac{5}{12}$.

21. $\pm(2x^2 - 3x + 2)$.
an hour.

22. First, 27 mi. an hour; second, 36 mi.

Page 352. — 1. -1 . 2. $-\sqrt{-1}$. 3. -1 . 4. $\sqrt{-1}$.
5. -1 . 6. $-\sqrt{-1}$. 7. i . 8. 1. 9. -2 . 10. -9 .
11. -12 . 12. -4 .

Page 353. — 14. $9i$. 15. $11i$. 16. $7i$. 17. $6i\sqrt{3}$.
18. $9i\sqrt{2}$. 19. $-5i$. 20. -120 . 21. $-30\sqrt{3}$.
22. -72 . 23. $-20\sqrt{6}$. 24. $\sqrt{6}$. 25. $3i$.

Page 355. — 3. Real, unequal, irrational, the positive root numerically greater. 4. Real, unequal, negative, rational. 5. Real, unequal, rational, the negative root numerically greater. 6. Real, equal, positive, rational. 7. Real, unequal, rational, the negative root numerically greater. 8. Imaginary. 9. Imaginary. 10. Real, unequal, irrational, the negative root numerically greater. 11. Real, unequal, negative, rational. 12. Real, equal, negative, rational. 14. $m = \pm 6$; $m > 6$ numerically or $m = \pm 6$; $m < 6$ numerically. 15. $a = 1$ or 5 ; $a > 5$ or $a < 1$; for all values of a between 1 and 5. 16. $m = 3$ or -5 . 17. $n = 3$ or $-\frac{1}{2}$. 18. $a = 1$; $x = \pm\sqrt{-1}$. 19. $d = \pm 3$; when $d = 3$, $x = 0$ or 1 ; when $d = -3$, $x = 0$ or -5 . 20. $m = \pm 2$. 21. $x = \frac{1}{18}(25k \pm 2\sqrt{75k^2 - 104})$, $y = \frac{5}{28}(6k \pm \sqrt{75k^2 - 104})$; when $k = \pm\frac{2}{15}\sqrt{78}$ or when $k > \frac{2}{15}\sqrt{78}$ numerically; when $k < \frac{2}{15}\sqrt{78}$ numerically; when $k = \pm\frac{2}{15}\sqrt{78}$.

Page 357. — 2. $x^2 - 10x + 24 = 0$. 3. $x^2 - 2x - 15 = 0$.
4. $3x^2 - 8x - 3 = 0$. 5. $9x^2 - 21x + 10 = 0$.
6. $2x^2 + 5x + 2 = 0$. 7. $4x^2 + 8x + 3 = 0$.
8. $x^2 + 2ax - 3a^2 = 0$. 9. $x^2 - 2ax + a^2 - 4 = 0$.
10. $x^2 - 2bx + b^2 - 1 = 0$. 11. $x^2 - 2ax + a^2 - b^2 = 0$.
12. $x^2 - 9ax + 19.61a^2 = 0$. 13. $4x^2 - 4ax + a^2 - b = 0$.
14. $x^2 - 6x + 11 = 0$. 15. $x^2 - 4x - 1 = 0$.
16. $x^2 - 4x + 7 = 0$. 17. $4x^2 + 12x + 3 = 0$.
18. $4x^2 + 4x - 1 = 0$. 19. $x^2 - 4ax + 24a^2 = 0$.
20. $\frac{5m-1}{2m^2}$; $m = 1$ or $\frac{1}{4}$. 22. $a^2 = 2b - 6$. 23. $b^2 - 9a^2 = 4ac$.
24. $a = c$. 25. $ax^2 - bx + c = 0$. 26. $c = 6$.

Page 358. — 28. 37. 29. $1\frac{1}{2}$. 30. $\pm\frac{7}{12}$. 31. ± 5 .
32. $-\frac{b}{c}$. 33. $\pm\frac{3}{4}$.

Page 363. — 1. $62\frac{1}{2}$ mi.; 230 mi.; 310 mi. 3. 25; the same.

Page 364. — 1. 40; 80; 104; 160; 168; 176. 2. $1\frac{1}{4}$; $1\frac{7}{8}$; $2\frac{1}{2}$; $3\frac{3}{4}$; $4\frac{1}{4}$; $5\frac{1}{2}$. 3. v increases as t increases. 4. 36; 64; 100; 256. 5. $\frac{5}{4}\sqrt{2}$; $\frac{5}{4}\sqrt{3}$; $2\frac{1}{2}$; $\frac{5}{4}\sqrt{6}$. 6. t varies as $\sqrt{s}$.

Page 365. — 8. Twice as great; base is tripled. 10. Values of r from 0 to 2; $r = 2$.

Page 366. — 11. 13, 50, 113; 13, 25, 38. 12. 16 times; 4 times; 25 times; 5 times. 14. (1) $p = 12$, $A = 9$; (2) $p = 16$, $A = 16$; (3) $p = 10$, $A = 6\frac{1}{4}$; (4) $s = 5$, $A = 25$; (5) $s = 6$, $p = 24$; (6) $s = 3\frac{1}{2}$, $A = 12\frac{1}{4}$.

Page 367. — 1. $1\frac{1}{4}$ cords; 2 cords; $3\frac{3}{4}$ cords; 44 ft.; 30 ft. 2. 2 sec. 3. 25.1 in.

Page 368. — 4. $19\frac{5}{8}$ sq. in.

Page 369. — 2. 20 min. 3. $40\frac{5}{8}$ mi. an hour. 4. $\frac{V_1}{V_2} = \frac{P_2}{P_1}$. 5. 200 cu. cm. 6. $\frac{t_1}{t_2} = \frac{m_2}{m_1}$. 7. $1\frac{7}{8}$ da. 8. $4\sqrt{2}$ ft. or 5.7 ft.

Page 370. — 1. (a) 4 hr.; (b) 10 hr.; (c) 4 mi. an hour; (d) $t_1 = 12$; (e) $r_1 = 20$.

Page 373. — 1. $x^2 + 3x + 2$. 2. $4x^2 - x + 3$, rem. -2 . 3. $5v^2 - 7v + 2$. 4. $y^3 - y^2 + 8$. 5. $4x^3 + x^2 + x + 1$. 6. $m^4 - 2m^3 + 4m^2 - 8m - 3$. 7. $x^4 - 2x^3 + 4x^2 - 8x + 16$. 8. $y^3 + 3y^2 + 9y + 27$. 9. $A^2 - 5A + 25$. 10. $x^3 + x^2 - 2x - 3$. 11. $2x^4 + 4x^3 + 5x^2 + 10x + 22$, rem. 4. 12. $x^2 - 5x - 24$, rem. 5.

Page 374. — 1. $9(x + 2y)(x - 2y)$. 2. $(2x + 3y)(2x - 3y)(2x + 3y)(2x - 3y)$. 3. $.75(10a - b)(10a - b)$. 4. $5(a - 9b)(a + 2b)$. 5. $7(a + 9bc)(a - 4bc)$. 6. $(5a - 2b)(2a + 3b)$. 7. $5(4x^2 + 9y^2)(2x + 3y)(2x - 3y)$. 8. $(7a + 5b)(2a - 15b)$. 9. $(2a - 3b + 3c)(2a - 3b - 3c)$. 10. $(a + b - 3c)(a - b + 3c)$. 11. $3(2x + 3)(2x - 1)$. 12. $10(2x^2 - 5)(x^2 + 1)$. 13. $y(y^2 + 1)(y + 1)$. 14. $(9x^{2n} - 2)(x^{2n} + 5)$. 15. $2(2x^2 - 5)(x^2 - 5)$. 16. $3x(17x - 9)(x + 2)$. 17. $a(5a + b)(2a - 3b)$. 18. $(3a^2 + b^2)(3a + 2b)(3a - 2b)$. 19. $(5a - 3b^2)(5a - 3b^2)$. 20. $(3a^m + 4b^{n+1})(3a^m + 4b^{n+1})$. 21. $(2a + b + 8)(2a + b - 3)$. 22. $(b - ac)(x - by)$. 23. $9(2a - b)(2b - c)$. 24. $3x(5x - 3y^2)(5x - 3y^2)$.

Page 376. — 3. $(x + y)(x^2 - xy + y^2)$. 4. $(x - y)(x^2 + xy + y^2)$. 5. $(m - 1)(m^2 + m + 1)$. 6. $(1 + 5m)(1 - 5m + 25m^2)$. 7. $(x - y^2)(x^2 + xy^2 + y^4)$. 8. $8(2r^3 + 3s^3)(4r^6 - 6r^3s^3 + 9s^6)$. 9. $x(1 - x)(1 + x + x^2)$. 10. $v(v^2 + 3)(v^4 - 3v^2 + 9)$. 11. $(ab - cd)(a^2b^2 + abcd + c^2d^2)$. 12. $(r^2 + 4s)(r^4 - 4r^2s + 16s^2)$. 13. $27(xy^2 - 2)(x^2y^4 + 2xy^2 + 4)$. 14. $(7n + 10)(49n^2 - 70n + 100)$. 15. $(r^x - 9s^z)(r^{2x} + 9r^xs^z + 81s^{2z})$. 16. $(1 + a + b)(1 - a - b + a^2 + 2ab + b^2)$. 17. $(x - y - 2)(x^2 - 2xy + y^2 + 2x - 2y + 4)$.

- Page 376 (continued).**—2. $(m+n)(m^4 - m^3n + m^2n^2 - mn^3 + n^4)$.
 3. $(m-n)(m^4 + m^3n + m^2n^2 + mn^3 + n^4)$.
 4. $(2x+3y)(16x^4 - 24x^3y + 36x^2y^2 - 54xy^3 + 81y^4)$.
 5. $x(x-1)(x^4 + x^3 + x^2 + x + 1)$. 6. $(x^2 - y)(x^4 + x^2y + y^2)$.
 7. $(x+y^2)(x^4 - x^3y^2 + x^2y^4 - xy^6 + y^8)$.
 8. $(a+2)(a^4 - 2a^3 + 4a^2 - 8a + 16)$.
 9. $2(2-a)(16 + 8a + 4a^2 + 2a^3 + a^4)$.
 10. $ab(a-b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$.
 11. $(x^m - y^n)(x^{4m} + x^{3m}y^n + x^{2m}y^{2n} + x^my^{3n} + y^{4n})$.
 12. $(1 - ab^2c^3)(1 + ab^2c^3 + a^2b^4c^6 + a^3b^6c^9 + a^4b^8c^{12})$.
 13. $(x^2 + 3a)(x^8 - 3ax^6 + 9a^2x^4 - 27a^3x^2 + 81a^4)$.

- Page 377.**—2. $(x+y)(x^2 - xy + y^2)(x-y)(x^2 + xy + y^2)$.
 3. $(2x+1)(4x^2 - 2x+1)(2x-1)(4x^2 + 2x+1)$.
 4. $(a^4 + b^4)(a^2 + b^2)(a+b)(a-b)$. 5. $(x^2 + 4)(x+2)(x-2)$
 6. $(8x^2 + 9)(8x^2 - 9)$. 7. $(a^2 + 25)(a+5)(a-5)$
 8. $(3+b)(9 - 3b + b^2)(3-b)(9 + 3b + b^2)$.
 9. $(2+y)(4 - 2y + y^2)(2-y)(4 + 2y + y^2)$.
 10. $(1+x^4)(1+x^2)(1+x)(1-x)$.

- Page 379.**—3. $(x-y)(x+2y)(2x-y)$. 4. $(x-1)(x+2)(x-5)$.
 5. $(x-6)(x+3)(x-3)$. 6. $(x+1)(x-3)(x-5)$.
 7. $(x-1)(x-5)(x-6)$. 8. $(x+1)(x-4)(2x+3)$.
 9. $(x-13)(x^2 + 13)$. 10. $(x-1)(x-7)(x-8)$.
 11. $(x-2)(x-4)(2x+3)$. 12. $(n+2)(n+3)(n+7)$.
 13. $(x+2y)(x+y)(x-y)$. 14. $(x+y)(x+y)(x+2y)$.
 15. $(x-1)(x-2)(x+3)$. 16. $(x-2)(x-3)(x+5)$.
 17. $(x+2)(x+5)(x-7)$. 18. $(x+2)(x+7)(x-9)$.
 19. $(x-y)(x-4y)(x+5y)$. 20. $(x-y)(x-3y)(x+4y)$.
 21. $(a+2)(a-3)(a+5)$. 22. $(a+2)(a+3)(a+4)$.
 23. $(n+2)(n-3)(2n-5)$. 24. $(x-y)(x-2y)(x^2 + 3xy - 2y^2)$.
 25. $(x-1)(x-2)(x-2)(x^2 + x - 3)$.
 26. $(x-1)(x-3)(x^2 + 5x + 17)$.
 27. $(x-1)(x-3)(x^2 + 4x + 1)$.

- Page 380.**—2. $(a^2 + 3ab - 6b^2)(a^2 - 3ab - 6b^2)$.
 3. $(a^2 + 2a + 2)(a^2 - 2a + 2)$. 4. $(x^2 + xy + y^2)(x^2 - xy + y^2)$.
 5. $(a^2 + ab + b^2)(a^2 - ab + b^2)(a^4 - a^2b^2 + b^4)$.
 6. $(3x^2 + 2xy + 4y^2)(3x^2 - 2xy + 4y^2)$.
 7. $(2a^2 + ab + 3b^2)(2a^2 - ab + 3b^2)$.
 8. $(4a-x)(a+x)(4a+x)(a-x)$.
 9. $(5x-2y)(x+y)(5x+2y)(x-y)$.
 10. $(7a^2 + 13ab + 11b^2)(7a^2 - 13ab + 11b^2)$.
 11. $3(a^2 + 2ab - 4b^2)(a^2 - 2ab - 4b^2)$.
 12. $(4x^2 + 2xy + y^2)(4x^2 - 2xy + y^2)$.
 13. $(2a+3)(a-1)(2a-3)(a+1)$.
 14. $(5a^2 + 2ab^2 - b^4)(5a^2 - 2ab^2 - b^4)$.
 15. $(3a^2 + 2ab + 5b^2)(3a^2 - 2ab + 5b^2)$.
 16. $(b^2 + 4b + 8)(b^2 - 4b + 8)$.
 17. $(a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2)$.
 18. $(m^4 + 2m^2 + 2)(m^4 - 2m^2 + 2)$.
 19. $(a^2 + 6a + 18)(a^2 - 6a + 18)$.
 20. $(a^2 + 2a + 2)(a^2 - 2a + 2)(a^2 + 2)(a^2 - 2)$.

21. $m(m^2 + 2mn + 2n^2)(m^2 - 2mn + 2n^2)$.
 22. $(x^2 + 4xy + 8y^2)(x^2 - 4xy + 8y^2)$.
 23. $(2a^2 + 6a + 9)(2a^2 - 6a + 9)$.
 24. $xy^2(x^2 + 2x + 2)(x^2 - 2x + 2)$.
 25. $(3m^4 + 7m^2n + 5n^2)(3m^4 - 7m^2n + 5n^2)$.
 26. $a(b^2 + 2b + 4)(b^2 - 2b + 4)(b^4 - 4b^2 + 16)$.
 27. $(5x^2 + 9xy - 9y^2)(5x^2 - 9xy - 9y^2)$.
 28. $(2c - d)(c + 3d)(2c + d)(c - 3d)$.
 29. $(5x^2 + 4xy - 4y^2)(5x^2 - 4xy - 4y^2)$.

Page 381. — 2. $(a + b + 5c)(a + b + 3c)$.

3. $(x - 3y + 5z)(x - 3y + z)$. 4. $(m - n + 10p)(m - n - 3p)$.
 5. $(m + 4n - 11)(m + 4n - 5)$. 6. $(3m^2 + k + 15)(3m^2 + k - 2)$.
 7. $(5a + y - 5x)(5a + y - 2x)$.

Page 382. — 1. $x(x - 3)$.

2. $x(x + 1)$. 3. $4a(a + 1)$.
 4. $2x^2(4x - 1)$. 5. $(x + 2y^2)(x^2 - 2xy^2 + 4y^4)$.
 6. $(y^n - x^b)(y^{2n} + x^by^n + x^{2b})$. 7. $(1 + x^4)(1 + x^2)(1 + x)(1 - x)$.
 8. $(x + 1)(x^2 - x + 1)(x - 1)(x^2 + x + 1)$.
 9. $a(1 + a)(1 - a + a^2)(1 - a)(1 + a + a^2)$.
 10. $(x + 1)(x^4 - x^3 + x^2 - x + 1)(x - 1)(x^4 + x^3 + x^2 + x + 1)$.
 11. $b(b^2 + 1)(b^4 - b^2 + 1)$. 12. $(p^2 + 2p + 2)(p^2 - 2p + 2)$.
 13. $(1 + x^4)(1 - x^4 + x^8)$. 14. $(a^3 - b)(a^6 + a^3b + b^2)$.
 15. $a^2(a + 3)(a^2 - 3a + 9)$. 16. $(5 - 2x^2)(25 + 10x^2 + 4x^4)$.
 17. $.004(10m + 1)(100m^2 - 10m + 1)$.
 18. $x^2(5x + 8y)(5x - 8y)$. 19. $8(a - 2)(a^2 + 2a + 4)$.
 20. $9d(d + 3)(d - 3)$. 21. $a^2(a - y)$.

Page 383. — 22. $4x(x - 1)(x^2 + x + 1)$. 23. $5(5 + 3x^2)(5 - 3x^2)$.

24. $(3a^2 + b - c)(3a^2 - b + c)$. 25. $5(x + 3)(x - 2)$.
 26. $(b^a + 14)(b^a - 14)$. 27. $(3a^2 + 13)(3a^2 - 13)$.
 28. $2(2 - y)(16 + 8y + 4y^2 + 2y^3 + y^4)$.
 29. $6(b^2 + 2b + 2)(b^2 - 2b + 2)$.
 30. $a(a^2 + b^2)(a^4 - a^2b^2 + b^4)(a + b)(a^2 - ab + b^2)(a^2 + ab + b^2)$.
 31. $7n(n^2 + 1)(n^4 - n^2 + 1)$. 32. $7(y^2 + 5)(y^2 - 5)$.
 33. $(2 - 3ax)(4 + 6ax + 9a^2x^2)$. 34. $(3a - p)(p - 2a)$, or
 $(2a - p)(p - 3a)$. 35. $(3x^n - 2)(x^n + 7)$. 36. $2(3x - 8y)(x + 3y)$.
 37. $(ax - 4b)(ax + b)$. 38. $2(2xy - 5)(xy + 4)$.
 39. $(4q + 3p)(4q - p)$. 40. $(3x + 5d)(3x + 2d)$.
 41. $(6b^2 + 5a)(3b^2 - 2a)$. 42. $5(2a + 3b^2)(a - 4b^2)$.
 43. $(5ab + 4cd)(ab + cd)$. 44. $3(x + 3)(x + 2)$.
 45. $(5x + 3 + 2y)(5x + 3 - 2y)$. 46. $(x - 1)(x - 1)(x^2 + 2x + 3)$.
 47. $(2x + 3)(7x - 1)$. 48. $(x - 1)(2x + 1)(2x + 1)$.
 49. $3(x^{n+2} + 3)(x^{n+2} - 3)$. 50. $(4a + 3b - 2c)(4a - 3b + 2c)$.
 51. $x(6 + x)(7 - x)$. 52. $(1 + 22x)(1 - 3x)$.
 53. $(x - 3)(x + 1)(x - 1)$. 54. $(x + 3)(x - 3)(x + 2)(x^2 - 2x + 4)$.
 55. $(a - x)(b + c)$. 56. $(3x^2 + xy + 2y^2)(3x^2 - xy + 2y^2)$.
 57. $(a^2 + a + 1)(a^2 - a + 1)$. 58. $(a^2 + 2a + 9)(a^2 - 2a + 9)$.
 59. $2a^2(4 - 5a)(16 + 20a + 25a^2)$.
 60. $(a + 2)(a - 1)(a^4 + 2a^3 + 3a^2 + 2a + 4)$.
 61. $(n^2 + nab^2 + a^2b^4)(n^2 - nab^2 + a^2b^4)$.
 62. $5(x^5 + 3)(x - 1)(x^4 + x^3 + x^2 + x + 1)$.
 63. $(4x^2 + 2xy + y^2)(4x^2 - 2xy + y^2)$.

- Page 383 (*continued*). — 64. $x(x^4 + 2x^2 + 2)(x^4 - 2x^2 + 2)$.
 65. $a^2(5a + 1)(2a - 7)$. 66. $3x(x + 2)(x^4 - 2x^3 + 4x^2 - 8x + 16)$.
 67. $(a + b + 1)(a^2 + 2ab + b^2 - a - b + 1)(a + b - 1)(a^2 + 2ab + b^2 + a + b + 1)$. 68. $a(a + 2x)(a^2 + 2ax + 2x^2)$.
 69. $(x + 2)(x^2 + x + 1)$. 70. $(a - 1)(a^2 - a - 1)$.
 71. $(b - 2)(b^2 - 2b - 4)$. 72. $-b(3a^2 + 3ab + b^2)$.
 73. $x(x - 1)(x^5 - x^4 - x^3 - x^2 - x - 1)$. 74. $(a^2b + 4)(a^2b - 3)$.
 75. $(10x - 3y)(100x^2 + 30xy + 9y^2)$. 76. $x^2(x - 1)(x^2 + 1)$.
 77. $(17x - 9)(x + 2)$. 78. $(x - y)(x^2 - y)$. 79. $2x(x^2 + 3y^2)$.
 80. $(2a - 3)(a^2 - 3a + 3)$. 81. $(4x + 1)(x^2 - 2)$. 82. $(x + 5)(x + a)$.
 83. $(x^2 + 11xy + y^2)(x^2 - 11xy + y^2)$.
 84. $(a^2 + 2ab + 2b^2 - 2bc + c^2)(a + 2b - c)(a + c)$.
 85. $m(m - n)(m + n + 1)$.

- Page 384. — 86. $(a + b + c)(a + b - c)(a - b + c)(a - b - c)$.
 87. $y(x - y)(x - y)(2x + y)$. 88. $(a + b)(a + b)(a + b)$.
 89. $(x^{n-1} + by)(x^{n-1} + by)$. 90. $(x - 3)(x - 5)(x - 5)$.
 91. $(x - 1)(x - 2)(x + 3)$. 92. $(x + 2)(2x - 1)(2x - 3)$.
 93. $(x + 1)(2x - 1)(3x + 2)$. 94. $x(3x + 5)(2x - 1)$.
 95. $x^2(5x + 2)(3x + 5)$. 96. $(2x - 1)(2x - 1)(2x - 1)$.
 97. $(x + 1)(x - 2)(2x - 1)$. 98. $(a - x^n)(b - y^m)$.
 99. $(a + b)(a - b)(a^2 + b^2 - 1)$. 100. $(x + 5)(x + 5)(x + 5)$.
 101. $(x - 3a)(x + 4b)$. 102. $(x^2 + 2xy - y^2)(x^2 - 2xy - y^2)$.
 103. $(7a + 6b - 2)(7a - 6b - 2)$.
 104. $(r - 1)(r^2 + r + 1)(r + 2)(r^4 - 2r^3 + 4r^2 - 8r + 16)$.
 105. $(7x + y)(x - 3y)$. 106. $3(y - \frac{1}{2})(y - \frac{1}{2})$.
 107. $(3a + b - c)(3a - b + c)$. 108. $(x - 1)(x + 1)(x - 2)(2x - 1)$.
 109. $x(x - 3)(2x + 1)(5x - 1)$. 110. $a(x - 2)(x - 3)(x - 4)$.
 111. $(3a - 2b)(4x - 3y)$. 112. $(5x + 3y + 4z)(5x - 3y - 4z)$.
 113. $ab(x + 3)(x + 1)(x - 1)$. 114. $(3x + 2)(bm - an)$.
 115. $a(5x - 7)(4x^2 + a)$. 116. $(x - y + a - z)(x - y - a + z)$.
 117. $b(b + x)(2m - 3a)$. 118. $(2 - x)(16 + 8x + 4x^2 + 2x^3 + x^4)$.
 119. $(1 + x)(16 - 11x)$.
 120. $(x - a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \dots + a^{n-1})$.
 121. $(x + r)(x^{n-1} - x^{n-2}r + x^{n-3}r^2 - \dots + r^{n-1})$.
 122. $(x - 2b)(x - 2b)(x - 2b)$. 123. 2, 4, 8, 16.

- Page 386. — 2. $a^3 - 3a^2c + 3ac^2 - c^3$. 3. $a^3 + 3a^2b + 3ab^2 + b^3$.
 4. $c^5 + 5c^4d + 10c^3d^2 + 10c^2d^3 + 5cd^4 + d^5$. 5. $x^6 - 6x^5y$
 $+ 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6$. 6. $x^8 + 8x^7y$
 $+ 28x^6y^2 + 56x^5y^3 + 70x^4y^4 + 56x^3y^5 + 28x^2y^6 + 8xy^7 + y^8$.
 7. $m^7 - 7m^6n + 21m^5n^2 - 35m^4n^3 + 35m^3n^4 - 21m^2n^5 + 7mn^6 - n^7$.
 8. $r^8 - 8r^7s + 28r^6s^2 - 56r^5s^3 + 70r^4s^4 - 56r^3s^5 + 28r^2s^6 - 8rs^7 + s^8$.
 9. $c^7 + 7c^6d + 21c^5d^2 + 35c^4d^3 + 35c^3d^4 + 21c^2d^5 + 7cd^6 + d^7$.
 11. $x^3 + 6x^2 + 12x + 8$. 12. $a^4 - 12a^3 + 54a^2 - 108a + 81$.
 13. $m^4 - 4m^3pn + 6m^2p^2n^2 - 4mp^3n^3 + p^4n^4$.
 14. $x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4$. 15. $8x^3 - 12x^2y$
 $+ 6xy^2 - y^3$. 16. $8x^3 - 60x^2 + 150x - 125$. 17. $x^8 - 40x^6$
 $+ 600x^4 - 4000x^2 + 10,000$. 18. $81a^4 + 108a^3b^2 + 54a^2b^4$
 $+ 12ab^6 + b^8$. 19. $1 - 12x^2 + 54x^4 - 108x^6 + 81x^8$.
 20. $x^6 + 15x^4y + 75x^2y^2 + 125y^3$. 21. $1 + 4a^2b^2 + 6a^4b^4$
 $+ 4a^6b^6 + a^8b^8$. 22. $125x^6 - 75abx^4 + 15a^2b^2x^2 - a^3b^3$.

23. $32 a^5 x^5 - 80 a^4 b x^4 + 80 a^3 b^2 x^3 - 40 a^2 b^3 x^2 + 10 a b^4 x - b^5$.
 24. $1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7$.
 25. $1 - 12x + 60x^2 - 160x^3 + 240x^4 - 192x^5 + 64x^6$.
 26. $a^{10}x^5 + 20 a^8x^4 + 160 a^6x^3 + 640 a^4x^2 + 1280 a^2x + 1024$.
 27. $x^6 - 3x^5 + \frac{15}{4}x^4 - \frac{5}{2}x^3 + \frac{15}{16}x^2 - \frac{3}{16}x + \frac{1}{64}$.
 $+ \frac{1}{6}x^2y^2 - \frac{2}{27}xy^3 + \frac{1}{81}y^4$.
 28. $\frac{1}{16}x^4 - \frac{1}{8}x^3y$
 $+ \frac{1}{6}x^2y^2 - \frac{2}{27}xy^3 + \frac{1}{81}y^4$.
 29. $\frac{1}{27} + \frac{1}{3}x + x^2 + x^3$.
 30. $32a^5 + 40a^4$
 $+ 20a^3 + 5a^2 + \frac{5}{8}a + \frac{1}{32}$.
 31. $\frac{x^4}{y^4} - \frac{4x^2}{y^2} + 6 - \frac{4y^2}{x^2} + \frac{y^4}{x^4}$.
 32. $\frac{x^6}{y^6} - \frac{6x^4}{y^4} + \frac{15x^2}{y^2} - 20 + \frac{15y^2}{x^2} - \frac{6y^4}{x^4} + \frac{y^6}{x^6}$.
 $+ \frac{a^2b^2}{4} + \frac{b^3}{216}$.
 33. $27a^6 + \frac{9a^4b}{2}$.
 34. $\frac{81}{256} - \frac{27x}{8} + \frac{27x^2}{2} - 24x^3 + 16x^4$.
 35. $1 + \frac{15x}{2} + \frac{45x^2}{2} + \frac{135x^3}{4} + \frac{405x^4}{16} + \frac{243x^5}{32}$.
 $+ 6x^2 + \frac{100x^3}{9} + \frac{625x^4}{81}$.
 36. $\frac{81}{625} + \frac{36x}{25}$.
 37. $\frac{1}{a^5} - \frac{5}{a^3} + \frac{10}{a} - 10a + 5a^3 - a^5$.
 38. $x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$.
 $+ \frac{5}{x} - 20x + 40x^3 - 32x^5$.
 39. $\frac{1}{32x^5} - \frac{5}{8x^3}$.
 40. $8a^6 - 6a^4b^3 + \frac{3a^2b^6}{2} - \frac{b^9}{8}$.

- Page 387. — 41. $a^3 + 3a^2b + 3ab^2 + b^3 - 3a^2c - 6abc - 3b^2c$
 $+ 3ac^2 + 3bc^2 - c^3$.
 42. $r^3 - 3r^2s + 3rs^2 - s^3 - 3r^2t + 6rst$
 $- 3s^2t + 3r^2v - 6rsv + 3s^2v + 3rt^2 - 6rtv + 3rv^2 - 3st^2 + 6stv$
 $- 3sv^2 - t^3 + 3t^2v - 3tv^2 + v^3$.
 43. $a^3 - 3a^2b + 3ab^2 - b^3$
 $+ 3a^2c - 6abc + 3b^2c + 3ac^2 - 3bc^2 + c^3$.
 44. $x^3 + 3x^2y + 3xy^2$
 $+ y^3 + 6x^2 + 12xy + 6y^2 + 12x + 12y + 8$.
 45. $b^3 - 3b^2c + 3bc^2$
 $- c^3 - 3b^2d + 6bcd - 3c^2d + 3bd^2 - 3cd^2 - d^3$.
 46. $x^3 + 9x^2y$
 $+ 27xy^2 + 27y^3 - 6x^2z - 36xyz - 54y^2z + 12xz^2 + 36yz^2 - 8z^3$.
 47. $a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3a^2d + 6abd$
 $+ 3b^2d + 3ac^2 + 6acd + 3ad^2 + 3bc^2 + 6bcd + 3bd^2 + c^3 + 3c^2d$
 $+ 3cd^2 + d^3$.
 48. $a^3 + 3a^2b + 3ab^2 + b^3 - 3a^2x - 6abx$
 $- 3b^2x - 3a^2y - 6aby - 3b^2y + 3ax^2 + 6axy + 3ay^2 + 3bx^2$
 $+ 6bxy + 3by^2 - x^3 - 3x^2y - 3xy^2 - y^3$.
 2. 1.0816. 3. 1.071225. 4. 1.092025. 5. 1.157625.
 6. 1.126162419264. 7. 1.1716593810022656. 8. 1.21550625.
 9. 1.31079601. 10. 1.26247696. 11. \$255.26. 12. \$552.04.

Page 388. — 13. \$52.31. 14. \$107.24.

- Page 389. — 2. $960 a^7$. 3. $- 330 x^4 y^7$. 4. $7920 x^8 y^4$.
 5. $42,504 x^{19}$. 6. $- 149,422,080 x^{17}$. 7. $455 a^{-18}$. 8. $540 a^3 b^3$.
 9. 252. 10. 70. 11. $\frac{126 a}{b}; - \frac{126 b}{a}$. 12. $- 448 m^6 x^{\frac{5}{2}}$. 13. $15 x$.
 14. $\frac{n(n-1)(n-2) \dots (n-r+1)}{r!} (2a)^{n-r} b^r$.
 15. $\frac{n^4 + 2n^3 - n^2 - 2n}{15,000} (3a)^{n-2} b^4$.

Page 395. — 1. 3.0228. 2. 3.1045. 3. $8.2175 - 10$.
 4. 3.2801. 5. 1.3241. 6. 0.4857. 7. 2.6445. 8. $9.8951 - 10$.
 9. $8.9586 - 10$. 10. $9.0052 - 10$. 11. 1.7378. 12. $8.9948 - 10$.
 13. $9.9932 - 10$. 14. 1.3288. 15. 0.9259. 16. 1.8010. 17. 2.8583.
 18. $9.6411 - 10$. 19. $8.3353 - 10$. 20. 0.6397.

Page 398. — 3. 2. 4. 40. 5. 800. 6. 46.44. 7. 626.7.
 8. 9006. 9. 7.575. 10. 865.2. 11. 6.752. 12. 80.15.
 13. .2336. 14. .09845. 15. .08984. 16. .00521. 17. .000503.
 18. .07021. 19. .163. 20. 439.6. 21. .3853. 22. .004618.
 23. 4.891. 24. 58.06. 25. .02477. 26. 365.9. 27. $n = 8$.
 28. $B = 128$. 29. $\log_{10} A = x$; $\log_{10} B = y$; $AB = 10^{x+y}$;
 $\log_{10} AB = x + y$; $\frac{A}{B} = 10^{x-y}$; $\log_{10} \frac{A}{B} = x - y$.

Page 399. — 2. 212.8. 3. 2808. 4. 52.7. 5. 57.4. 6. 192.1.
 7. 1740. 8. 245,400. 9. .7182. 10. 227. 11. 10. 12. 2120.
 13. 3.999.

Page 400. — 3. 54.99. 4. 127.8. 5. 57.01. 6. .3825.
 7. 15,800. 8. 3.185. 9. .88. 10. .1251. 11. 767.8. 12. 384.
 13. .025. 14. .01333. 15. 381.9. 16. 17.32. 17. .6912.

Page 402. — 3. .5054. 4. 1.374. 5. -6.231 . 6. .05425.
 7. 197.0. 8. $-.132$. 9. 1.49. 10. 38.92. 11. .05826.
 12. 5. 13. 153.0. 14. 148.2.

Page 403. — 2. 49. 3. 121. 4. 2209. 5. 117.6. 6. 731.2.
 7. .2887. 8. 64.8. 9. 1.216. 10. 1.188. 11. .5098. 12. 67.42.
 13. .1601. 14. 2401. 15. 1.104. 16. 5.248. 17. .0225.
 18. .7834. 19. .005425. 20. .004394. 21. .0000000002868.

Page 404. — 2. 15. 3. 3.5. 4. .4498. 5. 18.05. 6. .8.
 7. .3438. 8. -11 . 9. 128. 10. .82. 11. 1.81. 12. .598.
 13. 1.1. 14. 1.414. 15. 1.316. 16. 1.71. 17. 1.431.
 18. 1.26. 19. 1.414. 20. -1.149 . 21. 3. 22. 5.55.
 23. .9654. 24. .8492. 25. .5023. 26. 3.735. 27. .2959.
 28. -1.542 . 29. 19.6. 30. 479.2 sq. m. 31. 270,200 cu. cm.

Page 405. — 32. 1378 ft. 34. $x = 57.4531$. 35. $x = 2.1705$.
 36. $x = 1.1984$. 37. $x = 1.6116$. 38. $x = 2.2620$. 39. $x = 8$.
 41. \$330.50. 42. \$936.80. 43. \$537.10. 44. \$2625.
 45. 12.94 yr., or 13 yr.; 12.79 yr., or 13 yr. 46. 10.24 yr., or $10\frac{1}{4}$ yr.
 47. 5%.

Page 407. — 2. 83. 3. 77. 4. 346. 5. 120. 6. 45. 7. 17.

Page 408. — 8. 17. 9. -31 . 10. 31 a. 11. $x + 3y$.
 12. $305\frac{7}{12}$ ft.
 2. 496. 3. 70.

Page 409. — 4. $43\frac{1}{2}$. 5. 64 a. 6. $3n^2 - 2n$. 7. $ax + a^3 - a^2$.
 8. 175. 9. -80 . 10. $-12\frac{1}{2}$. 11. 78 strokes. 12. \$29.
 13. $1608\frac{1}{3}$ ft. 14. 3720 ft. 15. 180 ft. 16. 8712 ft. 17. 432 ft.

Page 410. — 18. \$22.50. 19. \$487.50. 20. \$2.50;
 63.75 n dollars. 21. \$18.90. 22. \$72.

Page 411. — 3. 17. 4. 403. 5. 13. 6. $n = 13$; $s = 572$.
7. $l = 3\frac{1}{2}$; $s = -32\frac{1}{2}$. 8. $a = 2$; $l = 82$.

Page 412. — 9. $n = 11$. 10. -112 . 11. $a = -4$; $l = 34$.
12. 20, 18, 16, ..., -6 . 13. 5, 8, 11, ..., 47. 14. $l = \frac{2s - an}{n}$.
15. $s = \frac{n}{2}[2a + (n - 1)d]$. 16. $d = \frac{2s - 2an}{n^2 - n}$.
17. $a = \frac{2s - dn^2 + dn}{2n}$.

2. 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5, $5\frac{1}{2}$, 6. 3. 24, 22, 20, 18, 16, 14,
12, 10, 8, 6, 4, 2. 4. 10, 7, 4, 1, -2 , -5 , -8 , -11 , -14 .
5. $a - b$, $\frac{2a - b}{2}$, a , $\frac{2a + b}{2}$, $a + b$.

Page 413. — 6. $\frac{7}{12}$. 7. a . 8. $a^2 + b^2$. 9. $\frac{m^2 + n^2}{2mn}$. 10. $a + c$.

Page 414. — 2. 2, 6, 10. 3. 5, 7, 9. 4. 2, 5, 8 or -2 , -5 , -8 .
5. 2500. 6. 5 and 14. 7. 10 logs. 8. 21 yd. 9. 6.

Page 415. — 2. 512. 3. 32. 4. 1536. 5. 512. 6. 1458.
7. 12,500. 8. -1458 . 9. $\frac{1}{512}$. 10. $\frac{512}{19683}$. 11. $-\frac{64}{2187}$.
12. $a^{-1}b^4$. 13. $2^{\frac{3-n}{2}}$.

Page 416. — 14. \$128,000. 15. 1220.8 millions. 16. \$4882 $\frac{13}{16}$.
17. 50,625. 18. \$16. 19. \$233.80; \$234.40. 20. \$15.36.
21. \$11,802. 22. 45,000 bu.

Page 417. — 2. 255. 3. $1\frac{127}{128}$. 4. $113\frac{169}{512}$. 5. $1\frac{365}{728}$.

Page 418. — 6. $-\frac{1365}{4096}$. 7. $64x^3 + 32x^5 + 16x^4 + 8x^3 + 4x^2$
 $+ 2x + 1$. 8. $64x^6 - 32x^5 + 16x^4 - 8x^3 + 4x^2 - 2x + 1$.
9. $\frac{x^{2n} - 1}{x^2 - 1}$. 10. $2^n - 1$. 11. $\frac{1}{2}\left(\frac{3^n - 1}{3^{n-1}}\right)$. 12. 1093. 13. 381.
14. -147 . 15. 5. 16. \$10,719,999.99. 17. \$830.
18. \$54,000.

Page 419. — 19. \$6454.50. 20. \$1811.92. 21. \$1432.11.
22. \$3125.

Page 420. — 2. 2. 3. 4. 4. $\frac{3}{4}$. 5. $-5\frac{1}{3}$. 6. $-1\frac{2}{3}$.
7. $90\frac{10}{11}$. 8. 10. 9. $\frac{8}{11}$. 11. $\frac{11}{27}$. 12. $\frac{11}{11}$. 13. $1\frac{1}{8}$.
14. $\frac{67}{3300}$. 15. $\frac{7}{999}$. 16. $5\frac{13}{396}$. 17. $2\frac{16}{45}$. 18. $3\frac{19}{37}$.

Page 421. — 2. 1, 5, 25, 125, 625 or 1, -5 , 25, -125 , 625.
3. $4\frac{1}{2}$, 6, 8, $\frac{32}{3}$, $\frac{128}{9}$, $\frac{512}{27}$, $\frac{2048}{81}$ or $4\frac{1}{2}$, -6 , 8, $-\frac{32}{3}$, $\frac{128}{9}$, $-\frac{512}{27}$, $\frac{2048}{81}$.
4. $\frac{343}{16}$, $\frac{49}{4}$, 7, 4, $\frac{16}{7}$, $\frac{64}{49}$. 5. 5120, 1280, 320, 80, 20, 5. 6. $4\sqrt{2}$,
4, $2\sqrt{2}$, 2, $\sqrt{2}$, 1. 7. a^6 , a^5b , a^4b^2 , a^3b^3 , a^2b^4 , ab^5 , b^6 or a^6 , $-a^5b$, a^4b^2 ,
 $-a^3b^3$, a^2b^4 , $-ab^5$, b^6 . 8. x , $-x^{\frac{4}{3}}y^{\frac{1}{3}}$, $x^{\frac{3}{2}}y^{\frac{2}{3}}$, $-x^{\frac{2}{3}}y^{\frac{3}{2}}$, $x^{\frac{1}{2}}y^{\frac{4}{3}}$, $-y$.
9. ± 20 . 10. $\pm 1\frac{1}{3}$. 11. $\pm 1\frac{1}{8}$. 12. $\pm(a^2 - b^2)$. 13. $\pm \frac{a+b}{a-b}$.

- Page 422. — 1. $a = rl - s(r - 1)$. 2. 25. 3. $\frac{3}{16}$.
 4. $l = \frac{a + s(r - 1)}{r}$. 5. 80. 6. $\sqrt{2}$; 8. 7. $r = \frac{s - a}{s - l}$.
 8. $\frac{3}{2}$; 32, 48, 72, 108, 162, 243. 9. $\frac{4}{3}$; 81, 108, 144, 192, 256.
 10. $r = \sqrt[n-1]{\frac{l}{a}}$. 11. 3; 3, 9, 27, 81, 243, 729. 12. $l = \frac{r^{n-1}s(r - 1)}{r^n - 1}$.
 13. 500 ft.
- Page 423. — 14. 8190 letters. 15. $333\frac{1}{8}$ ft. 16. 12 da.
 18. 4, 6, 9. 19. 1, 2, 4 or - 1, 2, - 4. 20. 5, 10, 20, 40 or - 15, 30, - 60, 120. 21. $\frac{9}{25}$ water. 22. 7.8 lb.
- Page 425. — 2. Right angle; 90° . 3. 180° . 4. 75° .
- Page 427. — 2. 90° . 3. 60° . 4. 80° . 5. (1) 90° ; (2) 90° ; (3) 60° ; (4) 60° ; (5) 80° ; (6) 70° ; (7) 90° ; (8) 45° ; (9) 30° .
 6. 13. 7. 24. 8. 8.
- Page 430. — 5. .268. 6. 3.732. 7. 2.145.
- Page 431. — 3. 260 ft. 4. 35 ft. 5. 15 ft. 6. 57 ft.
- Page 432. — 7. 54.6 yd.
- Page 433. — 6. (a) .574; (b) .966.
- Page 434. — 3. 19 ft.; 23 ft. 4. 26 ft. 5. (a) 41 ft.; (b) 39 ft.; (c) 71.9 ft.; (d) 56.0 ft.; (e) 29.8 ft.
- Page 435. — 6. (a) .819; (b) .423.
- Page 436. — 2. 34 ft. 3. 84.2 ft. 4. 47 ft. 5. 313 ft.
- Page 438. — 1. $120\sqrt{3}$ ft. 2. 100 ft. 3. $200\sqrt{3}$ ft.
- Page 439. — 4. $25\sqrt{3}$ ft.
- Page 441. — 3. .5948. 4. .8878. 5. .3289. 6. .9025.
 7. .8845. 8. .7466. 9. 3.0970. 10. .6105. 11. .3307.
 12. .1581.
- Page 442. — 13. .3502. 14. .5195. 15. .9121. 16. .6687.
 17. 10.8554. 18. 18.1250. 21. 15.5° , or $15^\circ 30'$. 22. 50.3° , or $50^\circ 18'$. 23. 29.5° , or $29^\circ 30'$. 24. 35.9° , or $35^\circ 54'$. 25. 24.5° , or $24^\circ 30'$.
- Page 443. — 26. 60.6° , or $60^\circ 36'$. 27. 32 rd.; angle whose adjacent side is 25 rd. is 38.7° , or $38^\circ 42'$; angle whose adjacent side is 20 rd. is 51.3° , or $51^\circ 18'$. 28. Acute angle, 66.8° ; legs, 246 ft. and 574 ft.
 29. 6.99. 30. 57.2.
- Page 448. — 3. $a = 145.1$ ft.; $c = 218.9$ ft.; $B = 48^\circ 30'$.
 4. $b = 17.78$ ft.; $c = 30.39$ ft.; $A = 54^\circ 12'$. 5. $a = 157$ ft.; $b = 474.8$ ft.; $B = 71^\circ 42'$. 6. $a = 15.58$ ft.; $b = 7.528$ ft.; $A = 64^\circ 15'$. 7. $b = 26.96$ ft.; $c = 36.45$ ft.; $B = 47^\circ 44'$.
 8. 68.57 ft. 9. 375.3 ft.
- Page 449. — 10. $36^\circ 9'$. 11. 27.58 yd. 12. $b = 20.6$; $c = 32.4$; $B = 39^\circ 30'$. 13. $a = 67.78$; $c = 83.06$; $A = 54^\circ 40'$.

14. $b=117$; $c=139.5$; $A=33^\circ$. 15. $a=40.18$; $c=94.02$; $B=64^\circ 45'$.
 16. $a=42.04$; $c=108.5$; $A=22^\circ 50'$. 17. $a=223.7$; $b=111.5$;
 $B=26^\circ 30'$. 18. $a=80.52$; $b=330.3$; $A=13^\circ 40'$. 19. $b=38.38$;
 $c=129.8$; $B=17^\circ 15'$. 20. $a=30.96$; $b=10.18$; $B=18^\circ 10'$.
 21. $a=5.8$; $c=15.62$; $A=21^\circ 48'$. 22. 153.6 ft. 23. 566.3 yd.

- Page 450. — 24. 178.8 ft. 25. 792.7 ft. 26. 1510 ft.
 27. 350.7 ft. 28. 66.97 ft. 29. 59.06 ft.

- Page 451. — 30. 133.8 ft. 31. 221.3 yd. 32. 163.7 yd.
 33. 234.4 ft. 34. $61^\circ 36'$. 35. $32^\circ 51'$.

- Page 452. — 36. 64.76 ft. 37. 589.8 yd. 38. 73.48 ft.
 39. 187.5 ft. 40. 534.5 ft. 41. 726.7 ft.

- Page 453. — 4. (b) $x = -1$ or -5 ; minimum point $(-3, -4)$.
 5. Even powers; odd powers. 8. $5x^2 - 23x - 10 = 0$;
 $x^2 - 4x - 23 = 0$; $x^2 - 2x\sqrt{-2} - 3 = 0$; $x^2 - 6x - 11 = 0$.
 9. $\sqrt{7^2 - 4 \times 5 \times (-8)}$, or $\sqrt{209}$.

- Page 454. — 11. Sum, -4 ; product, $-\frac{7}{2}$. 12. $(x-1)(x-3)(x-4)$;
 $(x+a)(x-2a)(x^2+a^2)$. 14. $\frac{n(n-1)(n-2) \cdots (n-r+2)}{(r-1)!} a^{n-r+1} b^{r-1}$.

15. 0; 32. 17. $\log a + \log x$; $\log a - \log x$; $a \log x$; $\frac{1}{a} \log x$.

- Page 455. — 25. 2, 4, 6, 8, 10; 3, 5, 7, 9, 11; positive. 26. $\frac{81}{256}$; $\frac{3}{4}$.
 29. ± 5 ; 5; -2 . 34. $a^3 b^{-1} x^2$.

- Page 456. — 40. $\sqrt{\frac{2}{3}}$; $\sqrt{81bc}$; $\sqrt[3]{\frac{8}{343}x^2y}$. 41. $2b\sqrt{10c}$;
 $6n\sqrt{2m}$; $2\sqrt[3]{3}$. 45. (a) $x - \sqrt{y}$; (b) $\sqrt{x} + \sqrt{y}$;
 (c) $(\sqrt{x} + \sqrt{y} + \sqrt{z})(x - y + z - 2\sqrt{xz})$;
 (d) $(\sqrt{x} - \sqrt{y} + \sqrt{z})(x + y - z + 2\sqrt{xy})$.
 1. $10a + b - c$. 2. $7a^2 - 9ab + 4b^2$. 3. $(p-a)x^2 + qxy$
 $+(b-r)y^2 - cz^2$. 4. $2a^2 + 13a - 17$.

- Page 457. — 5. $5.3x^2 + 2.8xy + 4.5y^2$. 6. $5a^2 + 12ab - 5b^2$.
 7. $12xy - 3xz + 3yz$. 8. $2x^6 + x^4 - 3x^2 - 10x - 8$.
 9. $15a^{2n} - 7a^nb^{-n} - b^{-2n}$. 10. $-3a^{2n} - 31a^nb - 27b^2$.
 11. $2x^4 + 7x^2 - 16 + 34x^{-2} - 15x^{-4}$. 12. $6x^{2n+3} - 17x^{2n+2}$
 $+ 24x^{2n+1} - 18x^{2n}$. 13. $\frac{1}{12}a^3 + \frac{59}{144}a^2 - \frac{1}{12}a + \frac{3}{16}$.
 14. $15x^{-2} + x^{-1} - 30 + 22x - 12x^2$. 15. $2x^3 - 3x^2 + x - 5$.
 16. $x^3 + 2x^2 - 6x + 5$. 17. $a^2 - ab - ac + b^2 - bc + c^2$.
 18. $x^{-2} + 2x^{-1}y^2 + 4y^4$. 19. $a + b$. 20. $6x^{-1} + 10 + 2x$.
 21. $e^x + e^{-x}$. 22. $a^2 - 8a - 20$, rem. -162 . 23. $x^2 - 14x - 52$,
 rem. -275 . 24. $2x^2 - x + 8$, rem. -4 . 25. $2x^2 + 4x + 15$,
 rem. 50. 26. $m = 25$.

- Page 458. — 1. $(c+9)(c-8)$. 2. $(3y+2)(y-3)$.
 3. $(3x^{2a}+5)(3x^{2a}+5)$. 4. $3m(r+11m)(r-11m)$.
 5. $(m-4n+2R)(m-4n-2R)$. 6. $8(2a^3+1)(4a^6-2a^3+1)$.
 7. $(x-y)(x^4+x^3y+x^2y^2+xy^3+y^4)$. 8. $(x^4+y^4)(x^8-x^4y^4+y^8)$.

- Page 458 (continued). — 9. $(a+2)(a^2-2a+4)(a-2)(a^2+2a+4)$.
 10. $(m+4n+4)(m+4n-4)$. 11. $(y+1)(y-2)(y-3)(y-5)$.
 12. $.01(7x^a+5y^n)(2x^a-3y^n)$. 13. $2(m-n-1)(m-n-1)$.
 14. $(x+2)(x-1)(x^2+x+1)$. 15. $(x-3)(x^2+6x-12)$.
 16. $(x^2+2xy-y^2)(x^2-2xy-y^2)$. 17. $a(2+a)(5-6a)$.
 18. $(x^2+y)(x^4-x^2y+y^2)$. 19. $(15a-2)(a-5)$.
 20. $(x-1)(x-1)(x+1)$. 21. $(x-1)(x+1)(x^2-3)$.
 22. $(2a+b+3)(2a-b-3)$. 23. $(6x-4+y)(6x-4-y)$.
 24. $x(6ax+3bx+1)(6ax+3bx-1)$.
 25. $(5x^2-8)(8-3x^2)$. 26. $(x-2y)(x+2y-1)$.
 27. $(7x+8)(2x-3)$. 28. $(x-1)(x-2)(x+2)(x-3)$.
 29. $(x+y)(x^2+2xy+y^2+1)$. 30. $(2m+3n+4)(2m+3n-4)$.
 31. $(ax-b)(cx+d)$. 32. $(x+2)(x^4-2x^3+4x^2-8x+16)$.
 33. $(x^2+2x+2)(x^2-2x+2)$. 34. $(a+2)(a^2-2a+4)$.
 35. $(m-n)(m-n+5)$.
 36. $(2x^2+3xy-3y^2)(2x^2-3xy-3y^2)$.
 37. $(a-1)(a-2)(a+2)(a-3)$. 38. $x^{m+1}(x^m-1)(x^{m-1}+1)$.
 39. $(x+.3)(x-.2)$.
 40. $(x^a+y)(x^{2a}-x^ay+y^2)(x^a-y)(x^{2a}+x^ay+y^2)$.
 41. $(x-2)(x+3)(x+4)$. 42. $(2a+b)(c+3d)$.
 43. $3x^2(x^a+3)(x^a-2)$. 44. $(5a^n+2b+c)(5a^n-2b-c)$.
 45. $(3a^2-2.5b)(3a^2-2.5b)$. 46. $(7a+2b^2)(3a-5b^2)$.
 47. $3(a+1)(a-2)(a+3)$. 48. $(6a+5b)(4a-7b)$.
 49. $(a-2b)(3a+b)(2a-b)$.
 50. $(5x+3y-2z)(5x-3y+2z)$. 51. $(x-1)(x-2)(x-5)$.
 52. $(x+7)(x+1)(x-1)(x^2-6)$.
 53. $(2a-2b-3c)(2a-2b-3c)$.

- Page 459. — 1. $\frac{x^3-12x^2-3x+36}{x^2-12x+27}$. 2. $\frac{3}{c+2}$. 3. $\frac{5}{16}$.
 4. $\frac{ab^2-a^3}{b^3}$. 5. 1. 6. $x-2$. 7. $\frac{5}{16+3x-3x^2}$.
 8. $a-3+3b$. 9. $\frac{1}{a-2}$. 10. $\frac{15a+20b}{12a-30b}$. 11. $\frac{x^2+y^2}{x}$.
 12. $\frac{a^2+6}{a^2-3a}$. 13. $\frac{1+x}{1-x+x^2-x^3}$. 14. $\frac{2bx}{4x^2-1}$.
 15. $3a^4-15a^3$. 16. $\frac{8x^2-4x+9}{3x^2+5x-2}$. 17. $x-3$.

- Page 460. — 18. 1. 19. $\frac{x+5}{x+3}$. 20. $\frac{a+b}{a-b}$.
 21. $\frac{4a-10}{a^2+3a+9}$. 22. $\frac{16-a}{6}$. 23. $\frac{x-2}{2x^2-7x+5}$.
 1. -36. 2. 8.4. 3. $\frac{32}{2}\sqrt{2}$. 4. $2\frac{1}{3}$. 5. $-6\frac{1}{2}$. 6. 5.2 .
 7. $\frac{1}{16}$. 8. $26-4\sqrt{30}$. 9. $-\frac{1}{3}\sqrt{10}$. 10. $10\sqrt{6}$.
 11. $\frac{4\sqrt{15}+6\sqrt{3}-2\sqrt{5}-3}{11}$.

- Page 461. — 12. $-\sqrt[3]{6}$. 13. 0. 14. $\frac{1}{2}\sqrt{2}$. 15. 15.
 16. (a) $\frac{8\sqrt{10}-23}{6}$; (b) $\frac{5+12\sqrt{-1}}{13}$;
 (c) $\frac{6+3\sqrt{-3}-2\sqrt{-2}+\sqrt{6}}{11}$. 17. (a) $\pm(5+2\sqrt{2})$;
 (b) $\pm(\frac{3}{2}\sqrt{6}+\frac{1}{2}\sqrt{2})$; (c) $\pm(2\sqrt{2}-3\sqrt{3})$. 18. (a) $\frac{a^{\frac{1}{2}}}{b}$; (b) $\frac{b^{\frac{3}{2}}}{a^{\frac{8}{3}}c}$. 19. 0.
 20. $\sqrt{3}$. 21. $6+\sqrt{6}$. 22. 42. 23. 34. 24. $\sqrt{3}+2\sqrt{15}-\sqrt{5}-8$.
 25. $74-12\sqrt{30}$. 26. $2\sqrt[5]{2}$. 27. $\frac{1}{2}\sqrt[5]{4}$. 28. $20\sqrt{6}$.
 29. $14+8\sqrt{10}$. 30. .63. 31. 6.65. 32. 1.8. 33. .23.

- Page 462. — 34. $\frac{22}{3}\sqrt{3}$. 35. 5.125372. 36. .2. 37. .8
 38. -5.3. 39. $29-5\sqrt{30}$. 40. $\frac{21\sqrt{5}+19\sqrt{3}}{2}$. 41. .3.
 1. $x=9$. 2. $x=-1$. 3. $x=6$. 4. $x=10$. 5. $k=25$.
 6. $x=-9$. 7. $x=-640$. 8. $y=3$. 9. $x=-.2$.
 10. $x=1.4$.

- Page 463. — 11. A, \$2990; B, \$2300. 12. 14.4 mi. 13. 29, 13.
 14. $37\frac{4}{5}$ mi. 15. 8 single seats; 16 double seats. 16. 19, 7.
 17. $1\frac{3}{4}$ hr. 18. 12 nickels, 6 quarters, 4 dimes. 19. 224 children.
 20. $28\frac{1}{2}$ min. 21. 26, 27, 28.

- Page 464. — 1. $x=\frac{1}{8}$. 4. Mr. Ash, \$2400; Mr. Bond, \$3000;
 Mr. Cox, \$2400; Mr. Dunn, \$4200. 5. $d=st$. 6. $\frac{d_1}{d_2}=\frac{h_2}{h_1}$.
 7. $\frac{d_1}{d_2}=\frac{t_1^2}{t_2^2}$. 8. $V=lwh$. 9. $\frac{p_1}{p_2}=\frac{v_1}{v_2}$. 10. $\frac{f_1}{f_2}=\frac{d_2}{d_1}$. 11. $\frac{c_1}{c_2}=\frac{s_2t_1}{s_1t_2}$.

- Page 465. — 12. $\frac{t_1}{t_2}=\frac{\sqrt{l_1}}{\sqrt{l_2}}$. 13. 21, 28. 14. 24, 40.
 15. Milk, $14\frac{2}{7}$ gal.; cream, $25\frac{5}{7}$ gal. 16. 4 qt. 17. $\frac{1}{8}$ gal.
 1. $x=5, y=4$. 2. $x=2.5, y=3$. 3. $x=\frac{2bd}{ad+bc}$,
 $y=\frac{bc-ad}{ad+bc}$. 4. $x=3, y=2, z=1$. 5. $x=2, y=1.5$.
 6. $x=.2, y=.3$. 7. $x=15, y=25$. 8. $x=8, y=4, z=2$.

- Page 466. — 9. 760 men, 440 women. 10. Hard coal, \$13 per
 ton; soft coal, \$6 per ton. 11. Charge for first 10 words, \$.35;
 charge for each additional word, \$.03. 12. Better coffee, \$.32 per
 pound; cheaper coffee, \$.28 per pound. 13. Sugar, \$.08; coffee, \$.30.
 14. 225, 56. 15. \$800. 16. 15 nickels, 3 quarters, 6 half dollars.

Page 467. — 1. $n = \frac{cR}{e - cr}$; $R = \frac{en - cnr}{c}$; $R = 1.375$.

2. $S = \frac{CR}{E - bC}$; $C = .12$. 3. $A = 52.95$. 4. $d = \frac{13.9 \text{ IV}^2}{2500 H}$;

$d = 4.63$. 5. $L = \frac{\sqrt{E^2 - C^2 R^2}}{2 \pi C n}$. 6. $P = 31.39$. 7. $d = 8.08$.

8. $v = \sqrt{2gh}$; $v = 40$. 9. $c = \frac{w^2}{59.6^2 L}$; $c = .001239$. 10. 1.5.

Page 468. — 11. $M = \frac{Lt + g}{t}$. 12. $t = \frac{a + 2s}{2a}$; $t = 13.5$.

13. $l = \frac{a + s(r - 1)}{r}$; $l = 384$. 14. $F = \frac{PQ}{P + Q}$; $F = 2.71$.

15. $h = r \pm \sqrt{r^2 - s^2}$; $h = 4$ or 16. 16. $C = 2$. 17. $r = \frac{E - CR}{C}$;
 $r = 2.8$. 18. $l = \frac{r^{n_8} - r^{n-1}s}{r^n - 1}$.

1. $81a^4 - 756a^3b + 2646a^2b^2 - 4116ab^3 + 2401b^4$. 2. $32x^{10}$
 $+ 80x^7y + 80x^4y^2 + 40xy^3 + \frac{10y^4}{x^2} + \frac{y^5}{x^5}$. 3. $a^{\frac{3}{2}} - 6ab^{\frac{1}{2}} + 12a^{\frac{1}{2}}b^{\frac{3}{2}}$
 $- 8b$. 4. $625a^{-2} + 300a^{-\frac{3}{2}}b^{\frac{2}{3}} + 54a^{-1}b^{\frac{4}{3}} + \frac{108a^{-\frac{1}{2}}b^2}{25} + \frac{81b^{\frac{8}{3}}}{625}$.

5. $128 - \frac{448x}{3} + \frac{224x^2}{3} - \frac{560x^3}{27} + \frac{280x^4}{81} - \frac{28x^5}{81} + \frac{14x^6}{729} - \frac{x^7}{2187}$.

6. $\frac{32x^5}{y^5} - \frac{80x^3}{3y^3} + \frac{80x}{9y} - \frac{40y}{27x} + \frac{10y^3}{81x^3} - \frac{y^5}{243x^5}$. 7. $\frac{15625a^6}{729}$
 $- \frac{12500a^4}{81} + \frac{12500a^2}{27} - \frac{20000}{27} + \frac{2000}{3a^2} - \frac{320}{a^4} + \frac{64}{a^6}$. 8. $256a^{16}$

$+ 512a^{13}b + 448a^{10}b^2 + 224a^7b^3 + 70a^4b^4 + 14ab^5 + \frac{7b^6}{4a^2} + \frac{b^7}{8a^5}$
 $+ \frac{b^8}{256a^8}$. 9. $2401a^8 - 6860a^6b^{\frac{3}{2}} + 7350a^4b^3 - 3500a^2b^{\frac{5}{2}} + 625b^6$.

10. $243a^{10} + 2025a^8b + 6750a^6b^2 + 11,250a^4b^3 + 9375a^2b^4 + 3125b^5$.

11. $4096x^6 - 18,432x^5y^2 + 34,560x^4y^4 - 34,560x^3y^6 + 19,440x^2y^8$
 $- 5832xy^{10} + 729y^{12}$. 12. $7776r^{10} + 19,440r^5 + 19,440 + 9720r^{-5}$
 $+ 2430r^{-10} + 243r^{-15}$.

13. (a) $a^{20} - 50a^{18}b + 1125a^{16}b^2 - 15,000a^{14}b^3$;
(b) $a^{15} + 30a^{14}b + 420a^{13}b^2 + 3640a^{12}b^3$;
(c) $x^{60} - 60x^{57}y + 1710x^{54}y^2 - 30,780x^{51}y^3$.

Page 469. — 14. (a) $924a^{12}b^6$; (b) $\frac{1120b^2}{243a^2}$; (c) $672a^6b^3$.

15. (a) 1.276282; (b) 1.265319; (c) 1.229874; (d) 1.171659.

16. \$281.71. 17. \$84.02. 18. $\pm(x^2 - x + \frac{1}{4})$.

19. $\pm(2x^2 - \frac{3}{2}xy - y^2)$. 20. $\pm(x - y + 5)$.

21. $\pm(4x^3 + x^2 - 3x - 2)$. 22. 336; 324.

1. $x = 7.4$ or -1.4 . 2. $m = .09$ or $-.26$. 3. $k = \pm 10$;
 $k > 10$ or $k < -10$; all values between -10 and 10 .

Page 470. — 4. $r = -24$; $r = 6\frac{1}{4}$. 5. Real, unequal, irrational,
negative root numerically greater. 6. No. 7. $x = 2.64$ or $-.44$.

8. $x = 1$ or $-\frac{9}{2}$. 9. $x = \pm 3$. 10. $x = 4$. 11. $x = 4$.
 12. $x = 4$ or $\frac{20}{9}$. 13. $x = \frac{1}{3}(1 \pm \sqrt{10})$, $\frac{1}{2}$, or -2 . 14. $x = 0$,
 2, or $1 \pm 4\sqrt{-2}$. 15. $x = 1, -1, \frac{1}{2}$, or $-\frac{1}{2}$. 16. $x = 1$ or $\frac{1}{4}$.
 17. $x = 1$. 18. $x = 8$ or -2 . 19. $x = \pm 2$. 20. $x = \frac{a}{2}(1 \pm \sqrt{5})$ or
 $\frac{a}{2}(1 \pm \sqrt{5})$. 21. $x = 3n$ or $-\frac{1}{2}n$. 22. $x = 1$. 23. $x = 5$.
 24. $x = \pm \sqrt{a^2 + 1}$ or $\pm \sqrt{a^2 + 1}$. 25. $x = \frac{3}{2}$ or $\frac{3}{2}$. 26. $x = 4$ or 1 .
 27. $x = 0$. 28. $x = 5$ or $\frac{1}{3}$. 29. $3x^2 - 4x + 5 = 0$; $x^2 + 4x - 2 = 0$.

Page 471. — 30. $x^2 - 2ax - 2bx + 2a^2 + 2b^2 = 0$; $4x^2 - 4mx + m^2 - n = 0$. 31. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$; (a) when $b^2 = 4ac$;
 (b) when $b^2 > 4ac$; (c) when $b^2 - 4ac$ is a square or equals zero;
 (d) when $b^2 - 4ac$ is a rational number but is not a square or does
 not equal zero; (e) when $4ac > b^2$. 32. $k = \pm 2$; $x = -7$ or -7 or
 $x = 5$ or 5 . 33. $k = \pm 4\sqrt{2}$. 34. Real, unequal, irrational, positive
 root numerically greater; product, $-\frac{5}{3}$; sum, $\frac{7}{3}$. 35. Imaginary;
 product, $\frac{1}{4}$; sum, $\frac{1}{2}$. 36. Real, unequal, irrational, positive; product,
 .5; sum, 1.8. 37. Real, unequal, rational, positive; product, 1;
 sum, $\frac{1}{8}$. 38. Imaginary; product, $\frac{8}{3}$; sum, 3. 39. Real, unequal,
 irrational, positive root numerically greater; product, $-\frac{4.16}{3}$; sum, .4.
 40. $A = \frac{1}{50}(9 + 50E \pm 3\sqrt{9 + 100E})$. 41. 1.3 in. by 1.3 in.

1. $\begin{cases} x = 3, -3; \\ y = 2, -2. \end{cases}$ 2. $\begin{cases} x = 3, 1, -1, -3; \\ y = 1, 3, -3, -1. \end{cases}$
 3. $\begin{cases} x = 1, -1, \frac{19}{89}\sqrt{89}, -\frac{19}{89}\sqrt{89}; \\ y = 2, -2, \frac{4}{89}\sqrt{89}, -\frac{4}{89}\sqrt{89}. \end{cases}$ 4. $\begin{cases} x = 3, -1, 1, -3; \\ y = 1, -3, 3, -1. \end{cases}$

- Page 472. — 5. $\begin{cases} x = 3, -2; \\ y = 5, -10. \end{cases}$ 6. $\begin{cases} x = \frac{3}{2}, -3; \\ y = 3, -\frac{3}{2}. \end{cases}$
 7. $\begin{cases} x = 4, -3, 3, -4; \\ y = 3, -4, 4, -3. \end{cases}$ 8. $\begin{cases} x = 3, -3, 2\sqrt{2}, -2\sqrt{2}; \\ y = 1, -1, \sqrt{2}, -\sqrt{2}. \end{cases}$
 9. $\begin{cases} x = 1, -1, 2\sqrt{2}, -2\sqrt{2}; \\ y = 2, -2, \frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}. \end{cases}$ 10. $\begin{cases} x = 1 + 2\sqrt{3}, 1 - 2\sqrt{3}; \\ y = 2 - \sqrt{3}, 2 + \sqrt{3}. \end{cases}$
 11. $\begin{cases} x = \frac{1}{5}, -\frac{1}{5}; \\ y = \frac{1}{5}, -\frac{1}{5}. \end{cases}$ 12. $x = 7, y = 10$. 13. $\begin{cases} x = 4, -4; \\ y = 1, -1. \end{cases}$
 14. $\begin{cases} x = 1, -1, \frac{1}{2}, -\frac{1}{2}; \\ y = 1, -1, \frac{5}{4}, -\frac{5}{4}. \end{cases}$ 15. $\begin{cases} x = 1, -4; \\ y = 1, -\frac{2}{3}. \end{cases}$ 16. \$50; 30 sets.
 17. Length, 25 rd.; width, 16 rd. 18. 110 volts, 5 amperes or $3\frac{1}{2}$ volts,
 165 amperes. 19. 81.

- Page 473. — 20. $\frac{1}{2}(\sqrt{h^2 + 4k^2} + \sqrt{h^2 - 4k^2})$, $\frac{1}{2}(\sqrt{h^2 + 4k^2} - \sqrt{h^2 - 4k^2})$.
 21. \$125; 6%. 22. Charles, 3 mi. an hour; John, 4 mi. an hour.
 23. Length, 50 ft.; width, 40 ft. 24. 3 yr.; $5\frac{1}{2}\%$.
 1. 5 yr. 3. $\frac{1}{8}$; $\frac{7}{8}$. 4. 104,850.

- Page 474. — 5. (a) $s = \frac{n}{2}(1 + n)$; (b) $s = \frac{5n}{2}(1 + n)$;
 (c) $s = 2(2^n - 1)$; (d) $s = \frac{(b^{2n} - 1)(a - 2b)}{b^2 - 1}$. 6. 16 ft.
 7. $33\frac{103}{406}$. 8. 6; $38\frac{1}{2}$. 9. 7. 10. \$3219. 11. \$4860.
 12. 3; Friday. 13. 55. 14. \$433.70.

- Page 475. — 15. $x = 6$. 16. 8 yr. 17. $D = 31.68$.
 18. 14.2 yr.; 14.1 yr. 19. $n = 182.9$. 20. $T = .3733$.
 21. $x = 3.4653$. 22. 6.512. 23. 1.256. 24. 1.970.
 25. 1.522. 26. .12. 27. .272.

- Page 478. — 10. $\begin{cases} x = 7, -8; \\ y = -4, 1. \end{cases}$ 11. $x = 4.7$ or 1.3 .
 12. $\begin{cases} x = 3, -5; \\ y = 4, -4. \end{cases}$ 13. $x = 3.4$ or $.6$. 14. $\begin{cases} x = 3, 3; \\ y = 5, 5. \end{cases}$ 15. $x = 2, y = 3$.

- Page 479. — 1. $A = 50^\circ 18'$; $B = 39^\circ 42'$; $c = 19.88$.
 2. $B = 61^\circ 30'$; $b = 16.39$; $c = 18.65$. 3. $B = 55^\circ$; $a = 41.14$;
 $c = 71.72$. 4. $A = 52^\circ 30'$; $a = 41.70$; $c = 52.56$. 5. $A = 33^\circ 21'$;
 $B = 56^\circ 39'$; $a = 25.06$. 6. $B = 59^\circ 40'$; $a = 16.94$; $b = 28.99$.
 7. $A = 24^\circ 20'$; $a = 10.47$; $b = 23.19$. 8. $A = 49^\circ 57'$; $B = 40^\circ 3'$;
 $b = 13.19$. 9. $14^\circ 30'$; $75^\circ 30'$. 11. 56.59. 12. 327.6. 13. 889.
 14. 413.1. 15. 952.4 ft. 16. $1^\circ 36'$. 17. 255.9 ft. 18. 76.7 ft.
 19. $29^\circ 6'$. 20. 252.4 ft.

- Page 480. — 1. (a) $(a^2 + b)(a^4 - a^2b + b^2)(a^2 - b)(a^4 + a^2b + b^2)$;
 (b) $3(2x - 3)(x + 2)$; (c) $(x^2 - .9)(x^2 - .9)$;
 (d) $(y + 1)(y - 2)(y - 4)$; (e) $a^m(a^3 + b^2)(a^3 + b^2)$;
 (f) $x^2(x + 1)(x - 1)$; (g) $(m^n - 9)(m^n - 2)$; (h) $(3p + q)(2p - q)$;
 (i) $(3a + b + c)(3a + b - c)$; (j) $(a + b)(a^2 + 1)$. 2. (a) $-\frac{1}{8}$,
 $\frac{3}{8}, 4\sqrt{-1}$; (b) $\frac{\sqrt{6}}{4}$. 3. $\frac{x^2 + 4}{4x}$. 4. (a) $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$;
 (b) sum, $-\frac{b}{a}$; product, $\frac{c}{a}$; (c) $x^2 + 4x - 21 = 0$. 5. 11 yd.
 6. $x = 13.6$ or $-.2$. 7. \$238.60.

- Page 481. — 8. $3\frac{1}{2}$ ft., $\frac{1}{2}$ ft. 9. 55 sheep. 10. $x = 3$.
 11. $\begin{cases} x = -10, 6; \\ y = 0, 8. \end{cases}$ 12. (a) $(3x - 4)(x + 6)$;
 (b) $(x + 2)(x^4 - 2x^3 + 4x^2 - 8x + 16)$; (c) $(x + .3)(x - .2)$;
 (d) $(x^a + y)(x^{2a} - x^ay + y^2)(x^a - y)(x^{2a} + x^ay + y^2)$;
 (e) $(x - 2)(x + 3)(x + 4)$; (f) $5(3a^2 + 2b^2)(9a^4 - 6a^2b^2 + 4b^4)$.
 13. $2x + 10 + \frac{6}{x}$. 14. (a) $\frac{1}{y^2}\sqrt{2}$; (b) $a\sqrt[3]{18}$,
 $\frac{6\sqrt{3} + 2\sqrt{6} + 3\sqrt{2} + 2}{10}$. 15. $x = 2.64$ or $-.44$.

- Page 482.** — 16. (a) $\left(3q + \frac{s}{12}\right)$ ft.; (b) $(d + .045dn)$ dollars;
 (c) $(100s - 100d - c)$ cents. 17. $\begin{cases} x = 3, 1, -1, -3; \\ y = 1, 3, -3, -1. \end{cases}$
 18. $x = 1$. 19. 134.5. 20. (a) Real, unequal, irrational, the negative root numerically greater; (b) no. 21. $6\frac{3}{16}$ da. 22. $x = 2.7$
 or $-.7$. 23. (a) $\frac{5\sqrt{2} + 2\sqrt{5}}{3}$; (b) $x = 4$. 24. $x = 1.32$ or
 -1.26 . 25. $\begin{cases} x = 3, 2; \\ y = 2, 3. \end{cases}$

- Page 483.** — 26. (a) $-10 + 7x^{-\frac{1}{3}} + 5x^{-\frac{2}{3}} + 5x^{\frac{1}{3}} - x^{\frac{2}{3}}$;
 (b) $\frac{7}{x^{\frac{1}{3}}} + \frac{5}{x^{\frac{2}{3}}} - 10 + 5x^{\frac{1}{3}} - x^{\frac{2}{3}}$; (c) $\frac{7}{x}\sqrt[3]{x^2} + \frac{5}{x}\sqrt[3]{x} - 10 + 5\sqrt[3]{x} - \sqrt[3]{x^2}$.
 27. (a) Real, unequal, rational, negative root numerically greater;
 (b) $6x^2 - 11x + 4 = 0$; (c) $k = \frac{1}{3}$. 28. (b) $\frac{2}{3}, -\frac{1}{3}, \frac{1}{6}, -\frac{1}{12}, \frac{1}{24}, -\frac{1}{48}$.
 29. (a) 446.1; (b) .06033. 30. 2 mi. an hour.
 31. Length, 20 rd.; width, 10 rd. 32. $\begin{cases} x = 4, -6; \\ y = \frac{8}{3}, 6. \end{cases}$
 33. (a) $5x^{2a}(x^a - 6)(x^a + 3)$; (b) $(x + 1)(x - 2)(x - 6)$;
 (c) $(3x^{3a} - 5)(3x^{3a} - 5)$; (d) $(y - .5)(y^2 + .5y + .25)$;
 (e) $(x + 6)(x - 1)(x - 2)(x - 3)$; (f) $(.04x^2 + 1)(.2x + 1)(.2x - 1)$.

- Page 484.** — 34. (a) $3\sqrt{2} - 2\sqrt{3}$; (b) $\frac{1}{16}, 3\frac{1}{16}$. 35. $x = 4$.
 36. (a) Imaginary; (b) real, unequal, rational, the positive root numerically greater;
 (c) real, equal, irrational, positive; (d) imaginary. 37. (a) $h = \frac{aL + gL}{g - a}$; (b) $h = 9.3$. 40. (a) 132.4; (b) 2162.
 41. $\begin{cases} x = 3, -1; \\ y = 1, -3. \end{cases}$ 42. \$100.

- Page 485.** — 43. 4.83. 44. (b) $x = 0$ or 2 ; (c) $x = 3.6$ or -1.6 .
 45. $\frac{a^2 - 2ab + b^2}{a^2 + b^2}$. 46. $\frac{1}{b}\sqrt{ab}$. 47. $x = 7, y = 9$.
 48. $\begin{cases} x = 4, -1, 1, -4; \\ y = 1, -4, 4, -1. \end{cases}$ 49. 4, 8, 16. 50. 20 bbl.; \$2.50.
 51. $m = 20$. 52. (a) $p^{\frac{1-n}{m-n}}q^{\frac{m-1}{m-n}}$; (b) $\frac{243}{4}, \frac{81}{2}, 27, 18, 12, 8$.

- Page 486.** — 53. (a) $(2x + 1)(3x - 4)$;
 (b) $(5 + 2x - 3y)(5 - 2x + 3y)$; (c) $(x + 2)(x - 3)(x - 8)$;
 (d) $a(2a - 5c^2)(4a^2 + 10ac^2 + 25c^4)$; (e) $x^{2a}(6x^{2a} - 5)(6x^{2a} - 5)$;
 (f) $(1 - y)(1 - z)$. 54. $2 + 9a^{\frac{2}{3}} + 9a^{-\frac{2}{3}}$; $40\frac{1}{4}$.
 55. (a) $x^2 - 6x + 7 = 0$; (b) real, unequal, rational, the negative root numerically greater.
 56. $x = 4.1$ or $.9$. 57. $x = 3$.
 58. \$1165.50. 59. $-\frac{256}{729}$. 60. .3176. 61. $n = 12.267$.
 62. $\begin{cases} x = 6, -2; \\ y = -2, 6. \end{cases}$ 63. 20 articles.

Page 487. — 64. $\begin{cases} x = 2, 5.5; \\ y = -3, 2.25. \end{cases}$

65. (a) $-3432 a^7 b^7$,

(b) $\frac{x}{128 y^7}$. 66. $a^{10} + 10 a^8 b + 40 a^6 b^2 + 80 a^4 b^3 + 80 a^2 b^4 + 32 b^5$.

67. $1 - \frac{15}{x} + \frac{375}{4x^2} - \frac{625}{2x^3} + \frac{9375}{16x^4} - \frac{9375}{16x^5} + \frac{15625}{64x^6}$. 68. $m^{12} - 6 m^{10} n^3$

$+ 15 m^8 n^6 - 20 m^6 n^9 + 15 m^4 n^{12}$.

69. $x = 1, \frac{-1 + \sqrt{-3}}{2}$, or

$\frac{-1 - \sqrt{-3}}{2}$. 70. $l = \frac{r^{n-1} s (r - 1)}{r^n - 1}$.

71. $x^{-3} + y^3$. 72. \$15,510.

73. $x = -3, y = 1, z = 5$.

74. $x = 2, y = 3, z = 1$.

75. (a) $\frac{1}{b} \sqrt{ab}$; (b) $-\frac{\sqrt{5+x^2}}{5+x^2}$.

Page 488. — 76. $\sqrt{3}$.

77. -1 .

78. $\begin{cases} x = \frac{1}{8}(9 + \sqrt{201}), \frac{1}{8}(9 - \sqrt{201}); \\ y = \frac{1}{6}(9 - \sqrt{201}), \frac{1}{6}(9 + \sqrt{201}). \end{cases}$

79. $x = -2, 1 + \sqrt{-3}$,

or $1 - \sqrt{-3}$. diagonal, 29.58 in.

80. $553\frac{1}{8}$ mi.

81. 5 in. by 15 in. by 25 in.;

82. \$200; 35%.

83. $x = 6.88$ or -1.55 .

84. 36.

and

0

$$a^2 + b^2 = c^2$$

0

hyperbola

$$x^2 - y^2 = 1$$

Parabola

$$y^2 = 4ax$$

C

$$\sqrt{1/10 - 20x^4 - 1} = \sqrt{1/10 - 20x^4 - 1}$$

$$\begin{aligned} & -5x^3 \\ & -20x^6 + 20x^4 \\ & -20x^{11} + 20x^9 \end{aligned}$$

$$\sqrt{1/10 - 1}$$

$$\begin{aligned} & -4x^3 + 10x - 11 \\ & -4x^3 + 10x - 11 \end{aligned}$$

$$C = a(n-1)d$$

$$S = \frac{n}{2}(a + l)$$

$$C = a(n-1)d$$

$$S = \frac{n}{2}(a + l)$$

2nd

9-10

10-11 2nd

11-12 Eng

1-2 lat

2-3 alg

3-4 long

2nd

2nd

9-10 2nd

10-11 2nd

11-12 Eng

1-2 long

2-3

3-4 alg

9-10 2nd

10-11

11-12 Eng

1-2 lat

2-3

3-4 long

2nd

9-10

10-11 2nd

11-12 Eng

1-2 long

2-3

3-4 alg

100 sheep 100

85

82

10

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